Moduli spaces and their completion

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A *moduli space* is an algebraic variety (scheme, stack) whose points parametrize algebro-geometric objects up to some equivalence.

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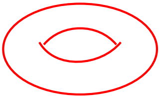
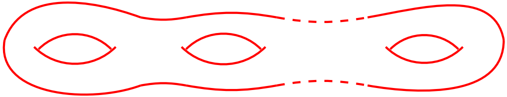
Over $\mathbb{C}$ a smooth curve is a Riemann surface:



$g = 1$



$g \geq 2$

- If $g = 0$ then M_0 is a point. Curves of genus 0 are called *rational*.

Moduli spaces of curves in a projective space

Moduli spaces of curves in a projective space

Moduli space of lines in $\mathbb{P}^2 = \mathbb{P}^2$

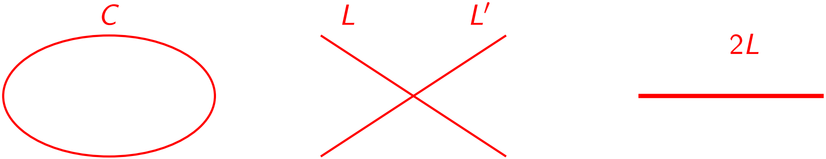
Moduli space of lines in $\mathbb{P}^r = G(1, r)$ (Grassmann variety)

Moduli space of hypersurfaces of degree d in $\mathbb{P}^r = \mathbb{P}^{\binom{r+d}{r}-1} = \mathbb{P}(V)$

(V = space of homogeneous polynomials of degree d)

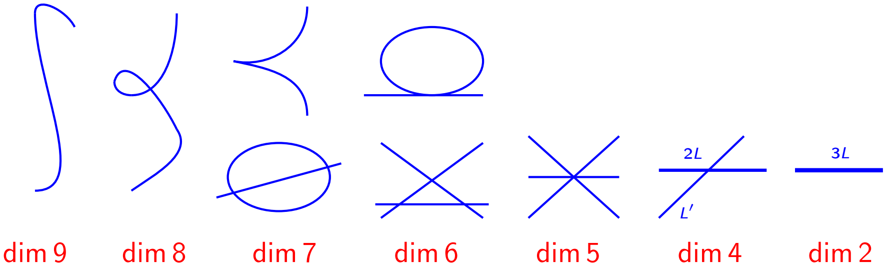
Moduli space of curves of degree d in $\mathbb{P}^2 = \mathbb{P}^{d(d+3)/2}$

Plane curves of degree 2 $= \mathbb{P}^5$



dimension 5 | dimension 4 | dimension 2

Plane curves of degree 3 $= \mathbb{P}^9$



Question: *In how many points do two plane curves intersect?*

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Theorem [É. Bézout (1730-1783)]

If $C, D \subset \mathbb{P}^2$ are curves of degrees c and d, then

$$\deg(C \cap D) = cd$$

where for "most" choices of C and D

$$\deg(C \cap D) = \# C \cap D.$$

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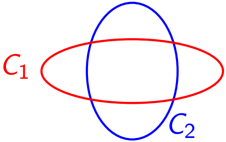
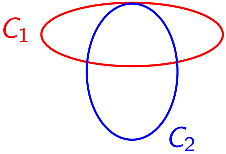
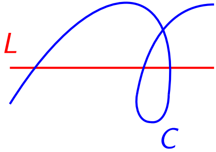
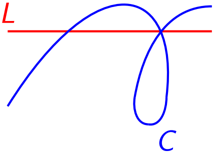
Two conics





A line and a singular cubic





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Therefore

$$\deg(C \cap D) = \deg(cL \cap dL) = cd \deg(L \cap L) = cd.$$

□

Counting plane curves of fixed degree

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Easy question: *How many curves of degree d pass through the appropriate number of points?*

1 line through 2 points
1 conic through 5 points
1 curve of degree d through $d(d+3)/2$ general points.

