Dynamic programming operators over noncommutative spaces: an approach to optimal control of switched systems

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References: SG, NS arXiv:1706.04471, in CDC2017; SG, NS arXiv:1805.03284, in Math. Control Related Fields (2020); NS PhD thesis; NS arXiv:1612.05664, in Proc. AMS; X. Allamigeon, SG, E. Goubault, S. Putot, NS; A scalable algebraic method to infer quadratic invariants of switched systems, ACM Transactions on Embedded Computing Systems (TECS), Volume 15 Issue 4, August 2016

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lattice order $\leqslant$	
probability measures	
Markov operator	
$P \geqslant 0$, $Pe = e$	
value function	
Bellman operator	
$[T(v)]_i = \max_j (A_{ij} + v_j)$	

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Bellman operator	How do we fill this box ?
$[T(v)]_i = \max_j (A_{ij} + v_j)$	what can it be used for?

The joint spectral radius

$\mathcal{A} = \{A_1, \ldots, A_m\} \subset \mathbb{R}^{n \times n}$, largest growth rate:

$$\rho(\mathcal{A}) \coloneqq \lim_{k \to \infty} \sup_{A_{i_1}, \ldots A_{i_k} \in \mathcal{A}} \|A_{i_1} \cdots A_{i_k}\|^{1/k}.$$

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Theorem (Blondel-Tsitsiklis - 2000)

Unless P = NP, there is no polynomial-time computable function $\hat{\rho}$ of $\mathcal{A}$ and ε satisfying

$$|\rho(\mathcal{A}) - \hat{\rho}(\mathcal{A},\varepsilon)| \leqslant \varepsilon\rho(\mathcal{A})$$

even if $\mathcal{A}$ consists of 2 matrices with entries in $\{0,1\}$.

Theorem (Barabanov, 1988)

If the set $\mathcal{A}$ is irreducible, then there is a norm ν such that

$$\max_{i\in[m]} \nu(A_i x) = \rho(\mathcal{A})\nu(x) \ , \ \forall x \, .$$

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Certifying a upper bound of the joint spectral radius

Find a norm ν such that

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Then $\rho(\mathcal{A}) \leqslant \rho$.

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Goal

Construct a sequence of such norms ν_k such that the corresponding upper bounds ρ_k of $\rho(\mathcal{A})$ do converge to $\rho(\mathcal{A})$.

This talk

Use ideas / techniques from:

- max-plus basis methods Fleming, McEneaney, Akian, Dower, Kaise, Qu, SG, . . .

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to obtain a decreasing sequence of upper approximations of the joint spectral radius.

$\rightarrow$ method little sensitive to the curse of dimensionality: can deal with instances up to dimension 500 (random matrices with real entries) and even up to dimension 5000 (random matrices with nonnegative entries)

Bounds arising from piecewise quadratic norms

Look for

$$\nu(x) = \max_{v \in V} \sqrt{x^T Q_v x}$$

with V finite set, such that

$$\max_{i \in [m]} \nu(A_i x) \leqslant \rho \nu(x) \ , \forall x \, .$$

Then $\rho(\mathcal{A}) \leqslant \rho$. (Ahmadi et al), related to McEneaney's max-plus basis method)

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Goal: find collection of matrices $(Q_v)_v$ such that

$$\max_{i \in [n], v \in V} x^T (A_i^T Q_v A_i) x \leqslant \max_{w \in V} x^T (\rho^2 Q_w) x$$

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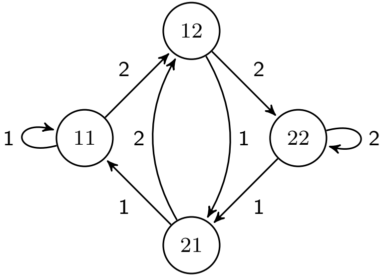
2 relaxations (Ahmadi et al.)

