The Tits Alternative

Jacques Tits (1930-2021)

— I —

Free groups

Free groups

- $\mathbf{F}_2 = \langle a, b \mid \text{ no relation } \rangle$, the **free group** on two generators.

- Elements of $\mathbf{F}_2$: reduced words in a, b, a^{-1}, b^{-1}.

$$w = a \cdot a \cdot b^{-1} \cdot b^{-1} \cdot b^{-1} \cdot a \cdot a \cdot b \cdot a^{-1} = a^2 b^{-3} a^2 b a^{-1}$$

$$w' = a b^{-1} a^7 b^2.$$

- Multiplication: concatenation, and then reduction

$$\begin{aligned} w \cdot w' &= (a^2 b^{-3} a^2 b a^{-1}) \cdot (a b^{-1} a^7 b^2) \\ &= (a^2 b^{-3} a^2 b) \cdot (a^{-1} \cdot a) \cdot (b^{-1} a^7 b^2) \\ &= (a^2 b^{-3} a^2 b) \cdot (b^{-1} a^7 b^2) \\ &= a^2 b^{-3} a^9 b^2. \end{aligned}$$

–Theorem.–

- X = *a set.*
- $f, g \colon X \to X$ = *two bijections of* X.
- A, B *two subsets of* X *such that*
    - $A \neq \emptyset$
    - $A \cap B = \emptyset$
    - $f^n(A) \subset B$ *for all* $n \in \mathbf{Z} \setminus \{0\}$
    - $g^n(B) \subset A$ *for all* $n \in \mathbf{Z} \setminus \{0\}$.

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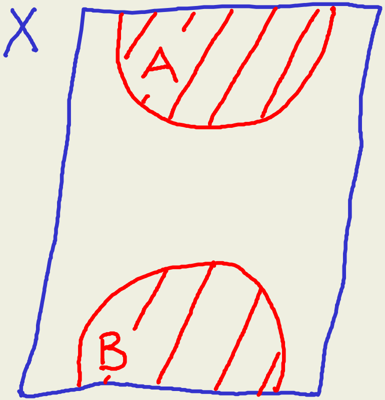

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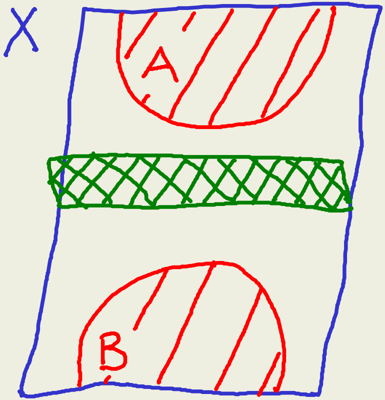

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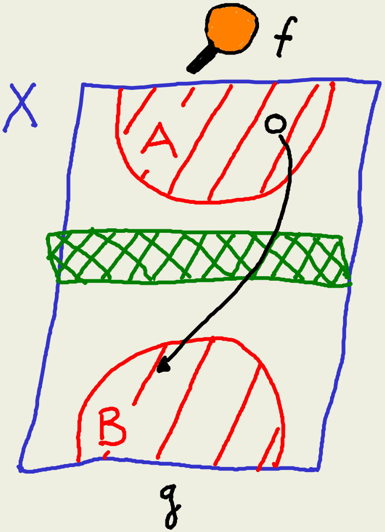


$w =$ a reduced word in f and g

$$w = f^2 g^{-3} f^9 g^{-1} fg.$$

Goal = prove that w is a non-trivial permutation of X.

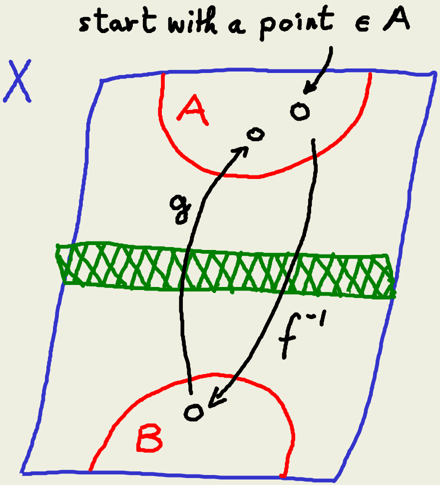
Write

$$fwf^{-1} = f^3 g^{-3} f^9 g^{-1} fgf^{-1}.$$

Then $w = Id_X$ iff $fwf^{-1} = Id_X$.

