An introduction on maximally hypoelliptic differential operators

Omar Mohsen
Université Paris-Saclay

8 June 2022

Differential operators

Let

$$D : C^\infty(M) \to C^\infty(M)$$

be a differential operator.

Differential operators

Let

$$D : C^\infty(M) \to C^\infty(M)$$

be a differential operator. Locally

$$D = \sum_{|I| \leq l} f_I \partial_{x_I}$$

Differential operators

Let

$$D : C^\infty(M) \to C^\infty(M)$$

be a differential operator. Locally

$$D = \sum_{|I| \leq l} f_I \partial_{x_I}$$

We are interested in

$$Du = v$$

Differential operators

Let

$$D : C^\infty(M) \to C^\infty(M)$$

be a differential operator. Locally

$$D = \sum_{|I| \leq l} f_I \partial_{x_I}$$

We are interested in

$$Du = v$$

Definition

We say that D is hypoelliptic if for any distribution u

Du is smooth $\Rightarrow$ u is smooth

Differential operators

Let

$$D : C^\infty(M) \to C^\infty(M)$$

be a differential operator. Locally

$$D = \sum_{|I| \leq l} f_I \partial_{x_I}$$

We are interested in

$$Du = v$$

Definition

We say that D is hypoelliptic if for any distribution u

$$Du \text{ is smooth} \Rightarrow u \text{ is smooth}$$

- ∂_x on $\mathbb{R}$ is hypoelliptic

Differential operators

Let

$$D : C^\infty(M) \to C^\infty(M)$$

be a differential operator. Locally

$$D = \sum_{|I| \leq l} f_I \partial_{x_I}$$

We are interested in

$$Du = v$$

Definition

We say that D is hypoelliptic if for any distribution u

Du is smooth $\Rightarrow$ u is smooth

- ∂_x on $\mathbb{R}$ is hypoelliptic
- ∂_x on $\mathbb{R}^2$ isn't

Sobolev spaces

$$H^0(\mathbb{R}^n) := L^2_{loc}(\mathbb{R}^n)$$
$$H^{k+1}(\mathbb{R}^n) := \{f \in H^k(\mathbb{R}^n) : \partial_{x_1}(f), \cdots, \partial_{x_n}(f) \in H^k(\mathbb{R}^n)\}$$

Sobolev spaces

$$H^0(\mathbb{R}^n) := L^2_{loc}(\mathbb{R}^n)$$
$$H^{k+1}(\mathbb{R}^n) := \{f \in H^k(\mathbb{R}^n) : \partial_{x_1}(f), \cdots, \partial_{x_n}(f) \in H^k(\mathbb{R}^n)\}$$

We have $\cdots \subseteq H^1(\mathbb{R}^n) \subseteq H^0(\mathbb{R}^n)$

Sobolev spaces

$$H^0(\mathbb{R}^n) := L^2_{loc}(\mathbb{R}^n)$$
$$H^{k+1}(\mathbb{R}^n) := \{f \in H^k(\mathbb{R}^n) : \partial_{x_1}(f), \cdots, \partial_{x_n}(f) \in H^k(\mathbb{R}^n)\}$$

We have $\cdots \subseteq H^1(\mathbb{R}^n) \subseteq H^0(\mathbb{R}^n)$

Theorem (Sobolev)

The intersection $\cap_k H^k(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$

Sobolev spaces

$$H^0(\mathbb{R}^n) := L^2_{loc}(\mathbb{R}^n)$$
$$H^{k+1}(\mathbb{R}^n) := \{f \in H^k(\mathbb{R}^n) : \partial_{x_1}(f), \cdots, \partial_{x_n}(f) \in H^k(\mathbb{R}^n)\}$$

We have $\cdots \subseteq H^1(\mathbb{R}^n) \subseteq H^0(\mathbb{R}^n)$

Theorem (Sobolev)

The intersection $\cap_k H^k(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$

Suppose that $\phi : \mathbb{N} \to \mathbb{N}$ such that

- $\phi(k) \to +\infty$ as $k \to +\infty$.