- For all v, i, there is w such that $A_i^T Q_v A_i \preccurlyeq \rho^2 Q_w$
- We enforce the choice of $w = \tau(v, i)$ for some transition map τ.

De Bruijn automaton, "concatenate and forget"

- Alphabet: $\Sigma := [m] = \{1, \ldots, m\}$, States: Σ^d
- Transition map τ_d:

$$\tau_d(v, i) = w \iff \begin{cases} v = i_1 i_2 \ldots i_d \\ w = i_2 \ldots i_d i \end{cases} .$$



Path-complete LMI automaton (Ahmadi et al.)

Solve family of LMIs:

$$(\mathcal{P}_\rho) \begin{cases} Q_v \succ 0 \ , \forall v \\ \rho^2 Q_w \succcurlyeq A_i^T Q_v A_i \ , \forall w = \tau_d(v, i) \end{cases}$$

Bisection:

$$\rho_d := \text{smallest } \rho \text{ such that } (\mathcal{P}_\rho) \text{ is feasible.}$$

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Theorem (Ahmadi et al. - SICON 2014)

An optimal solution $(Q_v)_v$ *provides a norm*

$$\nu(x) = \max_v (x^T Q_v x)^{1/2}$$

such that

$$\rho_d \geqslant \rho(A) \geqslant \frac{1}{n^{\frac{1}{2(d+1)}}} \rho_d$$

(asymptotically exact as $d \to \infty$*).*

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(asymptotically exact as $d \to \infty$).

Proof based on the Loewner-John theorem: the Barabanov norm can be approximated by an Euclidean norm up to a $\sqrt{n}$ multiplicative factor.

Before...

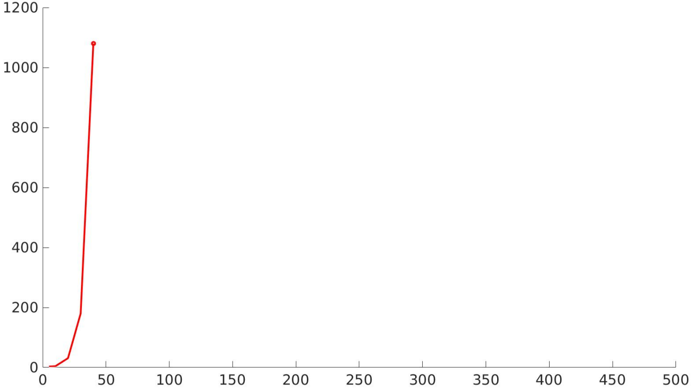

Figure: Computation time (s) vs dimension: red Ahmadi et al., ,

...Now

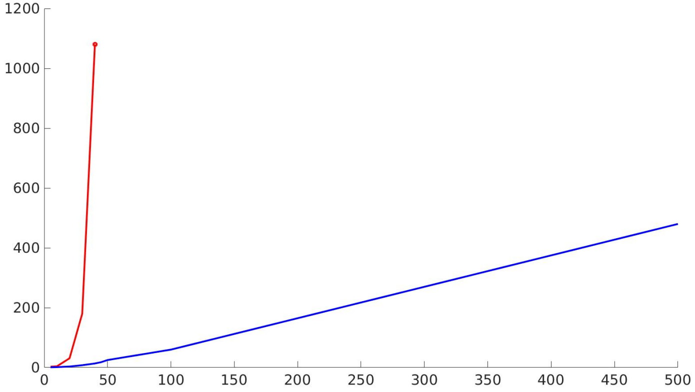

Figure: Computation time (s) vs dimension: red Ahmadi et al., blue "quantum" dynamic programming (this talk),