Counting plane curves of fixed degree

Easy question: *How many curves of degree d pass through the appropriate number of points?*

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Question: *How many singular curves of degree d pass through the appropriate number of points?*

$V_{d,\delta}$ =Moduli space of plane curves of degree d with δ nodes
Severi variety

$$\mathbb{P}^{d(d+3)/2} \supset V_{d,1} \supset V_{d,2} \supset \ldots \supset V_{d,\delta} \supset \ldots$$

$r_{d,\delta} = \dim V_{d,\delta} = d(d+3)/2 - \delta$

- *The set of curves with δ nodes passing through $r_{d,\delta}$ general points is finite*

$N_{d,\delta}$ = number of curves with δ nodes through $r_{d,\delta}$ general points.

Enumerative problem: *compute $N_{d,\delta}$.*

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$V_{d,\delta}$ is not complete, a completion is its closure:

$$\overline{V_{d,\delta}} \subset \mathbb{P}^{d(d+3)/2}$$

- $N_{d,\delta} = \deg \overline{V_{d,\delta}}$

But $\overline{V_{d,\delta}}$ is hard to deal with.

M_g and its completion by *stable* curves

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$$M_g = \{(\text{abstract}) \text{ smooth curves of genus } g\}/\cong$$

$$\dim M_1 = 1, \qquad \dim M_g = 3g-3, \quad \forall g \geq 2$$

- M_g is irreducible, not complete for $g \geq 1$.

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Deligne-Mumford (1969):

$$\overline{M}_g = \{\text{stable curves of genus } g\}/\cong$$

- $\overline{M}_g$ is complete and irreducible, $\dim \overline{M}_g = \dim M_g$.

- A stable curve has at most nodes as singularities.

Moduli spaces of maps and completion by *stable* maps

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Kontsevich (1995)

$\overline{M}_g(\mathbb{P}^r, d) = \{\text{stable degree } d\text{-maps from curves of genus } g \text{ to } \mathbb{P}^r\}/\sim$

a complete moduli space.

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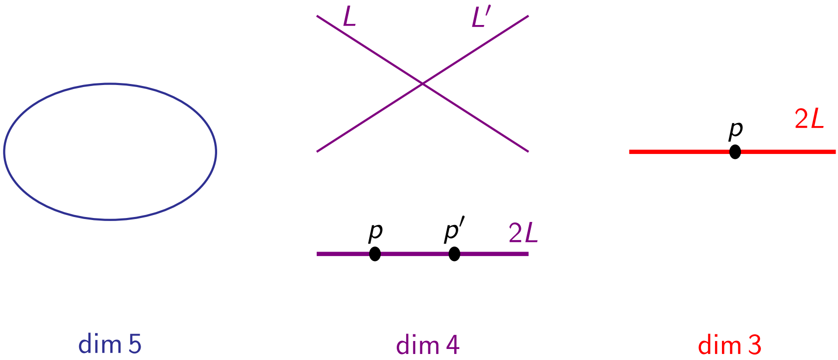
- If $g = 0$ then $\overline{M}_g(\mathbb{P}^r, d)$ is smooth and very useful in enumerative problems.

(Gromow-Witten theory and Quantum Cohomology)

$\overline{M}_0(\mathbb{P}^2, 2) \quad = \quad$ moduli space of stable degree-2 maps from rational curves to $\mathbb{P}^2$



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Theorem[Kontsevich's recursion (1993)] $N_1 = 1$ and for $d \geq 2$

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} d_1 d_2 \left[\binom{3d-4}{3d_1-2} d_1 d_2 - \binom{3d-4}{3d_1-3} d_2^2 \right]$$

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The general formula for $N_{d,\delta}$ was found by C.-Harris in 1998.

The completion $M_g \subset \overline{M}_g$

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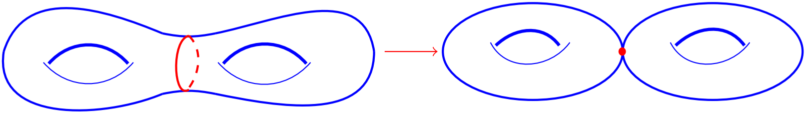
$\overline{M}_g = \{\text{stable curves of genus } g\}/\cong$

- A curve X is stable if it has only nodes as singularities and $\mathrm{Aut}(X)$ is finite (stability).



