Higher rho-numbers and metric with positive scalar curvature

ROMA
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$$\mathcal{R}^+(M) := \left\{ g \text{ Riemannian metrics on } M \text{ such that } \mathrm{scal}(g) > 0 \right\}$$

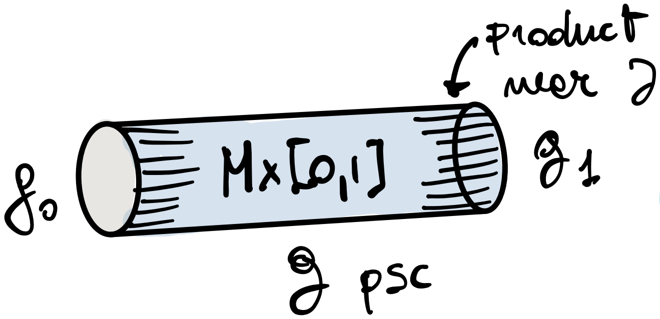
- SMOOTH and CLOSED
- $\dim M = n$ ODD
- M Spin

DEF

$$\frac{\mathrm{Vol}\left(B_x^{(M,g)}(r)\right)}{\mathrm{Vol}\left(B_0^{\mathbb{E}^n}(r)\right)} = 1 - \frac{\mathrm{scal}_x(g)}{6(n+2)} r^2 + O(r^4)$$

Q: How RICH is $\mathcal{R}^+(M)$?

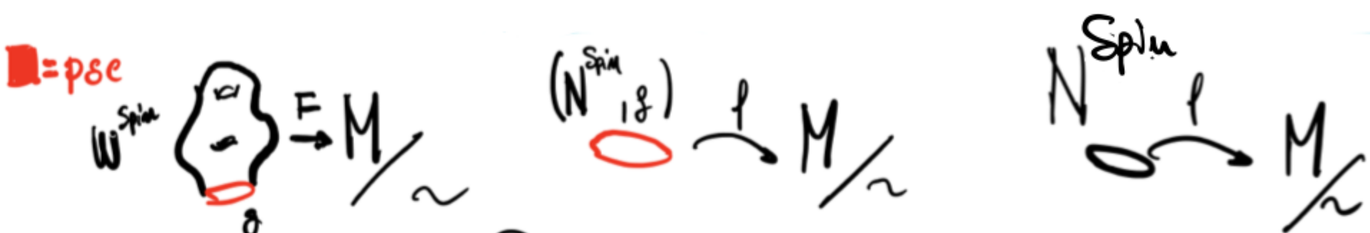
LET'S DEFINE THE FOLLOWING EQUIVALENCE RELATION



$\mathcal{P}^+(M) := \left\{ \text{CONCORDANCE CLASSES} \right\}$

$\mathcal{P}^+(M) \xleftrightarrow[\text{Stolz}]{} R^{Spin}_{n+1}(\pi_1(M))$ bordism group

Stolz sequence



$$\cdots \to R^{Spin}_{n+1}(M) \xrightarrow{\partial} Pos^{Spin}_{n}(M) \to \Omega^{Spin}_{n}(M) \to \cdots$$

$\mathrm{Diffeo}(M)$ acts on $\mathcal{P}^+(M)$

$$\mathrm{Diffeo}(M) \ni \varphi \rightsquigarrow \varphi \cdot [g] = [\varphi^* g]$$

$\boxed{\text{DEF}}$ $\widetilde{\mathcal{P}}^+(M) := \mathcal{P}^+(M) \big/ \mathrm{Diffeo}(M)$

CONCORDANCE MODULI SPACE

$$\widetilde{\mathcal{P}}^+(M) \twoheadrightarrow R^{Spin}_{n+1}(M)_{\mathrm{Diffeo}(M)}$$

U-Coinvariants

$$A_U := A \big/ \langle a - ua \rangle$$

DEF $\mathrm{vrk}\,\tilde{P}^{+}(M) := \sup_{[U : \mathrm{Diffeo}(M)] < +\infty} \mathrm{rk}\, R^{spin}_{n+1}(M)_{U}$

virtual rank

GOAL

ESTIMATING $\mathrm{vrk}\,\tilde{P}^{+}(M)$

TO THIS AIM WE WILL USE

HIGHER DELOCALIZED INVARIANTS

OF THE $\not{D}$IRAC OPERATORS ON M.

(M, g) Riemannian Spin $\rightsquigarrow$ $\mathcal{S}_g \to M$

spinor bundle

$\boxed{\text{DEF}}$ $\not{D}_g : s \mapsto \sum c(e_i) \nabla_{e_i} s \in C^\infty(M; \mathcal{S}_g)$

Elliptic differential $\Rightarrow$ Fredholm.

