Combinatorial Reid's Recipe for Dimer Models

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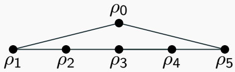
09/06/2022

Context: the McKay correspondence for finite groups $G \subset \mathrm{SL}(2, \mathbb{C})$.

Example: Consider the cyclic group $G = \left\{ \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \middle| \varepsilon^6 = 1 \right\} \subset \mathrm{SL}(2, \mathbb{C})$.

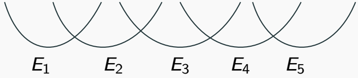
This group has 6 irreducible representations: $\rho_0, \ldots, \rho_5$.We form its McKay quiver:



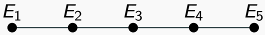
Vertices: the ρ_i.

Arrows: if $V \otimes \rho_i = \oplus_j a_{ij} \rho_j$, then we have a_{ij} arrows between ρ_i and ρ_j.

G acts on $\mathbb{C}^2$ by multiplication, and we form the quotient $\mathbb{C}^2/G$. This is the well-known hypersurface singularity A_5 with equation $\{xz - y^6 = 0\} \subset \mathbb{C}^3$. Its minimal resolution $Y \to \mathbb{C}^2/G$ has exceptional locus:



$\Leftrightarrow$



McKay correspondence

Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. There is a bijective correspondence between

$$\{\rho_i \text{ nontrivial irrep of } G\} \leftrightarrow \left\{\text{exceptional divisors of } Y \to \mathbb{C}^2/G\right\}.$$

Let $G \subset \mathrm{SL}(3, \mathbb{C})$ be finite.

Question: Does there exist a distinguished resolution which makes sense from the point of view of the McKay correspondence?

Answer for G abelian [Bridgeland-King-Reid]: Yes! This is the *G-Hilbert scheme* (or G-Hilb$(\mathbb{C}^3)$).

Classical Reid's recipe (Reid 1997, Craw 2005)

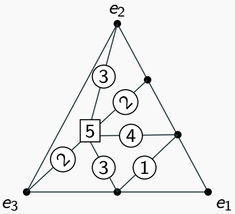
Given an affine Gorenstein toric simplicial singularity $X = \mathbb{C}^3/G$, with $G \subset \mathrm{SL}(3, \mathbb{C})$, there is a combinatorial algorithm that decorates the fan Σ of the distinguished crepant resolution G-Hilb$(\mathbb{C}^3)$ with irreducible representations of the group G.

Our aim is to generalise this construction for $X = \operatorname{Spec} R$ a Gorenstein toric variety.

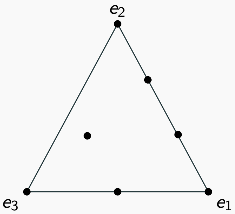
Example: Consider the group

$$G = \frac{1}{6}(1,2,3) = \left\{ \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^3 \end{pmatrix} \middle| \varepsilon^6 = 1, \varepsilon \text{ primitive} \right\} \subset \mathrm{SL}(3,\mathbb{C}).$$

To see the resolution, we first see $X = \mathbb{C}^3/G$ as an affine toric variety:



$\longrightarrow$



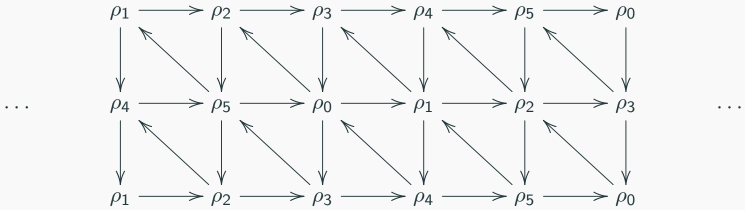
Fan of $Y = G\text{-Hilb}(\mathbb{C}^3)$ inside $\mathbb{Z}^3 + \frac{1}{6}(1,2,3)\mathbb{Z}$ Fan of X inside $\mathbb{Z}^3 + \frac{1}{6}(1,2,3)\mathbb{Z}$

G has six irreducible representations $\rho_0, \ldots, \rho_5$. We decorate the *interior edges* and *lattice points* in the fan of Y as above.

Reid's recipe for $\frac{1}{6}(1,2,3)$

Since G is abelian, its irreps are: $\rho_i : G \to \mathbb{C}$, $\rho_i \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^3 \end{pmatrix} = \varepsilon^i$, $i = 0, \ldots, 5$.

Maps between the ρ_i are encoded in the McKay quiver of $\frac{1}{6}(1,2,3)$:



which naturally lives on a real two-dimensional torus.

Remark

G-Hilb$(\mathbb{C}^3)$ is a moduli space of representations of this quiver (with relations).

