Dynamic programming operators over noncommutative spaces: an approach to optimal control of switched systems

Stéphane Gaubert* Nikolas Stott**

Stephane.Gaubert@inria.fr
*: INRIA and CMAP, Ecole polytechnique, IP Paris, CNRS
**: LocalSolver

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<tr>
<td>probability measures</td>
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<td>Markov operator</td>
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<tr>
<td>value function</td>
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<td>Bellman operator</td>
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<td>“noncommutative” dynamic programming</td>
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<td>$\mathbb{R}^n$</td>
<td>$S_n$, symmetric matrices</td>
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<tr>
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<td>Loewner order $(X \succeq 0 \iff \lambda_{\min}(X) \geq 0)$</td>
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<tr>
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<td></td>
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<tr>
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<tr>
<td>probability measures</td>
<td>density matrices</td>
</tr>
<tr>
<td>Markov operator</td>
<td>Quantum channel</td>
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<tr>
<td>$P \succeq 0$, $Pe = e$</td>
<td>$K(X) = \sum_i A_i^* X A_i$, $\sum_i A_i A_i^* = I$</td>
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<td>what can it be used for?</td>
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The joint spectral radius

\[ A = \{A_1, \ldots, A_m\} \subset \mathbb{R}^{n \times n}, \text{ largest growth rate:} \]

\[ \rho(A) := \lim_{k \to \infty} \sup_{A_{i_1}, \ldots, A_{i_k} \in A} \| A_{i_1} \cdots A_{i_k} \|^1/k. \]
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**Theorem (Blondel-Tsitsiklis - 2000)**

Unless \( P = NP \), there is no polynomial-time computable function \( \hat{\rho} \) of \( A \) and \( \varepsilon \) satisfying

\[ |\rho(A) - \hat{\rho}(A, \varepsilon)| \leq \varepsilon \rho(A) \]

even if \( A \) consists of 2 matrices with entries in \( \{0, 1\} \).
Theorem (Barabanov, 1988)

If the set $\mathcal{A}$ is irreducible, then there is a norm $\nu$ such that

$$\max_{i \in [m]} \nu(A_ix) = \rho(\mathcal{A})\nu(x) , \forall x .$$
Theorem (Barabanov, 1988)

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Special case of ergodic control problem. Continuous time version: reduction to an ergodic HJ PDE (Calvez, SG, Gabriel 2014).
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**Certifying a upper bound of the joint spectral radius**

Find a norm $\nu$ such that

$$
\max_{i \in [m]} \nu(A_i x) \leq \rho \nu(x) \quad \forall x.
$$

Then $\rho(\mathcal{A}) \leq \rho$. 
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Then $\rho(\mathcal{A}) \leq \rho$.

Goal

Construct a sequence of such norms $\nu_k$ such that the corresponding upper bounds $\rho_k$ of $\rho(\mathcal{A})$ do converge to $\rho(\mathcal{A})$. 
This talk

Use ideas / techniques from:

- max-plus basis methods Fleming, McEneaney, Akian, Dower, Kaise, Qu, SG, . . .
- path-complete automata Ahmadi, Parrilo, Jungers, Roozbehani
- polyhedral approximation: Guglielmi, Kozyakin, Protasov . . .
- geometry of the Loewner order
- non-linear Perron-Frobenius theory, nonexpansive mappings Nussbaum, Baillon, Bruck, SG, Gunawardena, . . .
- risk-sensitive control Anantharam, Borkar

→ method little sensitive to the curse of dimensionality: can deal with instances up to dimension 500 (random matrices with real entries) and even up to dimension 5000 (random matrices with nonnegative entries)
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...to obtain a decreasing sequence of upper approximations of the joint spectral radius.
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To obtain a decreasing sequence of upper approximations of the joint spectral radius.

→ Method little sensitive to the curse of dimensionality: can deal with instances up to dimension 500 (random matrices with real entries) and even up to dimension 5000 (random matrices with nonnegative entries)
Bounds arising from piecewise quadratic norms

Look for

$$\nu(x) = \max_{v \in V} \sqrt{x^T Q_v x}$$

with $V$ finite set, such that

$$\max_{i \in [m]} \nu(A_i x) \leq \rho \nu(x), \forall x.$$ 

Then $\rho(\mathbb{A}) \leq \rho$. (Ahmadi et al), related to McEneaney's max-plus basis method)
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Then \( \rho(\mathcal{A}) \leq \rho. \) (Ahmadi et al), related to McEneaney’s max-plus basis method)

Goal: find collection of matrices \((Q_v)_v\) such that

\[ \max_{i \in [n], v \in V} x^T (A_i^T Q_v A_i) x \leq \max_{w \in V} x^T (\rho^2 Q_w) x \]
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2 relaxations (Ahmadi et al.)