$g = 2$, a smooth curve degenerating to a singular stable curve.

- Nodes require a minimum amount of data.

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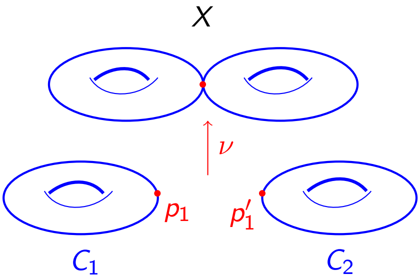
A curve X with δ nodes can be presented as follows

$$\bigsqcup_{i=1}^{\gamma} C_i \xrightarrow{\nu} X = \Big(\bigsqcup_{i=1}^{\gamma} C_i\Big)/\{p_j = p'_j,\ j = 1,\ldots,\delta\}$$

C_i smooth curves

p_j, p'_j distinct points



Stable graphs

$$G_X = \text{the dual (weighted) graph of } X$$

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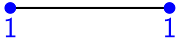
$V(G_X)$= components of X
$= \{C_1, \ldots, C_\gamma\}$

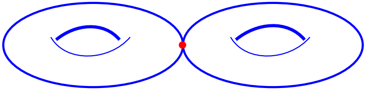
weight (C_i) = genus of C_i

$E(G_X)$ = nodes of X

G_X

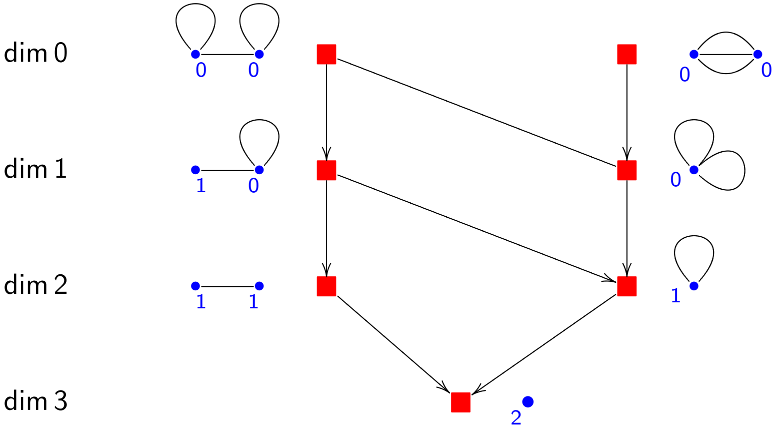


X



- Stability: every weight zero vertex has valency ≥ 3.

Stratification of $\overline{M}_2$ by stable graphs.



The stratification of $\overline{M}_g$ by stable graphs

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To each stable graph G we associate the stratum

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M_G is irreducible of codimension $\#E(G)$.

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The closure of a stratum is a union of strata.

The closures of the codimension-one strata, $D_0, \dots, D_h$, are the irreducible components of the boundary, $\overline{M}_g \setminus M_g$.

The boundary complex, an important invariant

The boundary $\overline{M}_g \setminus M_g$ is a *normal crossings divisor* (union of irreducible divisors meeting transversally).

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$\Delta(\overline{M}_g)$ = the boundary complex of $\overline{M}_g$:

Vertices $=$ irr. components $D_0, \ldots, D_h$ of $\overline{M}_g \setminus M_g$

$=$ closures of all strata M_G with $\#E(G) = 1$

Edges between D_i and D_j = irr. components of $D_i \cap D_j$

Faces on D_i, D_j, D_k = irr. components of $D_i \cap D_j \cap D_h$

. . . . . .

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- Tropical curves of genus g have a "nice" moduli space.

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Theorem Δ_g is isomorphic to the boundary complex of $\overline{M}_g$, i.e.

$$\Delta_g \cong \Delta(\overline{M}_g)$$

Follows from [Abramovich-C.-Payne (2015)]