Sobolev spaces

$$H^0(\mathbb{R}^n) := L^2_{loc}(\mathbb{R}^n)$$
$$H^{k+1}(\mathbb{R}^n) := \{f \in H^k(\mathbb{R}^n) : \partial_{x_1}(f), \cdots, \partial_{x_n}(f) \in H^k(\mathbb{R}^n)\}$$

We have $\cdots \subseteq H^1(\mathbb{R}^n) \subseteq H^0(\mathbb{R}^n)$

Theorem (Sobolev)

The intersection $\cap_k H^k(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$

Suppose that $\phi : \mathbb{N} \to \mathbb{N}$ such that

- $\phi(k) \to +\infty$ as $k \to +\infty$.
- for any distribution u and any $k \in \mathbb{N}$

$$Du \text{ is in } H^k(M) \Rightarrow u \text{ is in } H^{\phi(k)}(M)$$

Sobolev spaces

$$H^0(\mathbb{R}^n) := L^2_{loc}(\mathbb{R}^n)$$
$$H^{k+1}(\mathbb{R}^n) := \{f \in H^k(\mathbb{R}^n) : \partial_{x_1}(f), \cdots, \partial_{x_n}(f) \in H^k(\mathbb{R}^n)\}$$

We have $\cdots \subseteq H^1(\mathbb{R}^n) \subseteq H^0(\mathbb{R}^n)$

Theorem (Sobolev)

The intersection $\cap_k H^k(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$

Suppose that $\phi : \mathbb{N} \to \mathbb{N}$ such that

- $\phi(k) \to +\infty$ as $k \to +\infty$.
- for any distribution u and any $k \in \mathbb{N}$

$$Du \text{ is in } H^k(M) \Rightarrow u \text{ is in } H^{\phi(k)}(M)$$

Then D is hypoelliptic.

Sobolev spaces

$$H^0(\mathbb{R}^n) := L^2_{loc}(\mathbb{R}^n)$$
$$H^{k+1}(\mathbb{R}^n) := \{f \in H^k(\mathbb{R}^n) : \partial_{x_1}(f), \cdots, \partial_{x_n}(f) \in H^k(\mathbb{R}^n)\}$$

We have $\cdots \subseteq H^1(\mathbb{R}^n) \subseteq H^0(\mathbb{R}^n)$

Theorem (Sobolev)

The intersection $\cap_k H^k(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$

Suppose that $\phi : \mathbb{N} \to \mathbb{N}$ such that

- $\phi(k) \to +\infty$ as $k \to +\infty$.
- for any distribution u and any $k \in \mathbb{N}$
$$Du \text{ is in } H^k(M) \Rightarrow u \text{ is in } H^{\phi(k)}(M)$$

  Then D is hypoelliptic.

If D is of order l, then $\phi(k) \leq k + l$.

Sobolev spaces

$$H^0(\mathbb{R}^n) := L^2_{loc}(\mathbb{R}^n)$$
$$H^{k+1}(\mathbb{R}^n) := \{f \in H^k(\mathbb{R}^n) : \partial_{x_1}(f), \cdots, \partial_{x_n}(f) \in H^k(\mathbb{R}^n)\}$$

We have $\cdots \subseteq H^1(\mathbb{R}^n) \subseteq H^0(\mathbb{R}^n)$

Theorem (Sobolev)

The intersection $\cap_k H^k(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$

Suppose that $\phi : \mathbb{N} \to \mathbb{N}$ such that

- $\phi(k) \to +\infty$ as $k \to +\infty$.
- for any distribution u and any $k \in \mathbb{N}$

$$Du \text{ is in } H^k(M) \Rightarrow u \text{ is in } H^{\phi(k)}(M)$$

  Then D is hypoelliptic.

If D is of order l, then $\phi(k) \leq k + l$. If $\phi(k) = k + l$, we say D is elliptic.