- For all $v, i$, there is $w$ such that $A_i^T Q_v A_i \preceq \rho^2 Q_w$
- We enforce the choice of $w = \tau(v, i)$ for some transition map $\tau$. 
De Bruijn automaton, “concatenate and forget”

• Alphabet: $\Sigma := \{m\} = \{1, \ldots, m\}$, States: $\Sigma^d$

• Transition map $\tau_d$:

$$\tau_d(v, i) = w \iff \begin{cases} v = i_1i_2\ldots i_d \\ w = i_2\ldots i_di_1 \end{cases}.$$
Path-complete LMI automaton (Ahmadi et al.)

Solve family of LMIs:

\[
\begin{align*}
(\mathcal{P}_\rho) \begin{cases} 
  Q_v & \succ 0, \quad \forall v \\
  \rho^2 Q_w & \succeq A_i^T Q_v A_i, \quad \forall w = \tau_d(v, i)
\end{cases}
\end{align*}
\]

Bisection:

\[\rho_d := \text{smallest } \rho \text{ such that } (\mathcal{P}_\rho) \text{ is feasible.}\]
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Theorem (Ahmadi et al. - SICON 2014)

An optimal solution \((Q_v)_v\) provides a norm

\[
\nu(x) = \max_v (x^T Q_v x)^{1/2}
\]

such that

\[
\rho_d \geq \rho(A) \geq \frac{1}{n^{2(d+1)}} \rho_d
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(asymptotically exact as \(d \to \infty\)).
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(asymptotically exact as \(d \to \infty\)).

Proof based on the Loewner-John theorem: the Barabanov norm can be approximated by an Euclidean norm up to a \(\sqrt{n}\) multiplicative factor.
Before...

Figure: Computation time (s) vs dimension: red Ahmadi et al.,
...Now

Figure: Computation time (s) vs dimension: red Ahmadi et al., blue "quantum" dynamic programming (this talk),
Figure: Computation time (s) vs dimension: red Ahmadi et al., blue “quantum” dynamic programming (this talk), green specialization to nonnegative matrices (this talk - MCRF, 2020)
A closer look at simplified LMIs

\[
Q \succ 0 \quad \rho^2 Q \succeq A_i^T Q A_i , \; \forall i \in [m].
\]
How do we get there?

A closer look at simplified LMIs

\[
Q \succ 0 \quad \rho^2 Q \succeq A_i^TQA_i, \forall i \in [m].
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Solving a wrong equation

We would like to write:

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\text{"} \rho^2 Q \succeq \sup_{i \in [m]} A_i^TQA_i \text{"}.
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The supremum of several quadratic forms does not exist!

⇒ will replace supremum by a minimal upper bound
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Fast computational scheme

Interior point methods are relatively slow

→ Replace optimization by a fixed point approach. For nonnegative matrices, reduces to a risk-sensitive eigenproblem.
Minimal upper bounds

$x$ is a minimal upper bound of the set $A$ iff

\[ A \preceq x \quad \text{and} \quad (A \preceq y \preceq x \implies y = x). \]

The set of minimal upper bounds: $\bigvee A$. 

**Minimal upper bounds**

$x$ is a minimal upper bound of the set $\mathcal{A}$ iff

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**Theorem (Krein-Rutman - 1948)**

An *cone induces a lattice structure iff it is simplicial* ($\cong \mathbb{R}^+_n$).
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Theorem (Krein-Rutman - 1948)

A cone induces a lattice structure iff it is simplicial ($\cong \mathbb{R}_n^+$).

Theorem (Kadison - 1951)

The Löwner order induces an anti-lattice structure: two symmetric matrices $A, B$ have a supremum if and only if $A \preceq B$ or $B \preceq A$. 
The **inertia** of the symmetric matrix $M$ is the tuple $(p, q, r)$, where

- $p$: number of positive eigenvalues of $M$,
- $q$: number of negative eigenvalues of $M$,
- $r$: number of zero eigenvalues of $M$.