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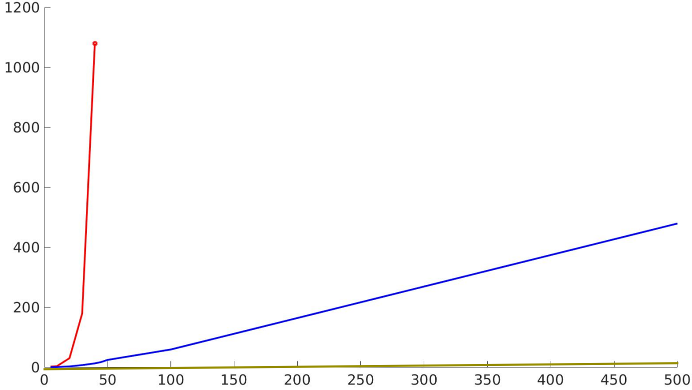

Figure: Computation time (s) vs dimension: red Ahmadi et al., blue "quantum" dynamic programming (this talk), green specialization to nonnegative matrices (this talk - MCRF, 2020)

How do we get there ?

A closer look at simplified LMIs

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Fast computational scheme

Interior point methods are relatively slow

$\rightarrow$ Replace optimization by a fixed point approach. For nonnegative matrices, reduces to a risk-sensitive eigenproblem.

Minimal upper bounds

x is a minimal upper bound of the set $\mathcal{A}$ iff

$$\mathcal{A} \preccurlyeq x \quad \text{and} \quad \left(\mathcal{A} \preccurlyeq y \preccurlyeq x \implies y = x\right).$$

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Theorem (Krein-Rutman - 1948)

A cone induces a lattice structure iff it is simplicial ($\cong \mathbb{R}_n^+$).

Theorem (Kadison - 1951)

The Löwner order induces an anti-lattice structure: two symmetric matrices A, B have a supremum if and only if $A \preccurlyeq B$ or $B \preccurlyeq A$.

The inertia of the symmetric matrix M is the tuple (p, q, r), where

- p: number of positive eigenvalues of M,
- q: number of negative eigenvalues of M,
- r: number of zero eigenvalues of M.

Definition (Indefinite orthogonal group)

$\mathcal{O}(p, q)$ is the group of matrices S preserving the quadratic form $x_1^1 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$:

$$S\begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}S^T = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix} =: J_{p,q}$$

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$\mathcal{O}(1,1)$ is the group of hyperbolic isometries $\begin{pmatrix} \epsilon_1 \operatorname{ch} t & \epsilon_2 \operatorname{sh} t \\ \epsilon_1 \operatorname{sh} t & \epsilon_2 \operatorname{ch} t \end{pmatrix}$, where $\epsilon_1, \epsilon_2 \in \{-1, 1\}$

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where $\epsilon_1, \epsilon_2 \in \{-1, 1\}$
$\mathcal{O}(p) \times \mathcal{O}(q)$ is a maximal compact subgroup of $\mathcal{O}(p, q)$.

Theorem (Stott - Proc AMS 2018, Quantitative version of Kadison theorem)

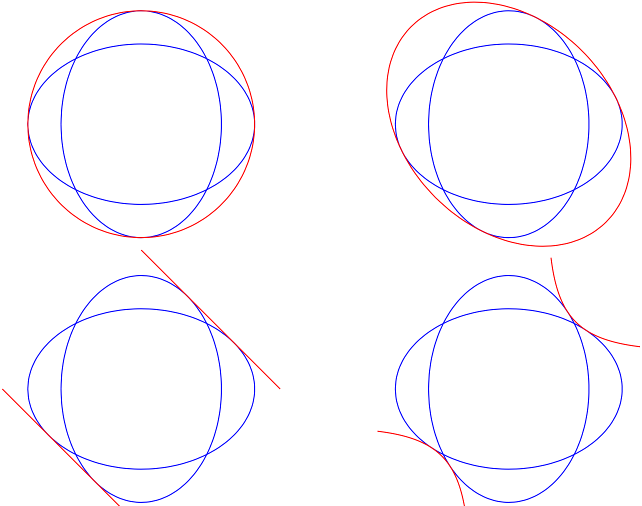
If the inertia of $A - B$ is $(p, q, 0)$, then

$$\bigvee\{A\,,B\} \;\cong\; \mathcal{O}(p,q)\Big/\mathcal{O}(p)\times\mathcal{O}(q) \;\cong\; \mathbb{R}^{pq}\,.$$

Example $p = q = 1$.