Principal symbol

Locally

$$D = \sum_{|I| \leq l} f_I \partial_{x_I}$$

Principal symbol

Locally

$$D = \sum_{|I|\leq l} f_I \partial_{x_I}$$

Principal symbol

$$\sigma(x,\xi) = i^l \sum_{|I|=l} f_I(x)\xi_I$$

replace each ∂_x with $i\xi$

Principal symbol

Locally

$$D = \sum_{|I| \leq l} f_I \partial_{x_I}$$

Principal symbol

$$\sigma(x, \xi) = i^l \sum_{|I| = l} f_I(x) \xi_I$$

replace each ∂_x with $i\xi$

Globally

$$\sigma : T^*M \to \mathbb{C}$$

Principal symbol

Locally

$$D = \sum_{|I| \leq l} f_I \partial_{x_I}$$

Principal symbol

$$\sigma(x, \xi) = i^l \sum_{|I|=l} f_I(x) \xi_I$$

replace each ∂_x with $i\xi$
Globally

$$\sigma : T^*M \to \mathbb{C}$$

Example $\sigma(\partial_x^2 + \partial_y^2) = -\xi^2 - \eta^2$

Principal symbol

Locally

$$D = \sum_{|I| \leq l} f_I \partial_{x_I}$$

Principal symbol

$$\sigma(x, \xi) = i^l \sum_{|I|=l} f_I(x) \xi_I$$

replace each ∂_x with $i\xi$
Globally

$$\sigma : T^*M \to \mathbb{C}$$

Example $\sigma(\partial_x^2 + \partial_y^2) = -\xi^2 - \eta^2 = \sigma(\partial_x^2 + \partial_y^2 + f(x)\partial_x)$

Principal symbol

Locally

$$D = \sum_{|I| \leq l} f_I \partial_{x_I}$$

Principal symbol

$$\sigma(x, \xi) = i^l \sum_{|I| = l} f_I(x) \xi_I$$

replace each ∂_x with $i\xi$

Globally

$$\sigma : T^*M \to \mathbb{C}$$

Example $\sigma(\partial_x^2 + \partial_y^2) = -\xi^2 - \eta^2 = \sigma(\partial_x^2 + \partial_y^2 + f(x)\partial_x)$

Theorem (Kohn, Nirenberg, Hörmander, $\cdots$)

Let D be of order l. Then TFAE

1. *for any k and any u, $Du \in H^k(M)$ implies $u \in H^{k+l}(M)$.*

Principal symbol

Locally

$$D = \sum_{|I| \leq l} f_I \partial_{x_I}$$

Principal symbol

$$\sigma(x, \xi) = i^l \sum_{|I|=l} f_I(x) \xi_I$$

replace each ∂_x with $i\xi$
Globally

$$\sigma : T^*M \to \mathbb{C}$$

Example $\sigma(\partial_x^2 + \partial_y^2) = -\xi^2 - \eta^2 = \sigma(\partial_x^2 + \partial_y^2 + f(x)\partial_x)$

Theorem (Kohn, Nirenberg, Hörmander, $\cdots$)

Let D be of order l. Then TFAE

1. *for any k and any u, $Du \in H^k(M)$ implies $u \in H^{k+l}(M)$.*
2. *for any x and $\xi \neq 0$, $\sigma(x, \xi) \neq 0$.*

Kolmogorov operator

We saw

$$\text{elliptic} \Rightarrow \text{hypoelliptic}$$

Kolmogorov operator

We saw

$$\text{elliptic} \Rightarrow \text{hypoelliptic}$$

Consider

$$D = \partial_x^2 + x^2 \partial_y^2$$

D is hypoelliptic but not elliptic.

Folland and Stein Sobolev spaces

$$\tilde{H}^0(\mathbb{R}^2) := L^2_{loc}(\mathbb{R}^2)$$
$$\tilde{H}^{k+1}(\mathbb{R}^2) := \{f \in H^k(\mathbb{R}^2) : \partial_x(f), x\partial_y(f) \in \tilde{H}^k(\mathbb{R}^2)\}$$

Folland and Stein Sobolev spaces

$$\tilde{H}^0(\mathbb{R}^2) := L^2_{loc}(\mathbb{R}^2)$$
$$\tilde{H}^{k+1}(\mathbb{R}^2) := \{f \in H^k(\mathbb{R}^2) : \partial_x(f), x\partial_y(f) \in \tilde{H}^k(\mathbb{R}^2)\}$$