**Definition (Indefinite orthogonal group)**

$O(p, q)$ is the group of matrices $S$ preserving the quadratic form

$$x_1^1 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

such that

$$S \begin{pmatrix} I_p & -I_q \\ & \end{pmatrix} S^T = \begin{pmatrix} I_p & -I_q \\ & \end{pmatrix} =: J_{p,q}$$
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$O(1, 1)$ is the group of hyperbolic isometries $\begin{pmatrix} \epsilon_1 \cosh t & \epsilon_2 \sinh t \\ \epsilon_1 \sinh t & \epsilon_2 \cosh t \end{pmatrix}$, where $\epsilon_1, \epsilon_2 \in \{-1, 1\}$
The inertia of the symmetric matrix $M$ is the tuple $(p, q, r)$, where

- $p$: number of positive eigenvalues of $M$,
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$\mathcal{O}(p) \times \mathcal{O}(q)$ is a maximal compact subgroup of $\mathcal{O}(p, q)$. 
Theorem (Stott - Proc AMS 2018, Quantitative version of Kadison theorem)

If the inertia of $A - B$ is $(p, q, 0)$, then

$$\bigvee\{A, B\} \cong \mathcal{O}(p, q) / \mathcal{O}(p) \times \mathcal{O}(q) \cong \mathbb{R}^{pq}.$$
Example $p = q = 1$.

$\mathcal{O}(1, 1) / \mathcal{O}(1) \times \mathcal{O}(1)$ is the group of hyperbolic rotations:

\[
\left\{ \begin{pmatrix} \text{ch} t & \text{sh} t \\ \text{sh} t & \text{ch} t \end{pmatrix} \mid t \in \mathbb{R} \right\}
\]
Ellipsoid: $\mathcal{E}(M) = \{x \mid x^T M^{-1} x \leq 1\}$, where $M$ is symmetric pos. def.

**Theorem (Löwner - John)**

There is a unique minimum volume ellipsoid containing a convex body $C$. 
Canonical selection of a minimal upper bound

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**Definition-Proposition (Allamigeon, SG, Goubault, Putot, NS, ACM TECS 2016)**

Let \( \mathcal{A} = \{ A_i \}_i \subset \mathcal{S}_{n}^{++} \) and \( C = \bigcup_i \mathcal{E}(A_i) \). We define \( \sqcup \mathcal{A} \) so that \( \mathcal{E}(\sqcup \mathcal{A}) \) is the Löwner ellipsoid of \( \bigcup_{A \in \mathcal{A}} \mathcal{E}(A) \), i.e.,

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(\sqcup \mathcal{A})^{-1} = \arg\max_X \{ \log \det X \mid X \preceq A_i^{-1}, i \in [m], \quad X \succ 0 \}.
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\[(\sqcup \mathcal{A})^{-1} = \arg\max_X \{ \log \det X | X \preceq A_i^{-1}, i \in [m], \ X \succ 0 \} \ .\]

Then, $\sqcup \mathcal{A}$ is a minimal upper bound of $\mathcal{A}$, and $\sqcup$ is the only selection that commutes with the action of invertible congruences:

\[L(\sqcup \mathcal{A})L^T = \sqcup(LAL^T) ,\]
Theorem (Allamigeon, SG, Goubault, Putot, NS, ACM TECS 2016)

Computing $X \sqcup Y$ reduces to a square root (i.e., SDP-free!).

Suppose $Y = I$: $X \sqcup I = \frac{1}{2}(X + I) + \frac{1}{2}|X - I|$.

General case reduces to it by congruence: add 1 Cholesky decomposition + 1 triangular inversion. Complexity: $O(n^3)$. 
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General case reduces to it by congruence: add 1 Cholesky decomposition + 1 triangular inversion. Complexity: $O(n^3)$.

The Loewner selection $\boxplus$ is
- continuous on $S_n^{++} \times S_n^{++}$ but does not extend continuously to the closed cone,
Theorem (Allamigeon, SG, Goubault, Putot, NS, ACM TECS 2016)

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*Computating* $X \sqcup Y$ *reduces to a square root (i.e., SDP-free!).*

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---

The Loewner selection $\sqcup$ is

- continuous on $S_n^{++} \times S_n^{++}$ but does not extend continuously to the closed cone,
- not order-preserving,
- not associative.
Reducing the search of a joint quadratic Lyapunov function to an eigenproblem

Goal
Compute norm $\nu(x) = \sqrt{x^T Q x}$ such that $\max_{i \in [m]} \nu(A_i x) \leq \rho \nu(x)$.

Computation: single quadratic form

Corresponding LMI:
$$\rho^2 Q \succeq A_i^T Q A_i, \ \forall i.$$ 

Eigenvalue problem for a multivalued map
$$\rho^2 Q \in \bigvee_i A_i^T Q A_i.$$
Quantum dynamic programming operators

Quantum channels (0-player games)

Completely positive trace preserving operators:

\[ K(X) = \sum_i A_i X A_i^*, \quad \sum_i A_i^* A_i = I_n. \]
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Propagation of "non-commutative probability measures" (analogue of Fokker-Planck).