$\mathcal{O}(1,1) \big/ \mathcal{O}(1) \times \mathcal{O}(1)$ is the group of hyperbolic rotations:

$$\left\{ \begin{pmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{pmatrix} \mid t \in \mathbb{R} \right\}$$



Canonical selection of a minimal upper bound

Ellipsoid: $\mathcal{E}(M) = \{x \mid x^T M^{-1} x \leqslant 1\}$, where M is symmetric pos. def.

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Let $\mathcal{A} = \{A_i\}_i \subset \mathcal{S}_n^{++}$ and $\mathcal{C} = \cup_i \mathcal{E}(A_i)$. We define $\sqcup\mathcal{A}$ so that $\mathcal{E}(\sqcup\mathcal{A})$ is the Löwner ellipsoid of $\cup_{A\in\mathcal{A}}\mathcal{E}(A)$, i.e.,

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$$(\sqcup \mathcal{A})^{-1} = \mathrm{argmax}_X \{\log\det X \mid X \preccurlyeq A_i^{-1}, i \in [m], \quad X \succ 0\} \ .$$

Then, $\sqcup \mathcal{A}$ is a minimal upper bound of $\mathcal{A}$, and $\sqcup$ is the only selection that commutes with the action of invertible congruences:

$$L(\sqcup \mathcal{A})L^T = \sqcup(L \mathcal{A} L^T) \, ,$$

Theorem (Allamigeon, SG, Goubault, Putot, NS, ACM TECS 2016)

Computating $X \sqcup Y$ reduces to a square root (i.e., SDP-free!).

$$\text{Suppose } Y = I: \qquad X \sqcup I = \tfrac{1}{2}(X + I) + \tfrac{1}{2}|X - I|\,.$$

General case reduces to it by congruence: add 1 Cholesky decomposition $+\,1$ triangular inversion. Complexity: $O(n^3)$.

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- continuous on $\mathcal{S}_n^{++} \times \mathcal{S}_n^{++}$ but does not extend continuously to the closed cone,
- not order-preserving,
- not associative.

Reducing the search of a joint quadratic Lyapunov function to an eigenproblem

Goal

Compute norm $\nu(x) = \sqrt{x^T Q x}$ such that $\max_{i\in[m]} \nu(A_i x) \leqslant \rho\nu(x)$.

Computation: single quadratic form

Corresponding LMI:

$$\rho^2 Q \succcurlyeq A_i^T Q A_i \ , \forall i \, .$$

Eigenvalue problem for a multivalued map

$$\rho^2 Q \in \bigvee_i A_i^T Q A_i \, .$$

Quantum dynamic programming operators

Quantum channels (0-player games)

Completely positive trace perserving operators:

$$K(X) = \sum_i A_i X A_i^* \, , \qquad \sum_i A_i^* A_i = I_n \, .$$

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Propagation of "non-commutative probability measures" (analogue of Fokker-Planck).

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with $\bigvee$ the set of least upper bounds in Löwner order (multivalued map).

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with $\bigvee$ the set of least upper bounds in Löwner order (multivalued map).
Propagation of norms (backward equation).

Quantum dynamic programming operator associated with an automaton

τ_d transition map of the De Bruijn automaton on d letters:

$$X \in (\mathcal{S}_n^+)^{(m^d)} \quad \text{and} \quad \mathcal{T}_w^d(X) := \bigvee_{w=\tau_d(v,i)} A_i^T X_v A_i$$

Reduces to the earlier $d = 1$ case by a block diagonal construction.