This studies the equivalence of categories appearing in works of Nakamura and proven by Bridgeland–King–Reid, in the case where G is abelian:

$$\Psi\colon D^b\big(G\text{-}\operatorname{coh}(\mathbb{C}^3)\big) \longrightarrow D^b\big(\operatorname{coh}(G\text{-Hilb}(\mathbb{C}^3))\big)$$

Key question (Cautis–Logvinenko 2009):

What can be said of the images of simple G-sheaves $\mathcal{O}_0 \otimes \rho$ on $\mathbb{C}^3$, for ρ a nontrivial irreducible representation?

Answer (Logvinenko 2010, Cautis–Craw–Logvinenko 2012):

They are pure sheaves which can be computed explicitly from Reid's recipe, i.e the image sheaf depends on what ρ marks in Σ.

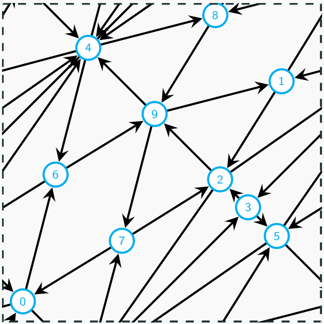
Any generalisation of the recipe from the simplicial case to an arbitrary Gorenstein affine variety $X = \operatorname{Spec} R$ would have to:

- Find an appropriate equivalent to the derived correspondence.

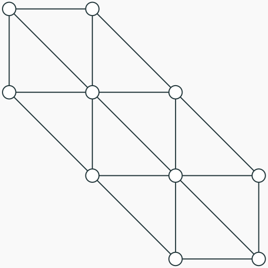
$\rightarrow$ Blocklandt–Craw–Quintero-Vélez (2014) used consistent dimer models as key instruments in writing this dictionary.

- Find a combinatorial way to deduce the markings.

$\rightarrow$ Craw–H.–Tapia Amador (2021) deduce the markings on points and line segments of the fan Σ.



(e) Construct a quiver coming from a dimer model on $\mathbb{T}$. Each vertex i of this quiver represents a line bundle L_i on Y.



(f) This determines an affine toric variety X and fixes a distinguished resolution $Y \to X$ associated to this triangulation.

Remark

1. The resolution Y is a moduli space of (θ-stable) representations of the quiver Q.
2. Ishii–Ueda (2008) showed that there is an equivalence of categories

$$\Psi(-)\colon D^b(\mathrm{rep}(Q,\mathcal{R})) \longrightarrow D^b(\mathrm{coh}\text{-}Y),$$

where $\mathcal{R} := \langle p_a^+ - p_a^- \mid a \in Q_1 \rangle$ are relations in Q, and $p_a^\pm$ are clockwise and anti-clockwise paths in Q with tail at $\mathrm{h}(a)$ and head at $\mathrm{t}(a)$.

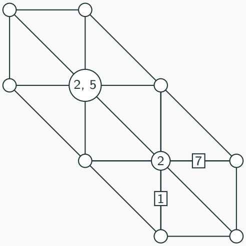
Bocklandt–Craw–Quintero-Vélez (2014) show:

Theorem (Geometric Reid's recipe)

For $i \in Q_0$, let $S_i := \mathbb{C}e_i$. If i is not the zero vertex then exactly one of the following happens:

$$\Psi(S_i) = \begin{cases} L_i^{-1}|_{D_i} & \text{for some divisor } D_i; \\ L_i^{-1}|_{C_\tau} & \text{for some } (-1,-1) \text{ curve } C_\tau; \\ F|_{Z_i}[1] & \text{for some sheaf } F \text{ and divisor } Z_i. \end{cases}$$

In the case of our running example, this suggests the following markings:

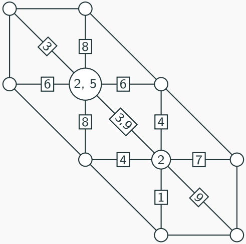


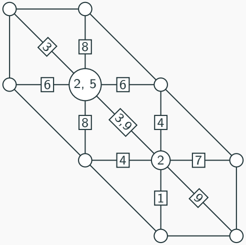
Theorem (Combinatorial Reid's recipe, Craw – H. – Tapia Amador (2021))

We introduce a marking for every interior lattice point and segment of Σ, such that:

1. *The recipe for points and vertices marking unique line segments coincides with the geometric recipe.*
2. *The recipe agrees with Reid's original recipe for marking cones in the toric fan of* G-Hilb *in the special case when Q is the McKay quiver of a finite abelian subgroup* $G \subset \mathrm{SL}(3, \mathbb{C})$.

Here is what we obtain for our running example:





Remark (New to the dimer case)

1. Interior lattice points can be marked with the same vertex $i \in Q_0$ (e.g. 2).
2. Interior line segments can be marked with more than one vertex (e.g. 3 and 9).
3. The marking of an interior line segment is not determined by the hyperplane containing it (e.g. 3 and 9).
4. The marking of an interior lattice point is not determined by the geometry of the toric surface D_ρ.
5. The Euler number of an irreducible component of the exceptional divisor is not bounded by 6 from above.

Thank you!