Quantum dynamic programming operator (1-player game)

\[ \mathcal{T}(X) = \bigvee_i A_i^T X A_i \]

with \( \bigvee \) the set of least upper bounds in Löwner order (multivalued map).
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Propagation of "non-commutative probability measures" (analogue of Fokker-Planck).

Quantum dynamic programming operator (1-player game)

\[ T(X) = \bigvee_{i} A_i^T X A_i \]

with \( \bigvee \) the set of least upper bounds in L"owner order (multivalued map). Propagation of norms (backward equation).
Quantum dynamic programming operator associated with an automaton

$\tau_d$ transition map of the De Bruijn automaton on $d$ letters:

$$X \in (S_n^+)^{m^d} \quad \text{and} \quad T_d^d(X) := \bigvee_{w=\tau_d(v,i)} \ A_i^T X_v A_i$$

Reduces to the earlier $d = 1$ case by a block diagonal construction.

**Theorem**

*Suppose that*

$$\rho^2 X \in T_d^d(X)$$

*with $\rho > 0$ and $X$ positive definite. Then,*

$$\rho(A) \leq \rho.$$
Theorem

Suppose that $A$ is irreducible. Then there exists $\rho > 0$ and $X$ such that $\sum_v X_v$ is positive definite and

$$\rho^2 X = T^d_{\Box}(X) \in \mathcal{T}^d(X)$$

where

$$[T^d_{\Box}(X)]_w := \bigcup_{w = \tau_d(v,i)} A^T_i X_v A_i.$$
Exercise: find the mistake in the following proof

We want to show that the following eigenproblem is solvable:

\[
[T^d_d(X)]_w := \bigcup_{w = \tau_d(v, i)} A_i^T X_v A_i = \rho^2 X_w
\]

1. suppose, w.l.g., \( d = 0 \).
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1. suppose, w.l.g., \( d = 0 \).

2. Consider the noncommutative simplex,
   \( \Delta := \{ X \succeq 0: \text{trace} \ X = 1 \} \). This set is compact and convex.
Exercise: find the mistake in the following proof

We want to show that the following eigenproblem is solvable:

\[
[T^d_{\sqcup}(X)]_w := \bigcup_{w = \tau_d(v,i)} A^T_i X_v A_i = \rho^2 X_w
\]

1. suppose, w.l.g., \( d = 0 \).

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   \( \Delta := \{ X \succcurlyeq 0: \text{trace } X = 1 \} \). This set is compact and convex.

3. Consider the normalized map \( \tilde{T}^d_{\sqcup}(X) = (\text{trace } T^d_{\sqcup}(X))^{-1} T^d_{\sqcup}(X) \). It sends \( \Delta \) to \( \Delta \)
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4. By Brouwer fixed point theorem, it has a fixed point
Exercise: find the mistake in the following proof

We want to show that the following eigenproblem is solvable:

\[
\left[ T_{\square}^d(X) \right]_w := \bigcup_{w=\tau_d(v,i)} \ A_i^T X_v A_i = \rho^2 X_w
\]

1. suppose, w.l.g., \( d = 0 \).
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3. Consider the normalized map \( \tilde{T}_{\square}^d(X) = (\text{trace} T_{\square}^d(X))^{-1} T_{\square}^d(X) \). It sends \( \Delta \) to \( \Delta \)
4. By Brouwer fixed point theorem, it has a fixed point.
5. This fixed point is an eigenvector of \( T^d \)
Exercise: find the mistake in the following proof

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4. By Brouwer fixed point theorem, it has a fixed point
5. This fixed point is an eigenvector of \(T^d\)
Exercise: find the mistake in the following proof

We want to show that the following eigenproblem is solvable:

\[
[T^d_{\square}(X)]_w := \bigsqcup_{w=\tau_d(v,i)} A_i^T X_v A_i = \rho^2 X_w
\]

1. suppose, w.l.g., \(d = 0\).
2. Consider the noncommutative simplex,
   \(\Delta := \{X \succcurlyeq 0: \text{trace } X = 1\}\). This set is compact and convex.
3. Consider the normalized map \(\tilde{T}^d_{\square}(X) = (\text{trace } T^d_{\square}(X))^{-1} T^d_{\square}(X)\). It sends \(\Delta\) to \(\Delta\)
4. By Brouwer fixed point theorem, it has a fixed point
5. This fixed point is an eigenvector of \(T^d\)

\(\nabla\) is continuous in \(\text{int } S^+_n \times \text{int } S^+_n\), but not on its closure.
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We want to show that the following eigenproblem is solvable:

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[T^d_{\sqcup}(X)]_w := \bigcup_{w=\tau_d(v,i)} A^T_i X_v A_i = \rho^2 X_w
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4. By Brouwer fixed point theorem, it has a fixed point.