The identity

$$[\partial_x, x\partial_y] = \partial_y$$

Folland and Stein Sobolev spaces

$$\tilde{H}^0(\mathbb{R}^2) := L^2_{loc}(\mathbb{R}^2)$$
$$\tilde{H}^{k+1}(\mathbb{R}^2) := \{f \in H^k(\mathbb{R}^2) : \partial_x(f), x\partial_y(f) \in \tilde{H}^k(\mathbb{R}^2)\}$$

The identity

$$[\partial_x, x\partial_y] = \partial_y$$

implies

$$\tilde{H}^2(\mathbb{R}^2) \subseteq H^1(\mathbb{R}^2)$$

Folland and Stein Sobolev spaces

$$\tilde{H}^0(\mathbb{R}^2) := L^2_{loc}(\mathbb{R}^2)$$
$$\tilde{H}^{k+1}(\mathbb{R}^2) := \{f \in H^k(\mathbb{R}^2) : \partial_x(f), x\partial_y(f) \in \tilde{H}^k(\mathbb{R}^2)\}$$

The identity

$$[\partial_x, x\partial_y] = \partial_y$$

implies

$$\tilde{H}^2(\mathbb{R}^2) \subseteq H^1(\mathbb{R}^2)$$

and by recurrence

$$\tilde{H}^{2k}(\mathbb{R}^2) \subseteq H^k(\mathbb{R}^2)$$

Folland and Stein Sobolev spaces

$$\tilde{H}^0(\mathbb{R}^2) := L^2_{loc}(\mathbb{R}^2)$$
$$\tilde{H}^{k+1}(\mathbb{R}^2) := \{f \in H^k(\mathbb{R}^2) : \partial_x(f), x\partial_y(f) \in \tilde{H}^k(\mathbb{R}^2)\}$$

The identity

$$[\partial_x, x\partial_y] = \partial_y$$

implies

$$\tilde{H}^2(\mathbb{R}^2) \subseteq H^1(\mathbb{R}^2)$$

and by recurrence

$$\tilde{H}^{2k}(\mathbb{R}^2) \subseteq H^k(\mathbb{R}^2)$$

Hence $\cap_k \tilde{H}^k(\mathbb{R}^2) = \cap_k H^k(\mathbb{R}^n)$

Folland and Stein Sobolev spaces

$$\tilde{H}^0(\mathbb{R}^2) := L^2_{loc}(\mathbb{R}^2)$$
$$\tilde{H}^{k+1}(\mathbb{R}^2) := \{f \in H^k(\mathbb{R}^2) : \partial_x(f), x\partial_y(f) \in \tilde{H}^k(\mathbb{R}^2)\}$$

The identity

$$[\partial_x, x\partial_y] = \partial_y$$

implies

$$\tilde{H}^2(\mathbb{R}^2) \subseteq H^1(\mathbb{R}^2)$$

and by recurrence

$$\tilde{H}^{2k}(\mathbb{R}^2) \subseteq H^k(\mathbb{R}^2)$$

Hence $\cap_k \tilde{H}^k(\mathbb{R}^2) = \cap_k H^k(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$

Folland and Stein Sobolev spaces

$$\tilde{H}^0(\mathbb{R}^2) := L^2_{loc}(\mathbb{R}^2)$$
$$\tilde{H}^{k+1}(\mathbb{R}^2) := \{f \in H^k(\mathbb{R}^2) : \partial_x(f), x\partial_y(f) \in \tilde{H}^k(\mathbb{R}^2)\}$$

The identity

$$[\partial_x, x\partial_y] = \partial_y$$

implies

$$\tilde{H}^2(\mathbb{R}^2) \subseteq H^1(\mathbb{R}^2)$$

and by recurrence

$$\tilde{H}^{2k}(\mathbb{R}^2) \subseteq H^k(\mathbb{R}^2)$$

Hence $\cap_k \tilde{H}^k(\mathbb{R}^2) = \cap_k H^k(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$

Theorem (Folland and Stein)