Theorem

Suppose that

$$\rho^2 X \in \mathcal{T}^d(X)$$

with $\rho > 0$ and X positive definite. Then,

$$\rho(\mathcal{A}) \leqslant \rho\,.$$

Theorem

Suppose that $\mathcal{A}$ is irreducible. Then there exists $\rho > 0$ and X such that $\sum_v X_v$ is positive definite and

$$\rho^2 X = T^d_{\sqcup}(X) \in \mathcal{T}^d(X)$$

where

$$[T^d_{\sqcup}(X)]_w := \bigsqcup_{w=\tau_d(v,i)} A_i^T X_v A_i \ .$$

Exercise: find the mistake in the following proof

We want to show that the following eigenproblem is solvable:

$$[T^d_\sqcup(X)]_w := \bigsqcup_{w=\tau_d(v,i)} A_i^T X_v A_i = \rho^2 X_w$$

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$\rightarrow$ cannot apply naively Brouwer.

Fixing the proof of existence of eigenvectors

Lemma

For $Y_i \succ 0$, we have

$$\frac{1}{m}\sum_{i=1}^{m} Y_i \preccurlyeq \bigsqcup_{i=1}^{m} Y_i \preccurlyeq \sum_{i=1}^{m} Y_i$$

Corollary

For all $X \in \mathcal{S}_n^+$, we have

$$\frac{1}{m}K^d(X) \preccurlyeq T_{\sqcup}^d(X) \preccurlyeq K^d(X)\,,$$

with

$$K_w^d(X) = \sum_{w=\tau_d(v,i)} A_i^T X_v A_i \qquad T_{\sqcup,w}^d(X) = \bigsqcup_{w=\tau_d(v,i)} A_i^T X_v A_i\,.$$

Proof

Reduction to $K\colon X \mapsto \sum_i A_i^T X A_i$ strictly positive:

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$$\frac{\alpha}{m} I \preccurlyeq T_{\sqcup}(X) \preccurlyeq \beta I\,,$$

so $T_{\sqcup}(\Delta) \subset$ compact subset of $\operatorname{int} \Delta$.

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so $T_{\sqcup}(\Delta) \subset$ compact subset of $\operatorname{int}\Delta$. Conclude by Brouwer's fixed point theorem. □

Computing an eigenvector

We introduce a damping parameter γ:

$$T_{\sqcup}^{\gamma}(X) = \bigsqcup_i \left(A_i^T X A_i + \gamma(\operatorname{trace} X) I_n\right).$$

Theorem

The iteration

$$X^{k+1} = \frac{T_{\sqcup}^{\gamma}(X)}{\operatorname{trace} T_{\sqcup}^{\gamma}(X)}$$

converges for a large damping: $\gamma > nm^{(3d+1)/2}$

Conjecture

The iteration converges if $\gamma > m^{1/2}n^{-1/2}$.

Experimentally: $\gamma \sim 10^{-2}$ is enough! Huge gap between conservative theoretical estimates and practice. How theoretical estimates are obtained?

Lipschitz estimations

Riemann and Thompson metrics

Two standard metrics on the cone $\mathcal{S}_n^{++}$

$$d_R(A,B) \coloneqq \| \log \operatorname{spec}(A^{-1}B)\|_2 \,.$$

$$d_T(A,B) \coloneqq \| \log \operatorname{spec}(A^{-1}B)\|_\infty \,.$$

They are invariant under the action of congruences:
$d(LAL^T, LBL^T) = d(A,B)$ for invertible L.

Lipschitz constant: $\operatorname{Lip}_M \sqcup \coloneqq \sup_{X_1,X_2,Y_1,Y_2 \succ 0} \frac{d_M(X_1 \sqcup X_2, Y_1 \sqcup Y_2)}{d_M(X_1 \oplus X_2, Y_1 \oplus Y_2)}$.