5. This fixed point is an eigenvector of \(T^d\).

\(\sqcup\) is continuous in \(\text{int } S_n^+ \times \text{int } S_n^+\), but not on its closure.

\(\rightarrow\) cannot apply naively Brouwer.
Fixing the proof of existence of eigenvectors

**Lemma**

For $Y_i > 0$, we have

$$
\frac{1}{m} \sum_{i=1}^{m} Y_i \lesssim \bigsqcup_{i=1}^{m} Y_i \lesssim \sum_{i=1}^{m} Y_i
$$

**Corollary**

For all $X \in S_n^+$, we have

$$
\frac{1}{m} K^d(X) \lesssim T^d_{\sqcup}(X) \lesssim K^d(X),
$$

with

$$
K^d_w(X) = \sum_{w=\tau_d(v,i)} A^T_i X_v A_i \\
T^d_{\sqcup,w}(X) = \bigsqcup_{w=\tau_d(v,i)} A^T_i X_v A_i.
$$
Proof

Reduction to \( K : X \mapsto \sum_i A_i^T X A_i \) strictly positive:
\[
X \succ 0 \implies K(X) \succ 0.
\]
Proof

Reduction to $K : X \mapsto \sum_i A_i^T X A_i$ strictly positive:

\[ X \succcurlyeq 0 \implies K(X) \succcurlyeq 0. \]

Let $X \in \Delta := \{X \succcurlyeq 0 : \text{trace } X = 1\}$. By compactness:

\[ \alpha I \preceq K(X) \preceq \beta I , \text{ with } \alpha > 0. \]
Proof

Reduction to $K : X \mapsto \sum_i A_i^T X A_i$ strictly positive:

$$X \succeq 0 \implies K(X) \succ 0.$$ 

Let $X \in \Delta := \{X \succeq 0: \text{trace } X = 1\}$. By compactness:

$$\alpha I \preceq K(X) \preceq \beta I, \text{ with } \alpha > 0.$$ 

Then

$$\frac{\alpha}{m} I \preceq T_\sqcup(X) \preceq \beta I,$$

so $T_\sqcup(\Delta) \subset \text{compact subset of int } \Delta$. 


Proof

Reduction to \( K : X \mapsto \sum_i A_i^T X A_i \) strictly positive:

\[ X \succcurlyeq 0 \implies K(X) \succcurlyeq 0. \]

Let \( X \in \Delta := \{ X \succcurlyeq 0 : \text{trace } X = 1 \} \). By compactness:

\[ \alpha I \preceq K(X) \preceq \beta I , \text{ with } \alpha > 0 . \]

Then

\[ \frac{\alpha}{m} I \preceq T_{\square}(X) \preceq \beta I , \]

so \( T_{\square}(\Delta) \subset \text{compact subset of } \text{int } \Delta \). Conclude by Brouwer’s fixed point theorem.
Computing an eigenvector

We introduce a damping parameter $\gamma$:

$$T^\gamma(X) = \bigcup_i (A_i^T X A_i + \gamma (\text{trace } X) I_n).$$

**Theorem**

*The iteration*

$$X^{k+1} = \frac{T^\gamma(X)}{\text{trace } T^\gamma(X)}$$

*converges for a large damping:* $\gamma > n m^{(3d+1)/2}$

**Conjecture**

The iteration converges if $\gamma > m^{1/2} n^{-1/2}$.

Experimentally: $\gamma \sim 10^{-2}$ is enough! Huge gap between conservative theoretical estimates and practice. How theoretical estimates are obtained?
Lipschitz estimations

Riemann and Thompson metrics

Two standard metrics on the cone $S_n^{++}$

$$d_R(A, B) := \| \log \text{spec}(A^{-1}B) \|_2.$$  

$$d_T(A, B) := \| \log \text{spec}(A^{-1}B) \|_{\infty}.$$  

They are invariant under the action of congruences:

$$d(LAL^T, LBL^T) = d(A, B) \text{ for invertible } L.$$  

Lipschitz constant:  

$$\text{Lip}_M \sqcup := \sup_{X_1, X_2, Y_1, Y_2 > 0} \frac{d_M(X_1 \sqcup X_2, Y_1 \sqcup Y_2)}{d_M(X_1 \oplus X_2, Y_1 \oplus Y_2)}.$$
Lipschitz estimations

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Theorem

$$\text{Lip}_T \sqcup = \Theta(\log n) \quad \text{Lip}_R \sqcup = 1$$

Proof.

