On the complex cobordism classes of hyper-Kähler manifolds

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- Complex cobordism ring $\mathrm{MU}^*(\mathrm{pt})$ invented by Milnor.

- **Construction.** Free abelian group $\mathcal{Z}^i$ with **generators** in degree i (diffeomorphism class) of (M, α)
- $M =$ compact differentiable manifold of dimension i.
- $\alpha =$ stable almost complex structure on M, i.e. the structure of a complex vector bundle on $T_M \oplus \mathbb{R}^k$ for some k.

- **Relations.**
- $N =$ compact differentiable manifold of dimension $i + 1$ with boundary.
- $\alpha =$ stable almost complex structure on N.
Observe that $T_{\partial N} \oplus \mathbb{R} \cong T_{N|\partial N}$, so α restricts to the boundary ∂N. This defines the boundary of (N, α).

Definition. $\mathrm{MU}^i(\mathrm{pt})$ is the quotient of $\mathcal{Z}^i$ by the subgroup generated by boundaries.

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• Any complex vector bundle E on a topological space X has Chern classes $c_i(E) \in H^{2i}(X, \mathbb{Z})$.

• Let M be a compact manifold and let $\alpha =$ stable almost complex structure on M. Get Chern classes $c_i(M, \alpha) \in H^{2i}(M, \mathbb{Z})$.

• M is oriented (using α), so for any polynomial P in the Chern classes, get a **Chern number** $\int_M P(c_i(M, \alpha))$.

• If N is a manifold with boundary, and α is a stable almost complex structure on N, we have $c_i(N, \alpha)_{|\partial N} = c_i(\partial N, \alpha_{|\partial N})$. Hence we get by Stokes $\int_{(\partial N, \alpha)} P(c_i(\partial(N, \alpha)) = 0$.
(Chern numbers of M depend only on the complex cobordism class of M.)

Theorem. (Milnor, Novikov) $\mathrm{MU}^*(\mathrm{pt})$ *has no torsion, is trivial in odd degree, and* $\mathrm{MU}^{2i}(\mathrm{pt}) \otimes \mathbb{Q}$ *is isomorphic by the Chern number map to the dual of the space of degree* $2i$ *weighted polynomials in the* c_j*.*

• **Explicit computation** $\Rightarrow$ *projective spaces* $\mathbb{CP}^k$ *form a (multiplicative) basis of* $\mathrm{MU}^*(\mathrm{pt}) \otimes \mathbb{Q}$*. (The* $\prod_i \mathbb{CP}^{k_i}$ *form an additive basis.)*

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Kähler metrics and canonical bundle

• Complex manifold $X \rightsquigarrow$ complex structure on tangent bundle T_X, called almost complex structure. I=operator of almost complex structure.

• Newlander-Nirenberg integrability condition needed to go back from almost complex structure to complex structure.

• Hermitian metric h on X= Hermitian metric on T_X equipped with this complex structure.

• Kähler form $\omega = \mathrm{Im}\, h$. Let $g := \mathrm{Re}\, h$. Related by $g(u,v) = \omega(u, Iv)$.

• g is a Riemannian metric, hence Levi-Civita connection and parallel transport.

Definition. The metric is Kähler if $d\omega = 0$.

• Equivalent property: *The operator I is parallel for Levi-Civita*. Hence: *Kähler metrics on X_{2n} are the metrics of holonomy contained in $U(n)$.*

• **Canonical bundle**. Holomorphic line bundle $\bigwedge^n \Omega_X$, trivialized in local holomorphic coordinates by $dz_1 \wedge \ldots \wedge dz_n$. K_X is trivial iff X has an everywhere nonzero holomorphic n-form η_X.

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Theorem. (Yau 1978) *Let X be a compact Kähler manifold whose canonical bundle is trivial. There exist* **Kähler-Einstein** *metrics on X, i.e. Kähler and Ricci flat*.

Definition: (Hyper-Kähler manifolds) *X = compact Kähler manifold of dimension $2n$ is hyper-Kähler if X is simply connected and has a holomorphic 2-form $\sigma_X = \sum_{ij} \alpha_{ij} dz_i \wedge dz_j$ which is everywhere nondegenerate : $\det \alpha_{ij} \neq 0$ (or $\sigma_X^n \neq 0$) at any point.*

- The $2n$-form σ_X^n trivializes the canonical bundle hence Yau's thm applies.

Thm. (Beauville 1984) *The holomorphic forms on X are parallel for any Kähler-Einstein metric. Thus K-E metrics on HK $2n$-folds have holonomy $Sp(2n) \cap U(2n)$.*

- **Quaternionic structure .** X HK, $g = \mathrm{Re}\, h$= Kähler-Einstein metric on X. $\rightsquigarrow$ Three real closed 2-forms on X, parallel for Levi-Civita : ω, $\mathrm{Re}\, \sigma_X$, $\mathrm{Im}\, \sigma_X$ $\rightsquigarrow$ action of the field of quaternions on T_X.