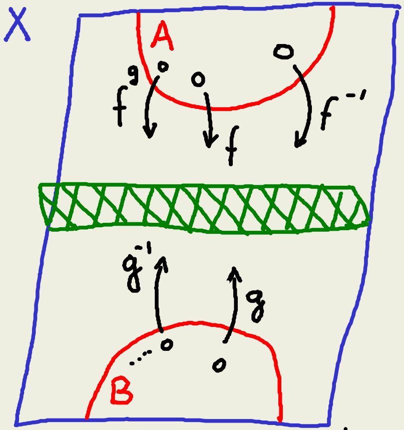
Goal

$$fwf^{-1} = f^3 g^{-3} f^9 g^{-1} f g f^{-1} \neq Id_X.$$



Goal

$$fwf^{-1} = f^3g^{-3}f^9g^{-1}fgf^{-1}$$
$$\neq Id_X.$$



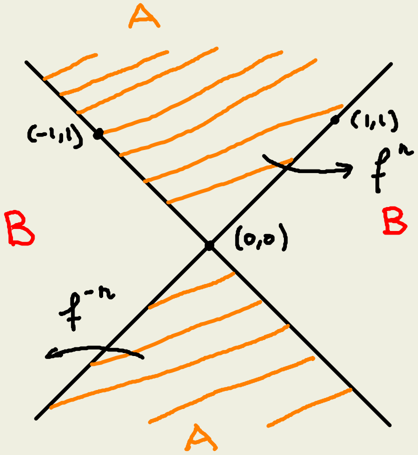
Finish with a point in B.

$$f = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$g = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a free subgroup on 2 generators in $SL_2(\mathbf{R})$.

$$f \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



— II —

Solvable groups, and the Tits alternative

- G = a group.
- G is **abelian** if and only if $uv = vu$,
  if and only if $uvu^{-1}v^{-1} = 1_G$ for all u, v in G.
- $D(G)$ = **derived subgroup**
  = group generated by the commutators $[u, v] = uvu^{-1}v^{-1}$.
- G is **solvable** if $D^n(G) = D(D(\cdots(D(G)))) = \{1_G\}$ for some $n \geq 1$.
- **Example.–** Group of affine transformations $z \mapsto az + b$ ($a, b \in \mathbf{C}$, $a \neq 0$)
  Group of upper triangular matrices.
- **Remark.–** The derived subgroup of $\mathbf{F}_2$ is a free group *on countably many generators*: $a^k b a^{-k} b^{-1}$, $k \in \mathbf{Z}^*$.

- **Tits alternative.–** *Let K be a field. Let m be a positive integer. Let G be a subgroup of $\mathrm{GL}_m(K)$ generated by finitely many matrices. Then,*
  - *either G contains a free group on 2 generators;*
  - *or G contains a solvable subgroup H of finite index in G.*

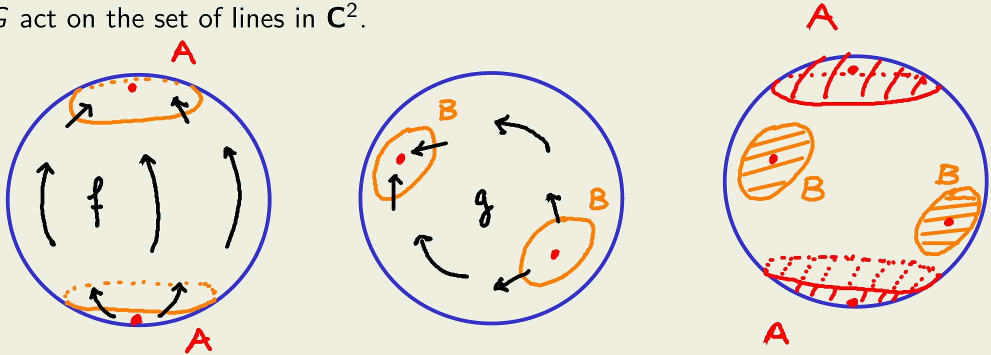
Variation:

- *either G contains two matrices that do not satisfy any non-trivial multiplicative relations;*
- *or G contains a subgroup H of finite index in G which preserves a full flag.*

- **Idea 1: Play ping-pong.–** Suppose $m = 2$ and G contains

$$f = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

Let G act on the set of lines in $\mathbf{C}^2$.