$d_T, d_R$ are Riemann/Finsler metrics $\rightarrow$ work locally $+$ Schur multiplier estimation (Mathias).
## Scalability: dimension

Table: big-LMI vs Tropical Kraus

<table>
<thead>
<tr>
<th>Dimension $n$</th>
<th>CPU time (tropical)</th>
<th>CPU time (LMI)</th>
<th>Error vs LMI</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.9 s</td>
<td>3.1 s</td>
<td>0.1 %</td>
</tr>
<tr>
<td>10</td>
<td>1.5 s</td>
<td>4.2 s</td>
<td>1.4 %</td>
</tr>
<tr>
<td>20</td>
<td>3.5 s</td>
<td>31 s</td>
<td>0.4 %</td>
</tr>
<tr>
<td>30</td>
<td>7.9 s</td>
<td>3 min</td>
<td>0.2 %</td>
</tr>
<tr>
<td>40</td>
<td>13.7 s</td>
<td>18 min</td>
<td>0.05 %</td>
</tr>
<tr>
<td>45</td>
<td>18.1 s</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>50</td>
<td>25.2 s</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>100</td>
<td>1 min</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>500</td>
<td>8 min</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>
Figure: Computation time vs dimension
Scalability: graph size

\[
A_1 = \begin{pmatrix}
-1 & 1 & -1 \\
-1 & -1 & 1 \\
0 & 1 & 1
\end{pmatrix} \quad A_2 = \begin{pmatrix}
-1 & 1 & -1 \\
-1 & -1 & 0 \\
1 & 1 & 1
\end{pmatrix}
\]

Table: big-LMI vs Tropical Kraus: 30 – 60 times faster.

<table>
<thead>
<tr>
<th>Order $d$</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size of graph</td>
<td>8</td>
<td>32</td>
<td>128</td>
<td>512</td>
<td>2048</td>
</tr>
<tr>
<td>CPU time (tropical)</td>
<td>0.03s</td>
<td>0.07s</td>
<td>0.4s</td>
<td>2.0s</td>
<td>9.0s</td>
</tr>
<tr>
<td>CPU time (LMI)</td>
<td>1.9s</td>
<td>4.0s</td>
<td>24s</td>
<td>1min</td>
<td>10min</td>
</tr>
<tr>
<td>Accuracy</td>
<td>1.1%</td>
<td>1.3%</td>
<td>0.4%</td>
<td>0.4%</td>
<td>0.6%</td>
</tr>
</tbody>
</table>
Special case of nonnegative matrices

Suppose $A_i \in \mathbb{R}_{+}^{n \times n}$, replace the quantum dynamic programming operator

$$X \in (S_n^+)^{(m^d)} \quad \text{and} \quad \mathcal{T}_w^d(X) := \bigvee_{w=\tau_d(v,i)} A_i^T X_v A_i$$

Theorem

Suppose the set of nonnegative matrices $A_i$ is positively irreducible.

Then, there exists $u \in (\mathbb{R}_{+}^n)^{(m^d)}\{0\}$ such that $\mathcal{T}_u^d = \lambda u$.

Follows from SG and Gunawardena, TAMS 2004.
**Special case of nonnegative matrices**

Suppose $A_i \in \mathbb{R}_{+}^{n \times n}$, replace the quantum dynamic programming operator

$$X \in (S_n^+)^{(m^d)} \quad \text{and} \quad T_w^d(X) := \bigvee_{w=\tau_d(v,i)} A_i^T X_v A_i$$

by the classical dynamic programming operator

$$x \in (\mathbb{R}_+^n)^{(m^d)} \quad \text{and} \quad T_w^d(x) := \sup_{w=\tau_d(v,i)} A_i^T x_v$$

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Suppose the set of nonnegative matrices $A_i$ is positively irreducible. Then, there exists $u \in (\mathbb{R}_+^n)^{(m^d)} \setminus \{0\}$ such that $T^d(u) = \lambda_d u$.

Follows from SG and Gunawardena, TAMS 2004.
Special case of nonnegative matrices

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$$X \in (S^+_n)^{(m^d)}$$ and

$$T^d_w(X) := \bigvee_{v=i} A^T_i X_v A_i$$

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Operators of this type arise in risk-sensitive control Anantharam, Borkar, also in games of topological entropy Asarin, Cervelle, Degorre, Dima, Horn, Kozyakin, Akian, SG, Grand-Clément, Guillaud.
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*Suppose the set of nonnegative matrices $\mathcal{A}$ is positively irreducible. Then, there exists $u \in (\mathbb{R}_+)^{(m^d)} \setminus \{0\}$ such that

$$T^d(u) = \lambda du.$$*

Follows from SG and Gunawardena, TAMS 2004.
A monotone hemi-norm is a map $\nu(x) := \max_{v \in V} \langle u_v, x \rangle$ with $u_v \geq 0$ such that $x \mapsto \nu(x) \vee \nu(-x)$ is a norm.