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Theorem

$$\operatorname{Lip}_T \sqcup = \Theta(\log n) \qquad \operatorname{Lip}_R \sqcup = 1$$

Proof.

d_T, d_R are Riemann/Finsler metrics $\to$ work locally + Schur multiplier estimation (Mathias). □

Scalability: dimension

Table: big-LMI vs Tropical Kraus

Dimension n	CPU time (tropical)	CPU time (LMI)	Error vs LMI
5	0.9 s	3.1 s	0.1 %
10	1.5 s	4.2 s	1.4 %
20	3.5 s	31 s	0.4 %
30	7.9 s	3min	0.2 %
40	13.7 s	18min	0.05 %
45	18.1 s	–	–
50	25.2 s	–	–
100	1min	–	–
500	8min	–	–

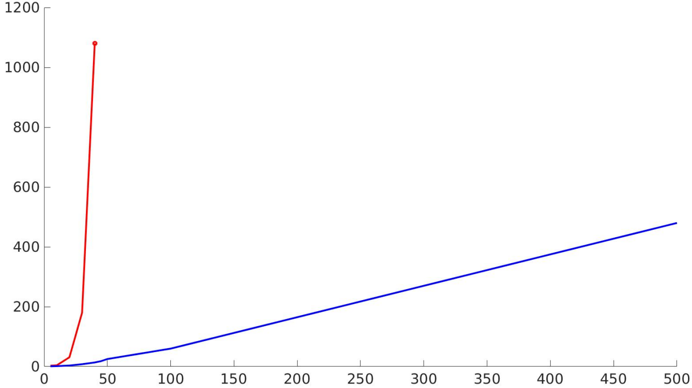

Figure: Computation time vs dimension

Scalability: graph size

$$A_1 = \begin{pmatrix} -1 & 1 & -1 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & 1 & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Table: big-LMI vs Tropical Kraus: $30 - 60$ times faster.

Order d	2	4	6	8	10
Size of graph	8	32	128	512	2048
CPU time (tropical)	0.03s	0.07s	0.4s	2.0s	9.0s
CPU time (LMI)	1.9s	4.0s	24s	1min	10min
Accuracy	1.1 %	1.3 %	0.4 %	0.4 %	0.6 %

Special case of nonnegative matrices

Suppose $A_i \in \mathbb{R}_+^{n\times n}$, replace the quantum dynamic programming operator

$$X \in (\mathcal{S}_n^+)^{(m^d)} \quad \text{and} \quad \mathcal{T}_w^d(X) := \bigvee_{w=\tau_d(v,i)} A_i^T X_v A_i$$

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Theorem

Suppose the set of nonnegative matrices $\mathcal{A}$ is positively irreducible. Then, there exists $u \in (\mathbb{R}_+)^{(m^d)} \setminus \{0\}$ such that

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Follows from SG and Gunawardena, TAMS 2004.

A monotone hemi-norm is a map $\nu(x) := \max_{v \in V} \langle u_v, x \rangle$ with $u_v \geqslant 0$ such that $x \mapsto \nu(x) \vee \nu(-x)$ is a norm.

Theorem (Coro. of Guglielmi and Protasov)

If $\mathcal{A} \subset \mathbb{R}_+^{n \times n}$ is positively irreducible, there is a monotone hemi-norm ν such that

$$\max_{i \in [m]} \nu(A_i x) = \rho(\mathcal{A}) \nu(x), \qquad \forall x \in \mathbb{R}_+^n$$

Theorem (Polyhedral monotone hemi-norms)

If $\mathcal{A} \subset \mathbb{R}_+^{n \times n}$ is positively irreducible, if $T^d(u) = \lambda_d u$, and $u \in (\mathbb{R}_+^n)^{(m^d)} \setminus \{0\}$, then

$$\|x\|_u := \max_{v \in [m^d]} \langle u_v, x \rangle$$

is a polyhedral monotone hemi-norm and

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Moreover, $\rho(\mathcal{A}) \leqslant \lambda_d \leqslant n^{1/(d+1)}\rho(\mathcal{A})$, in particular $\lambda_d \to \lambda$ as $d \to \infty$.