For any u and any k, $Du \in \tilde{H}^k(\mathbb{R}^2)$ implies $u \in \tilde{H}^{k+2}(\mathbb{R}^2)$

General Sobolev spaces

Let $X_1, \cdots, X_m$ be vector fields. We define

$$\tilde{H}^0(M) := L^2_{loc}(M)$$
$$\tilde{H}^{k+1}(M) := \{f \in \tilde{H}^k(M) : X_1(f), \cdots, X_m(f) \in \tilde{H}^k(M)\}$$

General Sobolev spaces

Let $X_1, \cdots, X_m$ be vector fields. We define

$$\tilde{H}^0(M) := L^2_{loc}(M)$$
$$\tilde{H}^{k+1}(M) := \{f \in \tilde{H}^k(M) : X_1(f), \cdots, X_m(f) \in \tilde{H}^k(M)\}$$

Hormander's condition for any $x \in M$,

$$X_1(x), \cdots, X_m(x), [X_i, X_j](x), [[X_i, X_j], X_k](x), \cdots$$

span T_xM.

General Sobolev spaces

Let $X_1, \cdots, X_m$ be vector fields. We define

$$\tilde{H}^0(M) := L^2_{loc}(M)$$
$$\tilde{H}^{k+1}(M) := \{f \in \tilde{H}^k(M) : X_1(f), \cdots, X_m(f) \in \tilde{H}^k(M)\}$$

Hormander's condition for any $x \in M$,

$$X_1(x), \cdots, X_m(x), [X_i, X_j](x), [[X_i, X_j], X_k](x), \cdots$$

span T_xM. Then $\cap_k \tilde{H}^k(M) = C^\infty(M)$

Main theorem

Theorem (Androulidakis, M., Yuncken 2022)

Let $X_1, \cdots, X_m$ *be vector fields satisfying Hörmander's condition,* D *be of order* l*.*

Main theorem

Theorem (Androulidakis, M., Yuncken 2022)

Let $X_1, \cdots, X_m$ be vector fields satisfying Hörmander's condition, D be of order l. Then TFAE

1. *for any k and any u, $Du \in \tilde{H}^k(M)$ implies $u \in \tilde{H}^{k+l}(M)$.*

Main theorem

Theorem (Androulidakis, M., Yuncken 2022)

Let $X_1, \cdots, X_m$ be vector fields satisfying Hörmander's condition, D be of order l. Then TFAE

1. *for any k and any u, $Du \in \tilde{H}^k(M)$ implies $u \in \tilde{H}^{k+l}(M)$.*
2. *for any x and for some irreducible unitary representations of G_x, $\tilde{\sigma}(x, \pi)$ is injective.*

Main theorem

Theorem (Androulidakis, M., Yuncken 2022)

Let $X_1, \cdots, X_m$ *be vector fields satisfying Hörmander's condition,* D *be of order* l*. Then TFAE*

1. *for any* k *and any* u*,* $Du \in \tilde{H}^k(M)$ *implies* $u \in \tilde{H}^{k+l}(M)$*.*
2. *for any* x *and for some irreducible unitary representations of* G_x*,* $\tilde{\sigma}(x, \pi)$ *is injective.*

Previously a conjecture by Helffer and Nourrigat (1979). In 1985, they prove $1 \Rightarrow 2$ and $2 \Rightarrow 1$ if G_x are of rank 2.

Index theory

$$D : C^\infty(M) \to C^\infty(M)$$

its index is

$$\mathrm{Ind}_a(D) = \dim(\ker(D)) - \mathrm{codim}(\mathrm{im}(D)).$$

Index theory

$$D : C^\infty(M) \to C^\infty(M)$$

its index is

$$\mathrm{Ind}_a(D) = \dim(\ker(D)) - \mathrm{codim}(\mathrm{im}(D)).$$

Atiyah and Singer (1960) a topological formula for the index for elliptic operators.

Index theory

$$D : C^\infty(M) \to C^\infty(M)$$

its index is

$$\mathrm{Ind}_a(D) = \dim(\ker(D)) - \mathrm{codim}(\mathrm{im}(D)).$$

Atiyah and Singer (1960) a topological formula for the index for elliptic operators.