**Theorem (Coro. of Guglielmi and Protasov)**

If $A \subset \mathbb{R}^{n \times n}_+$ is positively irreducible, there is a monotone hemi-norm $\nu$ such that

$$\max_{i \in [m]} \nu(A_i x) = \rho(A) \nu(x), \quad \forall x \in \mathbb{R}^n_+$$

**Theorem (Polyhedral monotone hemi-norms)**

If $A \subset \mathbb{R}^{n \times n}_+$ is positively irreducible, if $T_d(u) = \lambda_d u$, and $u \in (\mathbb{R}^n_+)^{md} \setminus \{0\}$, then

$$\|x\|_u := \max_{v \in [md]} \langle u_v, x \rangle$$

is a polyhedral monotone hemi-norm and

$$\max_{i \in [m]} \|A_i x\|_u \leq \lambda_d \|x\|_u .$$
A monotone hemi-norm is a map $\nu(x) := \max_{v \in V} \langle u_v, x \rangle$ with $u_v \geq 0$ such that $x \mapsto \nu(x) \vee \nu(-x)$ is a norm.

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$$\|x\|_{u} := \max_{v \in [m^d]} \langle u_v, x \rangle$$

is a polyhedral monotone hemi-norm and

$$\max_{i \in [m]} \|A_i x\|_{u} \leq \lambda_d \|x\|_{u}.$$

Moreover, $\rho(A) \leq \lambda_d \leq n^{1/(d+1)} \rho(A)$, in particular $\lambda_d \to \lambda$ as $d \to \infty$. 
How to compute $\lambda$ such that $T^d(u) = \lambda u$ for some $u \neq 0, u \neq 0$
How to compute $\lambda$ such that $T^d(u) = \lambda u$ for some $u \neq 0$, $u \neq 0$

- Policy iteration: Rothblum
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- Policy iteration: Rothblum
- Spectral simplex: Protasov
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policy iteration/spectral simplex requires computing eigenvalues (demanding), need to work with huge scale instances (dimension $N = n \times m^d$)
Krasnoselski-Mann iteration

\[ x_{k+1} = \frac{1}{2} (x_k + F(x_k)) \]

applies to a nonexpansive map \( F: \|F(x) - F(y)\| \leq \|x - y\| \).
Krasnoselski-Mann iteration

$$x_{k+1} = \frac{1}{2}(x_k + F(x_k))$$

applies to a nonexpansive map $F$: $\|F(x) - F(y)\| \leq \|x - y\|$.

Theorem (Ishikawa)

Let $D$ be a closed convex subset of a Banach space $X$, let $F$ be a nonexpansive mapping sending $D$ to a compact subset of $D$. Then, for any initial point $x^0 \in D$, the sequence $x^k$ converges to a fixed point of $F$. 
Krasnoselski-Mann iteration

\[ x_{k+1} = \frac{1}{2} (x_k + F(x_k)) \]

applies to a nonexpansive map \( F: \|F(x) - F(y)\| \leq \|x - y\| \).

Theorem (Ishikawa)

Let \( D \) be a closed convex subset of a Banach space \( X \), let \( F \) be a nonexpansive mapping sending \( D \) to a compact subset of \( D \). Then, for any initial point \( x^0 \in D \), the sequence \( x_k \) converges to a fixed point of \( F \).

Theorem (Baillon, Bruck)

\[ \|F(x_k) - x_k\| \leq \frac{2 \text{ diam}(D)}{\sqrt{\pi k}} \]
Definition (Projective Krasnoselskii-Mann iteration)

Suppose $f : \mathbb{R}_+^N \to \mathbb{R}_+^N$ is order preserving and positively homogeneous of degree 1. Choose any $v^0 \in \mathbb{R}_+^N$ such that $\prod_{i \in [N]} v^0_i = 1$,

$$ v^{k+1} = \left[ \frac{f(v^k)}{G[f(v^k)]} \circ v^k \right]^{1/2}, $$

where $x \circ y := (x_i y_i)$ and $G(x) = (x_1 \cdots x_N)^{1/N}$.

Theorem

Suppose in addition that $f$ has a positive eigenvector. Then, the projective Krasnoselskii-Mann iteration initialized at any positive vector $v^0 \in \mathbb{R}_+^N$ such that $\prod_{i \in [N]} v^0_i = 1$, converges towards an eigenvector of $f$, and $G[f(v^k)]$ converges to the maximal eigenvalue of $f$.