- **Idea 2: Play ping-pong.–** Assume G is contained in $SU_2(\mathbf{C})$, the group of unitary matrices, and contains

$$g = \begin{pmatrix} \frac{3+4\sqrt{-1}}{5} & 0 \\ 0 & \frac{3-4\sqrt{-1}}{5} \end{pmatrix}$$

- **Problem:** all eigenvalues have modulus 1.

- **Solution:** measure the size p-adically

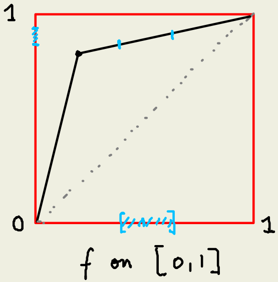
$$\left| \frac{3+4\sqrt{-1}}{5} \right|_5 = 5 > 1.$$

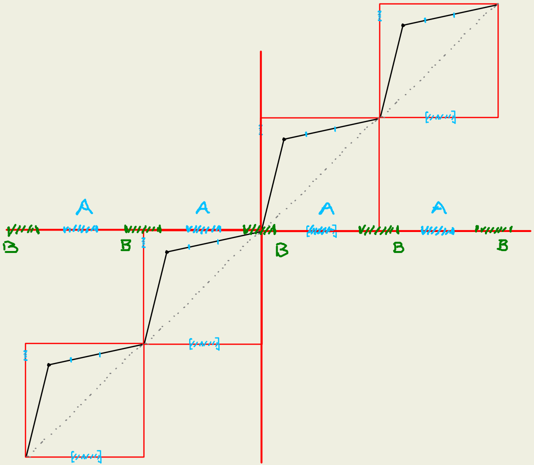
(with $\sqrt{-1} = 2 + 5 + 2 \cdot 5^2 + \cdots$ in $\mathbf{Z}_5$).

— III —

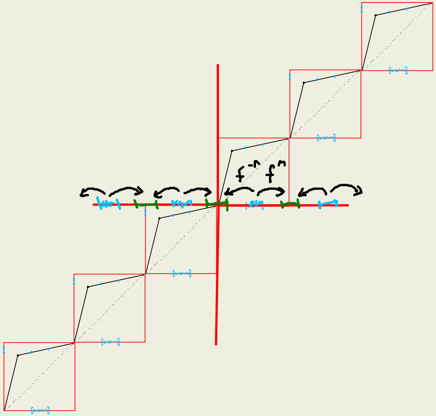
Homeomorphisms of the line, the interval, and the circle

- Homeo($\mathbf{R}$) = group of homeomorphisms of the real line.



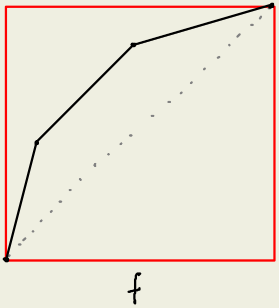


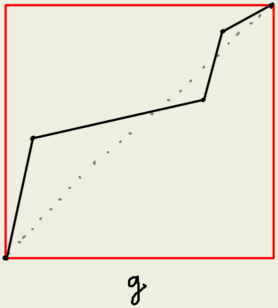
- **Theorem.–** *The group* Homeo($\mathbf{R}$) *contains a free group of rank* 2 *(made of piecewise linear homeomorphisms, with infinitely many pieces).*



- f^n maps A into B, $n \neq 0$
- $g = t \circ f \circ t^{-1}$ where $t(x) = x + \frac{1}{2}$.
- permutes A and B
- g^n maps A into B, $n \neq 0$

- $\mathrm{PL}^+([0,1])$ = piecewise linear, increasing homeomorphisms of $[0,1]$.
- **Brin-Squier Theorem.–** *The group* $\mathrm{PL}^+([0,1])$
  - *does not contain any free group on two generators,*
  - *but it satisfies no law.*





Goal

Find a non-trivial relation between f and g.