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Complex Tori. Real torus $\mathbb{R}^{2n}/\mathbb{Z}^{2n} \cong (\mathbb{S}^1)^{2n}$. An isomorphism $\mathbb{R}^{2n} \cong \mathbb{C}^n$ endows the torus with a complex structure. A hermitian metric on $\mathbb{C}^n$ induces a flat Kähler metric on the torus.

Remark. The holomorphic tangent bundle of a complex torus is trivial, hence all its Chern classes are trivial (trivial complex cobordism class).

Theorem. (Beauville, Bogomolov) *Any compact Kähler manifold X with $c_1(X) = 0$ in $H^2(X, \mathbb{R})$ has a finite cover which is isomorphic to a product of complex tori, hyper-Kähler manifolds, and simply connected compact Kähler manifolds Y of dimension n with $h^{i,0}(X) = 0$, $i = 1, n-1$.*

• The Y as above are Calabi-Yau in the strict sense.They admit Kähler-Einstein metrics with holonomy $SU(n)$. There are many examples, eg: hypersurfaces of degree d (or complete intersections) in projective space, of dimension $n \geq 2$, with trivial canonical bundle (condition $d = n + 2$).

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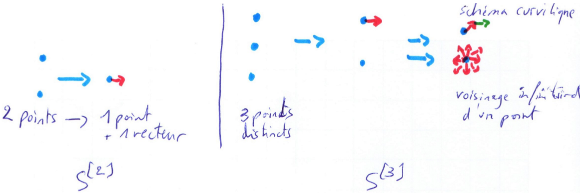
• **Dim. 2: only the $K3$ surfaces**. They are deformations of a smooth quartic surface $S \subset \mathbb{CP}^3$ defined by a homogeneous polynomial of degree 4.

• **Punctual Hilbert schemes.** S smooth complex surface. Quotient $S^{(k)} := S^k/\mathfrak{S}_k$=**singular** analytic space of dim $2k$.

• **Desingularisation** (Fogarty): *The Hilbert scheme of $S^{[k]}$ parameterizing* ***subschemes*** *of S of length k is smooth. This is a desingularization of $S^{(k)}$.*



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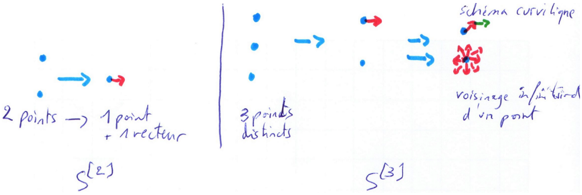

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• **Generalized Kummer.** (constructed by Beauville) Start with a 2-dimensional complex torus A. Then $A^{[n+1]}$ has trivial canonical bundle, but it is not simply connected since the Albanese map $\mathrm{alb}:\ A^{[n+1]} \to A,\ \{x_1, \ldots, x_{n+1}\} \mapsto \sum_i x_i$ is surjective.
Define $K_n(A) := \mathrm{alb}^{-1}(0) \subset A^{[n+1]}$.

Theorem. *This is a HK manifold*.

• The sporadic O'Grady examples OG_{10}, OG_6. Obtained by desingularizing a moduli space of sheaves on a $K3$ surface (resp. abelian surface).

• The topological types of $S^{[k]}$ and $K_k(A)$ are different for $k \geq 2$: Indeed $b_2(S^{[k]}) = 23$, $b_2(K_k(A)) = 7$ for $k \geq 2$. $K_1(A)$ is a Kummer K3 surface.

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• X a hyper-Kähler manifold with everywhere nondegenerate holomorphic 2-form σ_X. Then $\sigma_X \lrcorner : T_X \cong \Omega_X$, which implies $c_i(X) = 0$ for i odd.

• In particular, complex cobordism classes of HK belong to the subring $\mathrm{MU}^*(\mathrm{pt})_{\mathrm{even}}$ of classes α such that $\langle M_I, \alpha \rangle = 0$ for any monomial M_I in the Chern classes involving nontrivially an odd Chern class.

Theorem. (Oberdieck-Song-Voisin 2021) *The cobordism classes of $S^{[k]}$ form a multiplicative basis of $\mathrm{MU}^*(\mathrm{pt})_{\mathrm{even},\mathbb{Q}}$.*
The cobordism classes of $K_k(A)$ form a multiplicative basis of $\mathrm{MU}^*(\mathrm{pt})_{\mathrm{even},\mathbb{Q}}$.

• Equivalently, the products of $S^{[k]}$ or of $K_k(A)$ form an additive basis of $\mathrm{MU}^*(\mathrm{pt})_{\mathrm{even},\mathbb{Q}}$.

Question. Is there a simple way of computing the class of $S^{[n]}$ as a combination of the classes $K_J(A) := \prod_k K_{[j_k]}$?

Question. Are the coefficients so obtained positive ? (Yes for $n \leq 7$.)

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• Obvious property: $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$.

