Holomorphic 1-forms on moduli of curves

Sara Torelli
Leibniz Universität Hannover

YPATIA 2022

8 June 2022

General problem

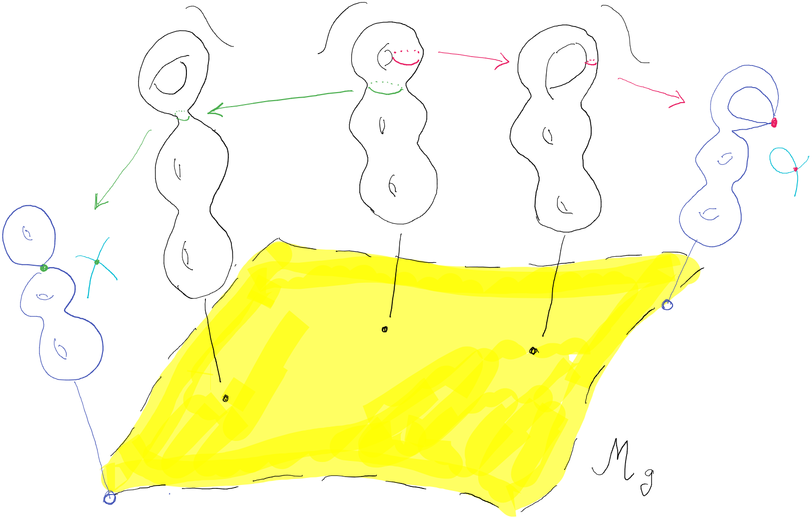
$\mathcal{M}_g$ (coarse) moduli space of smooth projective complex curves of genus g, $g \geq 2$

- It is a complex quasi-projective variety of dimension $3g-3$, a point $[C] \in \mathcal{M}_g$ is an ismomorphism class of a smooth projective curve C over $\mathbb{C}$ of genus g. Equivalently, a Riemann surface with g holes.
- It is singular on a sublocus of points $[C] \in \mathcal{M}_g$ such that $\mathrm{Aut}(C) \neq \{Id\}$. For $g > 3$, each component has dimension at most $2g-1$ (e.g. the hyperelliptic locus).

$\mathcal{M}_g^0 \subset \mathcal{M}_g$ smooth locus, $\Omega^1_{\mathcal{M}_g^0}$ holomorphic cotangent bundle.

Problem: Compute the dimension of the cohomology groups

- $H^k(\mathcal{M}_g^0, \mathbb{Z})$: k-forms with integer coefficients;
- $H^j(\mathcal{M}_g^0, \Omega^i_{\mathcal{M}_g^0})$: forms of type (i,j), i-holomorphic, j-antiholomorphic.


$\mathcal{M}_g$

From Pathology to Strategy

$\mathcal{M}_g^0$ smooth complex BUT not compact, so:

$$H^k(\mathcal{M}_g^0, \mathbb{C}) =^? \oplus_{i+j=k} H^j(\mathcal{M}_g^0, \Omega^i_{\mathcal{M}_g^0}) \oplus \oplus_{i+j=k} \overline{H^j(\mathcal{M}_g^0, \Omega^i_{\mathcal{M}_g^0})}$$

the Hodge decomposition doesn't follow from general theory!
Pathology on non-compact varieties:

$(\Omega^*_{\mathcal{M}_g^0}, d)$ holomorphic de Rham complex

1. $\ker\{d : H^0(\Omega^i_{\mathcal{M}_g^0}) \to H^0(\Omega^{i+1}_{\mathcal{M}_g^0})\} \subset H^0(\mathcal{M}_g^0, \Omega^i_{\mathcal{M}_g^0})$ might be proper.
   That is: i-holomorphic forms are not all closed.
2. $\mathrm{Im}\{d : H^0(\mathcal{O}_{\mathcal{M}_g^0}) \to H^0(\Omega^1_{\mathcal{M}_g^0})\} \neq 0$ a priori.
   That is: non constant holomorphic functions define forms.