D maximally hypoelliptic $\Rightarrow \dim(\ker(D))$ and $\mathrm{codim}(\mathrm{im}(D))$ are finite.

Index theory

$$D : C^\infty(M) \to C^\infty(M)$$

its index is

$$\mathrm{Ind}_a(D) = \dim(\ker(D)) - \mathrm{codim}(\mathrm{im}(D)).$$

Atiyah and Singer (1960) a topological formula for the index for elliptic operators.

D maximally hypoelliptic $\Rightarrow \dim(\ker(D))$ and $\mathrm{codim}(\mathrm{im}(D))$ are finite.

Theorem (M. 2022)

Let $X_1, \cdots, X_m$ be vector fields satisfying Hörmander's condition, D maximally hypoelliptic on any compact manifold M. Then

$$\mathrm{Ind}_a(D) = \mathrm{Ind}_{AS}(Ex(\mu^{-1}(\tilde{\sigma}(D))))$$

Example

Consider ∂_x and $x^k\partial_y$ on $\mathbb{R}^2$.

Example

Consider ∂_x and $x^k\partial_y$ on $\mathbb{R}^2$. Descend to $S^1 \times S^1$ to ∂_x and $\sin(x)^k\partial_y$.

Example

Consider ∂_x and $x^k\partial_y$ on $\mathbb{R}^2$. Descend to $S^1 \times S^1$ to ∂_x and $\sin(x)^k\partial_y$. Consider

$$D = (\partial_x^2 + (\sin(x)^k\partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

where $g : S^1 \times S^1 \to \mathbb{C}$ smooth.

Example

Consider ∂_x and $x^k \partial_y$ on $\mathbb{R}^2$. Descend to $S^1 \times S^1$ to ∂_x and $\sin(x)^k \partial_y$.
Consider

$$D = (\partial_x^2 + (\sin(x)^k \partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

where $g : S^1 \times S^1 \to \mathbb{C}$ smooth. Our symbol of D at (x, y)

- if $\sin(x) \neq 0$

Example

Consider ∂_x and $x^k\partial_y$ on $\mathbb{R}^2$. Descend to $S^1 \times S^1$ to ∂_x and $\sin(x)^k\partial_y$. Consider

$$D = (\partial_x^2 + (\sin(x)^k\partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

where $g : S^1 \times S^1 \to \mathbb{C}$ smooth. Our symbol of D at (x, y)

- if $\sin(x) \neq 0$, then $G_x = T_xM$ and our symbol=classical symbol

Example

Consider ∂_x and $x^k\partial_y$ on $\mathbb{R}^2$. Descend to $S^1 \times S^1$ to ∂_x and $\sin(x)^k\partial_y$. Consider

$$D = (\partial_x^2 + (\sin(x)^k\partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

where $g : S^1 \times S^1 \to \mathbb{C}$ smooth. Our symbol of D at (x,y)

- if $\sin(x) \neq 0$, then $G_x = T_xM$ and our symbol=classical symbol
- if $\sin(x) = 0$. Then the Lie algebra of G_x is generated by

$$\partial_x, x^k\partial_y, x^{k-1}\partial_y, \cdots, \partial_y$$

with Lie bracket

$$[\partial_x, x^j\partial_y] = jx^{j-1}\partial_y$$

Example

Consider ∂_x and $x^k\partial_y$ on $\mathbb{R}^2$. Descend to $S^1 \times S^1$ to ∂_x and $\sin(x)^k\partial_y$. Consider

$$D = (\partial_x^2 + (\sin(x)^k\partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

where $g : S^1 \times S^1 \to \mathbb{C}$ smooth. Our symbol of D at (x,y)

- if $\sin(x) \neq 0$, then $G_x = T_xM$ and our symbol=classical symbol
- if $\sin(x) = 0$. Then the Lie algebra of G_x is generated by

  $$\partial_x, x^k\partial_y, x^{k-1}\partial_y, \cdots, \partial_y$$

  with Lie bracket

  $$[\partial_x, x^j\partial_y] = jx^{j-1}\partial_y$$

  The two representations

  $$\begin{aligned} \pi_\pm : G_x &\to B(L^2\mathbb{R}) \\ \partial_x &\mapsto (f \mapsto \partial_t f) \\ x^j\partial_y &\mapsto (f \mapsto \pm it^j f) \end{aligned}$$