Definition. X (almost) complex manifold with $\dim_{\mathbb{C}}(X) = n$. The Milnor genus of X is the Chern number defined by $M(X) := \int_X \mathrm{ch}_n(T_X)$.

Lemma. *If X is a product of (almost) complex manifolds, $M(X) = 0$.*

Proposition. *Let $\alpha_i \in \mathrm{MU}^{4i}(\mathrm{pt})_{\mathrm{even},\mathbb{Q}}$ be given for all i. Then the α_i form a multiplicative basis of $\mathrm{MU}^{4i}(\mathrm{pt})_{\mathrm{even},\mathbb{Q}}$ iff $M(\alpha_i) \neq 0$.*

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Theorem. (Oberdieck-Song-Voisin 2021) *One has $M(S^{[k]}) \neq 0$, $M(K_k(A)) \neq 0$ for any $k \geq 1$.*

• More precisely, we show that the signs of these numbers are $(-1)^k$, suggesting

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• Pontryagin classes : $p_i(M) = c_{2i}(\Omega_M \otimes \mathbb{C}) \in H^{4i}(M, \mathbb{Z})$.
Theorem (Thom) *The oriented cobordism ring tensor $\mathbb{Q}$ is a polynomial algebra with one generator in each dimension 4 (isomorphism given by Pontryagin numbers)*.

Lemma. *If X is a compact complex manifold with $c_{2i+1}(X) = 0 \ \forall i$, the Chern classes $c_i(X)$ are (universal) polynomials in the $p_j(X)$.*

Proof. $\Omega_X \otimes \mathbb{C} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$ with $\Omega_X^{0,1} = (\Omega_X^{1,0})^*$. So by Whitney, we get under the assumptions equality of **total** Pont/Chern classes $P(X) = C(\Omega_X^{1,0})C((\Omega_X^{1,0})^*) = C(\Omega_X^{1,0})^2$. Hence $C(X) = P(X)^{\frac{1}{2}}$. **qed**

Corollary. (of lemma) *The oriented cobordism class of a hyper-Kähler manifold determines its complex cobordism class.*

Corollary. (of main thm) *The oriented cobordism classes of $S^{[n]}$ form a multiplicative basis of the oriented cobordism ring tensor $\mathbb{Q}$. Same for the generalized Kummer $K_n(A)$.*

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Question. What are the complex cobordism classes of hyper-Kähler manifolds?

Subquestion 1. Do they satisfy linear relations?

- In dimension 2, $\mathrm{MU}^4(\mathrm{pt})_{\mathrm{even},\mathbb{Q}}$ is of rank 1 (c_2)and we have $K3$ surfaces.
- In dimension 4, $\mathrm{MU}^8(\mathrm{pt})_{\mathrm{even},\mathbb{Q}}$ is of rank 2 (c_2^2, c_4) and we have $S^{[2]}$, $K_2(A)$ that can be shown to have independent classes.
- In dimension 6, $\mathrm{MU}^{12}(\mathrm{pt})_{\mathrm{even},\mathbb{Q}}$ is of rank 3 (c_2^3, c_4c_2, c_6) and we have $S^{[3]}$, $K_3(A)$, $\mathrm{OG6}$ that can be shown to have independent classes.
- What about higher dimension?

Subquestion 2. What inequalities do they satisfy? Two sign conjectures:

Question 1. Is it true that for any multiplicities $i_1, \ldots, i_k$, $(-1)^n \int_X \prod_r \mathrm{ch}_{2i_r}(T_X) \geq 0$ for any HK n-fold X?

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• **Holomorphic Euler-Poincaré characteristic**. X a compact complex manifold, $E \to X$ a holomorphic vector bundle $\rightsquigarrow$ cohomology groups $H^i(X, E)$= cohomology of the sheaf of holomorphic sections of E (computed as Dolbeault cohomology). $\chi(X, E) := \sum_i (-1)^i h^i(X, E)$.

• **Hirzebruch-Riemann-Roch formula**. $\chi(X, E)$ is given by integrating over X a universal polynomial expression in $c_i(X)$, $c_j(E)$.

Corollary. *Let Ω_X be the holomorphic cotangent bundle (basis dz_i in holomorphic coordinates z_i). Numbers $\chi(X, \Omega^i_X)$ are Chern numbers of X.*

Remark. In particular, $\chi(X, \mathcal{O}_X)$= Chern number of X. If X is a HK $2n$-fold, $h^i(X, \mathcal{O}_X) = h^{0,i}(X) = h^{i,0}(X) = 1$ for i even, 0 for i odd. So $\chi(X, \mathcal{O}_X) = n + 1$ (**affine relation between Chern numbers of HK's**).

Thm. (Ellingsrud-Göttsche-Lehn) *The Chern numbers of $S^{[n]}$, S a smooth compact complex surface, depend only on those of S.*

Corollary. *If $X = \mathrm{K3}^{[n]}$, the numbers $(-1)^i \chi(X, \Omega^i_X)$ are increasing for $i \leq n$.*

Question. Is this true for any hyper-Kähler $2n$-fold?

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