LICHNEROWICZ FORMULA

$$\not{D}_g^2 = \nabla^* \nabla + 1/4 \, \mathrm{scal}(g)$$

COR $\mathrm{scal}(g) > 0 \Rightarrow \not{D}_g$ invertible

$\Gamma \to \tilde{M} \to M$ UNIVERSAL COVERING

$\rightsquigarrow \tilde{D\!\!\!/}_g$ Γ-invariant lift

$\mathrm{scal}(g) > 0 \Rightarrow \dfrac{\tilde{D\!\!\!/}_g}{|\tilde{D\!\!\!/}_g|}$ well-defined unitary in $\Psi^0_\Gamma(\tilde{M})$ (Γ-equivariant ΨDOs)

Ex: Lott's delocalized η-invariants

$$\eta_{\langle x \rangle}(g) := \frac{1}{2} \sum_{\gamma \in \langle x \rangle} \int_{\mathcal{F}} \mathrm{Tr}\left(\frac{\tilde{D\!\!\!/}_g}{|\tilde{D\!\!\!/}_g|} (\tilde{m}\gamma, \tilde{m}) \right) d\,\mathrm{vol}_g(\tilde{m})$$

$\langle x \rangle$: conjugacy class of Γ

$\mathcal{F}$: fundamental domain

Q1: HOW DO WE REALIZE

$$\begin{aligned} \mathcal{P}^+(M) &\longrightarrow \mathbb{C} \\ [g] &\longmapsto \eta_{\langle x \rangle}(g) \end{aligned}$$

AS A "GROUP HOMOMORPHISM"?

Q2: HOW DO THESE (AND HIGHER ORDER) INVARIANTS BEHAVE W.R.T. THE ACTION OF $\mathrm{Diffeo}(M)$?

$$
\begin{tikzcd}
 & \text{Stolz} \arrow[d] & \text{Higson-Roe} \arrow[d] & \text{Connes} \arrow[d] & \\
 & R^{spin}_{n+1}(M) \arrow[r] \arrow[d] & K_{n+1}(C^*_r\Gamma) \arrow[r] \arrow[d] & HP_{n+1}(A\Gamma) \arrow[d] & {[\tau]} \\
\mathcal{R}^+(M) \arrow[r] \arrow[ur, bend left] & Pos^{spin}_n(M) \arrow[r, "\varrho"] \arrow[d] & S^\Gamma_n(\tilde{M}) \arrow[r, "Ch^{del}_*"] \arrow[d] & HP^{del}_{n+1}(A\Gamma) \arrow[d] & \times \; HP^{n+1}_{del}(\mathbb{C}\Gamma) \\
 & \Omega^{spin}_n(M) \arrow[r] \arrow[d] & K^\Gamma_n(\tilde{M}) \arrow[r] \arrow[d] & HP^e_n(A\Gamma) \arrow[d] & \\
 & {} & {} & {} &
\end{tikzcd}
$$

$$[g] \longmapsto \varrho(g) \longmapsto \langle Ch^{del}_*(\varrho(g)), \tau \rangle := \varrho_\tau(g) \in \mathbb{C}$$

DEFINITION/THEOREM (Z. 2019)

The Higson-Roe exact sequence is induced in K-theory by the following SES:

$$0 \to \Sigma \otimes C^*_\pi(\tilde{M} \times_\Gamma \tilde{M}) \to \left(C(M) \hookrightarrow \Psi^0_\Gamma(\tilde{M})\right) \to C_0(T^*M) \to 0$$

THEOREM/DEFINITION (BURGHELEA)

$$HP^*(\mathbb{C}\Gamma) \cong \bigoplus_{k \in \mathbb{N}} \left(H^{*+2k}(\Gamma) \times \prod_{\langle x \rangle \in \langle \Gamma \rangle^{fin}} H^{*+2k}(\Gamma_x) \right)$$

LOCALIZED COCYCLES

DELOCALIZED COCYCLES

centralizer

THEOREM (Piazza - Schick - Z. 2019 - 2021)

IF
- $\dim M \geq 5$ s.t. $\exists\, g$ with $\mathrm{scal}(g) > 0$
- $\Gamma := \pi_1(M)$ hyperbolic s.t. $|\mathrm{Out}(\Gamma)| < +\infty$

THEN
- $$\varrho_\tau(g) := \frac{1}{K!} \sum_{\substack{\gamma_0 \cdots \gamma_n \\ \cap \\ \langle c \rangle}} \int_{\tilde{M}^{n+1}} \mathrm{Tr}\left(\chi(\tilde{x}_0)\Pi_>(\tilde{x}_0\gamma_0, x_1)\chi(\hat{x}_1) \dots \chi(\tilde{x}_n)\Pi_>(\tilde{x}_n\gamma_n, \tilde{x}_0)\right) d\mathrm{vol}_g \cdot \tau(\gamma_0, \dots, \gamma_n)$$ is well-defined
- $$\mathrm{rk}\left(\tilde{P}^+(M)\right) \geq \sum_{\substack{k \geq 0 \\ p = 0,1}} \dim HC^{n+1-4k-2p}(\mathbb{C}\Gamma)$$

WORKING PROGRESS $(M, \mathcal{F})$ Foliation

Z. 2019

$$g_F \quad \text{PSC} \longrightarrow \rho(g_F) \in K_*\left(C^*_r\left(\mathrm{Mon}(M,\mathcal{F})^{[0,1)}_{ad}\right)\right)$$

$$\downarrow \mathrm{Ch}^{del}_S$$

complete transversal

étale groupoid

$$HP^{del}_*\left(\mathcal{A}\left(\mathrm{Mon}_S(M,\mathcal{F})\right)\right)$$

$$\times$$

$$HP^*_{del}\left(C^\infty_c\left(\mathrm{Mon}_S(M,\mathcal{F})\right)\right)$$

$\downarrow$

Well-defined?

$\rho_\tau(g_F) \in \mathbb{C}$ if the monodromy gpd is hyperbolic

- easy for Riemannian foliated bundles
- challenging for general foliations.

THANK YOU
FOR
THE ATTENTION