Unitary vertex algebras and Wightman conformal field theories

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Mathematical Quantum Field Theory

- Quantum mechanics: Hilbert space, Hamiltonian (a specific self-adjoint operator), spectral analysis, observables (self-adjoint operators)...
- Quantum field theory (QFT): infinite degrees of freedom on continuum configulation space (infrared and ultraviolet difficulties)
- Axiomatic approaches: **Wightman**, Osterwalder-Schrader, Araki-Haag-Kastler.
- Examples: free fields, $\mathcal{P}(\phi)_2$ models (and more "(super)renormalizable" models), some gauge theories in $d = 1+1, 1+2$, ϕ^4_3 model, integrable models in $d = 1+1$, **conformal field theories (CFT) in** $d = 1+1$.
- In physics, CFTs capture universal properties of larger classes of QFT.
- Two-dimensional CFTs have infinite dimensional symmetries, **many examples** and purely algebraic axiomatization (**Vertex (operator) algebras**).

Two-dimensional chiral conformal field theory

- In relativistic QFT in $d = 1 + 1$, one puts the Lorentzian metric $(x, y) = x_0 y_0 - x_1 y_1$ on $\mathbb{R}^2$.
- The conformal group (transformations of $\mathbb{R}^2$ which preserve the metric up to a function) is $\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$, acting on the lightrays $x_0 \pm x_1 = 0$.
- In a quantum theory, $\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ gets a (projective) unitary representation.
- There are observables that are invariant by $\iota \times \mathrm{Diff}(\mathbb{R})$ (or $\mathrm{Diff}(\mathbb{R}) \times \iota$): chiral observables.
- Chiral observables are **quantum fields living on the lightray** $\mathbb{R}$. By conformal symmetry, they **extend to** the one-point compactification (**the circle** S^1 under the stereographic projection, and have $\mathrm{Diff}(S^1)$ as the symmetry group).
- Many examples: free fields (boson/fermion), $\mathrm{Diff}(S^1)$-symmetry itself (the Virasoro algebra), the WZW models (loop groups).

Axiomatic approaches to 2d CFT

- Wightman fields:
  - Operator-valued distributions ϕ. For $f \in C^\infty(S^1, \mathbb{R})$, $\phi(f)$ gives an (unbounded) operator on a Hibert space $\mathcal{H}$.
  - **Locality**: $[\phi(f), \phi(g)] = 0$ if $\mathrm{supp}\, f \cap \mathrm{supp}\, g = \emptyset$, **Möbius** covariance, spectrum condition, vacuum...
- Vertex (operator) algebras:
  - Algebra generated by formal series $Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, V a linear space, $a \in V$ and $a_{(n)} \in \mathrm{End}(V)$.
  - **Locality**: $[Y(a, w), Y(b, z)](w - z)^N = 0$ where N depends on a, b, **Möbius** covariance, grading, vacuum...
- (Conformal (Araki-Haag-Kastler) nets:
  - Family of operator algebras $\mathcal{A}(I)$ parametrized by intervals $I \subset S^1$.
  - (Isotony), locality, covariance, grading, vacuum...)
- Many examples have been constructed in all of these axioms, separately.
- What are the relations between axioms? (cf. Kac, Fredenhagen-Jörß, Carpi-Kawahigashi-Longo-Weiner under some technical conditions)

Examples

- $\mathrm{Diff}(S^1)$: infinite dimensional Lie group with $\mathrm{Vect}(S^1)(\cong C^\infty(S^1,\mathbb{R}))$ as the Lie algebra: $[f,g] = f'g - fg'$. Complexification $\mathrm{Vect}(S^1,\mathbb{C})$ contains a dense subalgebra of trigonometric polynomials $L_n(e^{i\theta}) = e^{in\theta}$: $[L_m, L_n] = (m+n)L_{m+n}$ (the Witt algebra)
- The Witt algebra admits a (unique) central extension, the **Virasoro algebra**:

$$[L_m, L_n] = (m+n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}$$

- The Virasoro algebra admits the vacuum representation V (the lowest weight representation with the trivial lowest weight, with the lowest weight vector Ω) with positive-definite invariant sesquilinear form for some $c > 0$.
- Wightman field: $L(f) = \sum_n L_n \hat{f}_n$, $\hat{f}_n = \int f(e^{i\theta})e^{-in\theta}d\theta$
- Vertex (operator) algebra: $Y(\nu, z) = \sum_n L_n z^{-n-2}$.
- (Conformal net: $\mathcal{A}(I) = \{e^{iL(f)} : \mathrm{supp}\, f \subset I\}''$)

Equivalence between VA and W with UBO (Raymond-T.-Tener, to appear in *Commun. Math. Phys.*)

- Eigenspaces of L_0 are assumed to be finite-dimensional.
- From Unitary vertex algebras to Wightman fields:
  - Unitarity: scalar product $\langle \cdot, \cdot \rangle$.
  - Formal power series $Y(a, z) = \sum_n a_{(n)} z^{-n-1}$.
  - Convergence? $\sum_n a_{(n)} \hat{f}_n \Phi, \Phi \in V$
  - Automatic estimate (**uniformly bounded order** (UBO)): $|\langle a_{1,(n_1)} \cdots a_{k,(n_k)} \Phi, \Phi' \rangle| \leq p_{\Phi,\Phi',a_1,\cdots,a_k}(n_1, \cdots n_k)$, where p is a polynomial whose degree is independent of Φ, Φ' (cf. polynomial energy bounds, Carpi-Kawahigashi-Longo-Weiner).
  - (proof: conformal (Möbius) covariance, decomposition of V into irreducible representations (quasi-primary fields))
  - $\sum_n v_{(n)} \hat{f}_n \Phi, \Phi \in V$ converges in the Hilbert space completion.
- From Wightman fields **with UBO**:
  - $\{\phi\}$: generating quantum fields satisfying UBO
  - Fourier components $\phi_n = \phi(f_n), f_n(e^{i\theta}) = e^{in\theta}$.
  - Locality in W $[\phi^1(f), \phi^2(g)]\Phi = 0$ + UBO $\Rightarrow$ Locality in VA $[\phi^1(w), \phi^2(z)](w - z)^N = 0$ for some N.

From fields to conformal nets

- ϕ: conformal Wightman field on S^1.
- When does $e^{i\phi(f)}$ make sense, and when is it local?
- $\phi(f)$ should be **self-adjoint as an unbouded operator**, and $\phi(f)$ and $\phi(g)$ should **commute strongly** (their spectral projections should commute).
- Spectral problem!
- Usually solved by a "linear energy bound" (the Nelson-Glimm-Jaffe commutator theorem).
- For many conformal fields, linear energy bound fails.
- When ϕ satisfies **local energy bound**, then it is strongly local (Carpi-T.-Weiner '22, *Commun. Math. Phys.*).
- Local energy bounds can be derived from an optimal bound $\|\phi_n\Phi\| \leq C\|(L_0 + \mathbb{1})^{d-1}\Phi\|$, where d is the conformal dimension of ϕ (an **algebraic problem**).

Outlook

- Maybe VOA, Wightman, Araki-Haag-Kastler are equivalent?
    - Removing UBO?
    - Local energy bounds automatic?
- Axiomatization of full 2d CFT?
- Non-conformal QFT from 2d CFT?