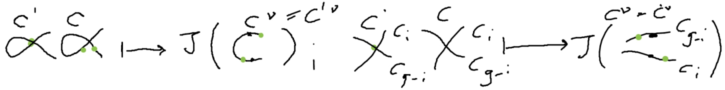
Aim: understand how far we are from having Hodge decomposition

From Pathology to Strategy

Strategy:

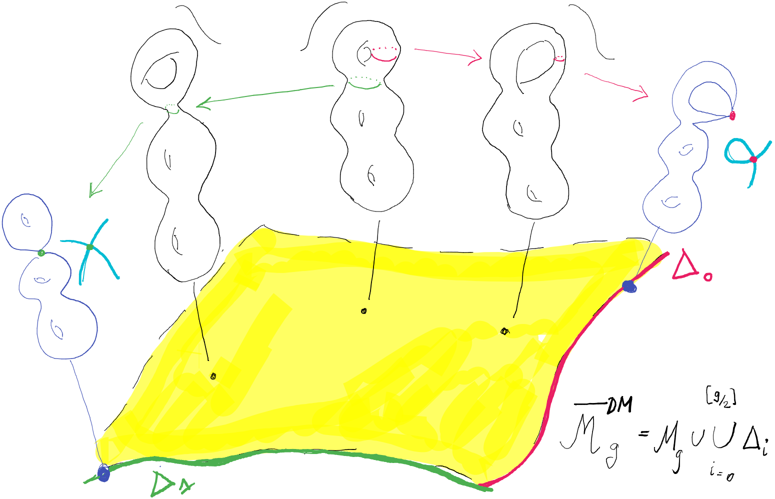
- Use a compactification $\overline{\mathcal{M}}_g = \mathcal{M}_g \cup \partial\mathcal{M}_g$ of $\mathcal{M}_g$ to recover the Hodge decomposition (at least on a resolution);
- Use intersection theory on its image $\mathcal{M}_g^L$ given by a big and nef line bundle L contracting the boundary $\partial\mathcal{M}_g$ to compare global sections with section defined outside the boundary.

$\overline{\mathcal{M}}_g^{DM}$ Deligne Mumford compactification, $\partial\mathcal{M}_g = \cup_{i=0}^{[g/2]}\Delta_i$,
A_g^{Sat} Satake compactification of A_g, $\psi : \mathcal{M}_g^{DM} \to A_g^{Sat}$,
$[C] \mapsto [JC^\nu]$, C^ν normalization of C, (extended) Torelli map.



$$\psi_L : \mathcal{M}_g^{DM} \to A_g^{Sat} \subset^H \mathbb{P}^n, \quad L = \psi_L^{-1}(\mathcal{O}_{\mathbb{P}^n}(1)) \in \mathrm{Pic}(\mathcal{M}_g^{DM})$$

is the composition of ϕ with the embedding given by the Hodge class H on A_g^{Sat}, is birational onto its image and contracts Δ_i as ψ does.



$$\overline{\mathcal{M}}_g^{DM} = \mathcal{M}_g \cup \bigcup_{i=0}^{[g/2]} \Delta_i$$

Result on comparison of sections and general attempt

Theorem (F.Favale, G.P. Pirola, – (2020))

X projective, $\mathcal{F}$ vector bundle over X, L big and nef line bundle, $D \in |L^a|$, $a >> 0$, E divisor contracted to points by $\phi_D : X \to \mathbb{P}^n$ (birational morphism onto its image). Then the following are equivalent:

1. $H^0(\mathcal{F}) \to H^0(\mathcal{F}_{|D})$ *isomorphism*
2. $H^0(\mathcal{F}) \to H^0(\mathcal{F}(mE))$ *isomorphism, for every* $m \geq 0$.

Attempt:

- apply the result to X $(i+1)$-subvariety of $\overline{\mathcal{M}}_g^{DM}$ intersection of multiples of H', $\mathcal{F} = \Omega^i_X$, $D = aH' \cap X$, $L = O_X(H')$
- prove the second condition $H^0(\Omega^i_X(mE)) = 0, \forall m \geq 0$ to get first $0 = H^0(\Omega^i_X) = H^0((\Omega^1_{X|D}))$
- start with a non zero $\eta \in H^0(\Omega^i_{\mathcal{M}^0_g})$ and use the above to find a contradiction.