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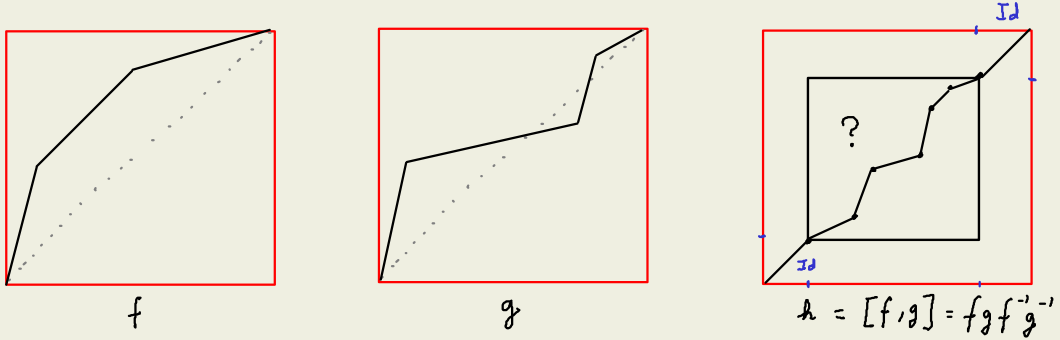

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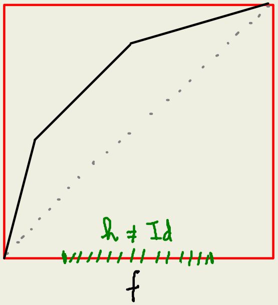

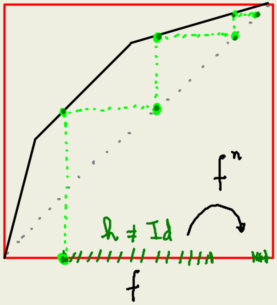

- $\exists\, n \geqslant 1$ such that
  - Support of h

  and
  - f^n(Support of h)

  are disjoint.
- $f^n \circ h \circ f^{-n}$ and h commute.

• **Ghys-Sergiescu Theorem.–** *The group G of piecewise linear homeomorphisms of the circle $\mathbf{R}/\mathbf{Z}$ such that*

- *the pieces are of type* $[\frac{a}{2^k}, \frac{b}{2^k}]$, $(a, b) \in \mathbf{Z}^2$, $k \in \mathbf{Z}_+$,
- *the slopes are powers of* 2 *(i.e. in* $2^{\mathbf{Z}}$*),*

is conjugate to a group of diffeomorphisms by some homeomorphism h of the circle:

$$h \circ G \circ h^{-1} \subset \mathrm{Diff}^{\infty}(\mathbf{R}/\mathbf{Z}).$$

• **Margulis Theorem.–** *Let H be a subgroup of* $\mathrm{Homeo}(\mathbf{R}/\mathbf{Z})$. *Then, either there is an H-invariant probability measure on $\mathbf{R}/\mathbf{Z}$, or H contains a non-abelian free group.*

— III —

Algebraic transformations

- K = a field.
- $\mathrm{Cr}_m(K)$ = Cremona group in m variables (over the field K)
  = group of birational transformations of the affine space $\mathbb{A}^m_K$
  = group of birational transformations of the projective space $\mathbb{P}^m_K$.

Examples in two variables (x, y)

$$f(x,y) = (x, y + p(x)) \text{ with inverse } f^{-1}(x,y) = (x, y - p(x))$$

$$g(x,y) = (x^2/y, y/x) \text{ monomial map associated to } \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$h(x,y) = (y + x^2 + c, -x) \text{ a Hénon map.}$$

- In one variable, the Cremona group is $\mathrm{Cr}_1(K) = \mathrm{PGL}_2(K)$. It satisfies the Tits alternative.

- **Theorem.–** *The Cremona group in two variables, over the field* $\mathbf{C}$*, satisfies the Tits alternative: if* $G \subset Cr_2(\mathbf{C})$ *is a subgroup, generated by finitely many elements, then*
  - *either* G *contains a finite index solvable subgroup,*
  - *or* G *contains a non-abelian free group.*

- **More generally.–** The group of birational transformations of a complex projective surface satisfies the Tits alternative.

- $\mathrm{End}(\mathbb{P}^1_K)$ = endomorphisms of the projective line $\mathbb{P}^1(K) = K \cup \{\infty\}$
  = rational fractions in one variable,
  = a semi-group for composition.

- **Theorem (Bell, Huang, Peng, Tucker) .–** *Let* $S \subset \mathrm{End}(\mathbb{P}^1_{\mathbf{C}})$ *be a finitely generated semi-group. Then,*
  - *either* S *has polynomial growth,*
  - *or* S *contains a non-abelian free semi-group.*