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policy iteration/spectral simplex requires computing eigenvalues (demanding), need to work with huge scale instances (dimension $N = n \times m^d$)

Krasnoselski-Mann iteration

$$x_{k+1} = \frac{1}{2}(x_k + F(x_k))$$

applies to a nonexpansive map F: $\|F(x) - F(y)\| \leqslant \|x - y\|$.

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Theorem (Ishikawa)

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Theorem (Baillon, Bruck)

$$\|F(x^k) - x^k\| \leqslant \frac{2 \operatorname{diam}(D)}{\sqrt{\pi k}} \ ,$$

Definition (Projective Krasnoselskii-Mann iteration)

Suppose $f : \mathbb{R}_+^N \to \mathbb{R}_+^N$ is order preserving and positively homogeneous of degree 1. Choose any $v^0 \in \mathbb{R}_{>0}^N$ such that $\prod_{i\in[N]} v_i^0 = 1$,

$$v^{k+1} = \left[\frac{f(v^k)}{G\big[f(v^k)\big]} \circ v^k\right]^{1/2}, \tag{1}$$

where $x \circ y := (x_i y_i)$ and $G(x) = (x_1 \cdots x_N)^{1/N}$.

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Suppose in addition that f has a positive eigenvector. Then, the projective Krasnoselskii-Mann iteration initialized at any positive vector $v^0 \in \mathbb{R}_+^N$ such that $\prod_{i\in[N]} v_i^0 = 1$, converges towards an eigenvector of f, and $G(f(v^k))$ converges to the maximal eigenvalue of f.

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Proof idea. This is Krasnoselski iteration applied to $F := \log \circ f \circ \exp$ acting in the quotient of the normed space $(\mathbb{R}^n, \|\cdot\|_\infty)$ by the one dimensional subspace $\mathbb{R}\mathbf{1}_N$.

Corollary

Take $f := T^d$*, the risk-sensitive dynamic programming operator, and let*

$$\beta_k := \max_{i \in [N]} (f(v^k))_i / v_i^k \ .$$

Then,

$$\log \rho(\mathcal{A}) \leqslant \log \beta_k \leqslant \log \rho(A) + \frac{4}{\sqrt{\pi k}} d_H(v^0, u) + \frac{\log n}{d+1}$$

where d_H *is Hilbert's projective metric.*

Level d	CPU Time (s)	Eigenvalue λ_d	Relative error
1	0.01	2.165	6.8%
2	0.01	2.102	3.7%
3	0.01	2.086	2.9%
4	0.01	2.059	1.6%
5	0.02	2.041	0.7%
6	0.05	2.030	0.1%
7	0.7	2.027	0.0%
8	0.32	2.027	0.0%
9	1.12	2.027	0.0%

Table: Convergence of the hierarchy on an instance with 5×5 matrices and a maximizing cyclic product of length 6

Dimension n	Level d	Eigenvalue λ_d	CPU Time
10	2	4.287	0.01 s
	3	4.286	0.03 s
20	2	8.582	0.01 s
	3	8.576	0.03 s
50	2	22.34	0.04 s
	3	22.33	0.16 s
100	2	44.45	0.17 s
	3	44.45	0.53 s
200	2	89.77	0.71 s
	3	89.76	2.46 s
500	2	224.88	5.45 s
	3	224.88	19.7 s
1000	2	449.87	44.0 s
	3	449.87	2.7 min
2000	2	889.96	4.6 min
	3	889.96	19.2 min
5000	2	2249.69	51.9 min
	3	2249.57	3.3 h

Table: Computation time for large matrices

MEGA

The Minimal Ellipsoid Geometric Analyzer, Stott - available from http://www.cmap.polytechnique.fr/~stott/

- implements the quantum dynamic programming approach
- 1700 lines of OCaml and 800 lines of Matlab
- uses BLAS/LAPACK via LACAML for linear algebra
- uses OSDP/CSDP for some semidefinite programming
- uses Matlab for other semidefinite programming

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Thank you !