Example

The operator

$$D = (\partial_x^2 + (\sin(x)^k \partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

evaluated at $\pi_\pm$

Example

The operator

$$D = (\partial_x^2 + (\sin(x)^k \partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

evaluated at $\pi_\pm$ gives

$$(\partial_t^2 - t^{2k})^{\frac{k+1}{2}} \mp g(x,y) Id_{L^2\mathbb{R}}$$

Example

The operator

$$D = (\partial_x^2 + (\sin(x)^k \partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

evaluated at $\pi_\pm$ gives

$$(\partial_t^2 - t^{2k})^{\frac{k+1}{2}} \mp g(x,y) Id_{L^2\mathbb{R}}$$

D maximally hypoelliptic iff

$$g(0,y), g(\pi,y) \notin \pm \mathrm{spec}((\partial_t^2 - t^{2k})^{\frac{k+1}{2}})$$

Example

The operator

$$D = (\partial_x^2 + (\sin(x)^k \partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

evaluated at $\pi_\pm$ gives

$$(\partial_t^2 - t^{2k})^{\frac{k+1}{2}} \mp g(x,y) Id_{L^2\mathbb{R}}$$

D maximally hypoelliptic iff

$$g(0,y), g(\pi,y) \notin \pm\mathrm{spec}((\partial_t^2 - t^{2k})^{\frac{k+1}{2}})$$

Consider $g(0,y) : S^1 \to \mathbb{C}$.

Example

The operator

$$D = (\partial_x^2 + (\sin(x)^k \partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

evaluated at $\pi_\pm$ gives

$$(\partial_t^2 - t^{2k})^{\frac{k+1}{2}} \mp g(x,y) Id_{L^2\mathbb{R}}$$

D maximally hypoelliptic iff

$$g(0,y), g(\pi,y) \notin \pm \mathrm{spec}((\partial_t^2 - t^{2k})^{\frac{k+1}{2}})$$

Consider $g(0,y) : S^1 \to \mathbb{C}$. Let $w(g(0,y),\lambda)$ be winding number.

Example

The operator

$$D = (\partial_x^2 + (\sin(x)^k \partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

evaluated at $\pi_\pm$ gives

$$(\partial_t^2 - t^{2k})^{\frac{k+1}{2}} \mp g(x,y) Id_{L^2\mathbb{R}}$$

D maximally hypoelliptic iff

$$g(0,y), g(\pi,y) \notin \pm \mathrm{spec}((\partial_t^2 - t^{2k})^{\frac{k+1}{2}})$$

Consider $g(0,y) : S^1 \to \mathbb{C}$. Let $w(g(0,y),\lambda)$ be winding number. Then

$$\mathrm{Ind}_a(D) = \sum_\lambda w(g(0,y),\lambda) - w(g(0,y),-\lambda)$$

Example

The operator

$$D = (\partial_x^2 + (\sin(x)^k \partial_y)^2)^{\frac{k+1}{2}} + ig(x,y)\partial_y$$

evaluated at $\pi_\pm$ gives

$$(\partial_t^2 - t^{2k})^{\frac{k+1}{2}} \mp g(x,y) Id_{L^2\mathbb{R}}$$

D maximally hypoelliptic iff

$$g(0,y), g(\pi,y) \notin \pm \mathrm{spec}((\partial_t^2 - t^{2k})^{\frac{k+1}{2}})$$

Consider $g(0,y) : S^1 \to \mathbb{C}$. Let $w(g(0,y),\lambda)$ be winding number. Then

$$\begin{aligned} \mathrm{Ind}_a(D) = \; & \textstyle\sum_\lambda w(g(0,y),\lambda) - w(g(0,y),-\lambda) \\ & + w(g(\pi,y),\lambda) - w(g(\pi,y),-\lambda) \end{aligned}$$

The End

Thank you for your attention