Fano varieties as zero loci

YPATIA 2022

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Rome, 10 June 2022

Fano varieties

A smooth complex algebraic variety X is called Fano if its anticanonical bundle

$$-K_X = \wedge^{\dim X} T_X$$

is ample, i.e., $-nK_X$ gives a closed immersion $X \hookrightarrow \mathbb{P}^N$ for large n.

Why are they interesting? In part because they are

- very common: projective spaces, quadrics, complete intersections of small degree...;
- key ingredients for the Minimal Model Program, the birational classification of algebraic varieties;
- limited in number up to deformation for every fixed dimension.

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Fano varieties in low dimension

There exist up to deformation

- 1 Fano variety in dimension 1, $\mathbb{P}^1$;
- 10 families in dimension 2, Del Pezzo surfaces: $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathrm{Bl}_k \mathbb{P}^2$ for $0 \leq k \leq 8$;
- 105 families in dimension 3.

Things escalate quickly: from dimension 4 on, no complete classification.

Partial classifications available for particular (big) values of the index ι_X, the maximal integer s.t. $K_X = \iota_X L$ for some ample line bundle L.

Big databases are currently being produced, e.g., for Fano 4folds. Far from being complete.

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A deeper look at dimension 3

Among the 105 families of Fano 3folds:

- 17 have Picard rank 1.
  - Iskovskikh: birational technique, via double projections from a line;
  - Mukai: biregular technique, via the vector bundle method.
- 88 have higher Picard rank.
  - Mori–Mukai: using birational Mori's theory of extremal rays.

Question: can we provide biregular models for all Fano 3folds?

Answer: yes. In '16 Coates–Corti–Galkin–Kasprzyk describe them as zero loci of vector bundles on GIT quotients by products of general linear groups.

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Zero loci of sections of vector bundles

Zero loci

$\pi : E \to X$ vector bundle over X and $s : X \to E$ a section, i.e., $\pi \circ s = id_X$.
The zero locus $\mathscr{Z}(s) \subset X$ is made up of the points p where $s(p) = 0$.

Ubiquitous in algebraic geometry. Very good control on the geometry of $\mathscr{Z}(s)$ when s is general and E is globally generated:

- $\text{smooth, codim}\, \mathscr{Z}(s) = \operatorname{rank} E$,
- $K_{\mathscr{Z}(s)} = (K_X + \det E)|_{\mathscr{Z}(s)}$,
- Koszul complex resolving $\mathcal{O}_{\mathscr{Z}(s)}$.

Moreover, combinatorics often occurs (toric world, homogeneous varieties).

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Question: do Fano 3folds admit a biregular description as $\mathscr{Z}(s)$ for E a homogeneous vector bundle over a product G of Grassmannians or flag varieties?

De Biase–Fatighenti–T. '21

Yes, to some extent.
For 102 families of Fano there exists a description with no weighted factor in G.
For 85 families of Fano there exists a description with E completely reducible.

Motivating example: $\mathrm{Bl}_p\,\mathbb{P}^3$ is $\mathscr{Z}(s)$ for $E = \mathcal{Q}_{\mathbb{P}^2} \otimes \mathcal{O}(0,1)$ on $\mathbb{P}^2 \times \mathbb{P}^3$.

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For any smooth projective variety X, consider its Hochschild cohomology $\mathrm{HH}^i(X)$. The Hochschild–Kostant–Rosenberg decomposition yields, for $i = 0 \dots 2 \dim X$,

$$\mathrm{HH}^i(X) \cong \oplus_{p+q=i} H^p(X, \wedge^q T_X).$$

Problem: determine the Hochschild cohomology of Fano 3folds.

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Used both biregular and birational descriptions of Fano 3folds to determine it.

So, having biregular models for Fano 3folds allows us to do something new.

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New results for old varieties...

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...and brand new examples!

Why don't we venture into the unknown, i.e., in higher dimension?

A Fano variety of K3 type (FK3) is a Fano variety X s.t. $\mathrm{H}^*(X, \mathbb{C})$ contains at least a sub-Hodge structure of K3 type.

Deep link with hyperkähler geometry!

Bernardara–Fatighenti–Manivel–T. '21

Found and studied 64 families of FK3 of dimension 4.
Most of them are related to cubic fourfolds, Gushel–Mukai fourfolds, or actual K3 surfaces. Some of them remain mysterious.

Examples extracted from a database under construction containing at least 634 different Fano fourfolds, obtained as zero loci inside products of flag varieties.

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Extend the search to other varieties: trivial canonical bundle?

Extend the search to wider classes of varieties: degeneracy loci of morphisms between vector bundles? Orbital degeneracy loci?

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