Results: Holomorphic 1 forms and first Cohomology of $\mathcal{M}_g^0$

Case $k, i = 1$, $j = 0$ (Holomorphic 1 forms and first cohomology):

$$H^1(\mathcal{M}_g^0, \mathbb{C}) =^? H^0(\mathcal{M}_g^0, \Omega^1_{\mathcal{M}_g^0}) \oplus \overline{H^0(\mathcal{M}_g^0, \Omega^1_{\mathcal{M}_g^0})}$$

1. $\ker\{d : H^0(\Omega^1_{\mathcal{M}_g^0}) \to H^0(\Omega^2_{\mathcal{M}_g^0})\} \subset H^0(\mathcal{M}_g^0, \Omega^1_{\mathcal{M}_g^0})$ might be proper: there exist non-closed 1-holomorphic forms.
2. $\mathrm{Im}\{d : H^0(\mathcal{O}_{\mathcal{M}_g^0}) \to H^0(\Omega^1_{\mathcal{M}_g^0})\} \neq 0$: there exist exact holomorphic 1 forms.

- Mumford : $H^1(\mathcal{M}_g^0, \mathbb{Z})$ is torsion;
- Oort: $H^0(\mathcal{O}_{\mathcal{M}_g}) \simeq \mathbb{C}$ (so: $\mathrm{Im}\{H^0(\mathcal{O}_{\mathcal{M}_g^0}) \to^d H^0(\Omega^1_{\mathcal{M}_g^0})\} = 0$)

Theorem (F.Favale, G.P. Pirola, – (2020))

For $g \geq 5$, $H^0(\mathcal{M}_g^0, \Omega^1_{\mathcal{M}_g^0}) = 0$.

Proof: the attempt works for $i = 1$ because X is a surface and D is fully contained in Δ_1.

Results: Hol. 1 forms and first Cohomology on coverings

Setting: $\pi : \mathcal{M} \to \mathcal{M}_g$ finite covering such that $H^1(\mathcal{M}^0, \mathbb{C}) = 0$.

$$H^1(\mathcal{M}_g^0, \mathbb{C}) = 0 \neq^? H^0(\mathcal{M}_g^0, \Omega^1_{\mathcal{M}_g^0}) \oplus \overline{H^0(\mathcal{M}_g^0, \Omega^1_{\mathcal{M}_g^0})}$$

1. $\ker\{d : H^0(\Omega^1_{\mathcal{M}_g^0}) \to H^0(\Omega^2_{\mathcal{M}_g^0})\} \subset H^0(\mathcal{M}_g^0, \Omega^1_{\mathcal{M}_g^0})$ proper (i.e. there exist non-closed 1-holomorphic forms) ?

Interesting cases

- $\pi : \mathcal{M}_g[m] \to \mathcal{M}_g$ curves plus m-level structure;
- $\pi : \mathcal{R}_g \to \mathcal{M}_g$ curves plus a two-torsion point;
- $\pi : \mathcal{S}_g^+ \to \mathcal{M}_g$ curves plus an even θ-characteristic.

Theorem (F.Favale, J.C. Naranjo, G.P. Pirola, – (2022))

$H^0(\mathcal{M}^0, \Omega^1_{\mathcal{M}^0}) = 0$ for $g \geq 5$, when there is a compactification $\overline{\mathcal{M}}$ of $\mathcal{M}$ and covering extension $\overline{\pi} : \overline{\mathcal{M}} \to \overline{\mathcal{M}}_g^{DM}$ ramified at most outside Δ_1. For instance, for $\mathcal{M}$ one of the interesting cases.

What about higher k, i?

For higher k, i, j almost everything is open!

Generalities:

- $H^k(\mathcal{M}_g^0, \mathbb{Z})$ is known only for low k;
- The divisor D in the attempt is much more complicated so it is not clear how to find a contradiction;
- the attempt only solved the case $j = 0$.

Questions: Assume $H^k(\mathcal{M}_g^0, \mathbb{Z})$ known.

- Is the attempt just harder to compute or not conclusive?
- Ideas for $j \neq 0$?

....to be continued... Thanks for the attention!