Proof idea. This is Krasnoselski iteration applied to $F := \log \circ f \circ \exp$ acting in the quotient of the normed space $(\mathbb{R}^n, \|\cdot\|_\infty)$ by the one-dimensional subspace $\mathbb{R}^1 \mathbb{N}$.
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Corollary

Take \( f := T^d \), the risk-sensitive dynamic programming operator, and let

\[
\beta_k := \max_{i \in [N]} \frac{f(v^k)_i}{v^k_i}.
\]

Then,

\[
\log \rho(A) \leq \log \beta_k \leq \log \rho(A) + \frac{4}{\sqrt{\pi k}} d_H(v^0, u) + \frac{\log n}{d + 1}
\]

where \( d_H \) is Hilbert’s projective metric.
<table>
<thead>
<tr>
<th>Level</th>
<th>CPU Time (s)</th>
<th>Eigenvalue $\lambda$</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01</td>
<td>2.165</td>
<td>6.8%</td>
</tr>
<tr>
<td>2</td>
<td>0.01</td>
<td>2.102</td>
<td>3.7%</td>
</tr>
<tr>
<td>3</td>
<td>0.01</td>
<td>2.086</td>
<td>2.9%</td>
</tr>
<tr>
<td>4</td>
<td>0.01</td>
<td>2.059</td>
<td>1.6%</td>
</tr>
<tr>
<td>5</td>
<td>0.02</td>
<td>2.041</td>
<td>0.7%</td>
</tr>
<tr>
<td>6</td>
<td>0.05</td>
<td>2.030</td>
<td>0.1%</td>
</tr>
<tr>
<td>7</td>
<td>0.7</td>
<td>2.027</td>
<td>0.0%</td>
</tr>
<tr>
<td>8</td>
<td>3.3</td>
<td>2.027</td>
<td>0.0%</td>
</tr>
<tr>
<td>9</td>
<td>12.1</td>
<td>2.027</td>
<td>0.0%</td>
</tr>
</tbody>
</table>

Table: Convergence of the hierarchy on an instance with $5 \times 5$ matrices and a maximizing cyclic product of length 6.
<table>
<thead>
<tr>
<th>Level $d$</th>
<th>CPU Time (s)</th>
<th>Eigenvalue $\lambda_d$</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01</td>
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<td>7</td>
<td>0.7</td>
<td>2.027</td>
<td>0.0%</td>
</tr>
<tr>
<td>8</td>
<td>0.32</td>
<td>2.027</td>
<td>0.0%</td>
</tr>
<tr>
<td>9</td>
<td>1.12</td>
<td>2.027</td>
<td>0.0%</td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>Dimension $n$</th>
<th>Level $d$</th>
<th>Eigenvalue $\lambda_d$</th>
<th>CPU Time</th>
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</thead>
<tbody>
<tr>
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<td>0.01 s</td>
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<tr>
<td></td>
<td>3</td>
<td>4.286</td>
<td>0.03 s</td>
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<td>0.01 s</td>
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<td></td>
<td>3</td>
<td>8.576</td>
<td>0.03 s</td>
</tr>
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<td>2</td>
<td>22.34</td>
<td>0.04 s</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>22.33</td>
<td>0.16 s</td>
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<td>0.17 s</td>
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<td>0.53 s</td>
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<td>2</td>
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<td>0.71 s</td>
</tr>
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<td>3</td>
<td>89.76</td>
<td>2.46 s</td>
</tr>
<tr>
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<td>2</td>
<td>224.88</td>
<td>5.45 s</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>224.88</td>
<td>19.7 s</td>
</tr>
<tr>
<td>1000</td>
<td>2</td>
<td>449.87</td>
<td>44.0 s</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>449.87</td>
<td>2.7 min</td>
</tr>
<tr>
<td>2000</td>
<td>2</td>
<td>889.96</td>
<td>4.6 min</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>889.96</td>
<td>19.2 min</td>
</tr>
<tr>
<td>5000</td>
<td>2</td>
<td>2249.69</td>
<td>51.9 min</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2249.57</td>
<td>3.3 h</td>
</tr>
</tbody>
</table>

Table: Computation time for large matrices

- implements the quantum dynamic programming approach
- 1700 lines of OCaml and 800 lines of Matlab
- uses BLAS/LAPACK via LACAML for linear algebra
- uses OSDP/CSDP for some semidefinite programming
- uses Matlab for other semidefinite programming
Concluding remarks

- Reduced the approximation of the joint spectral radius to solving non-linear eigenproblems
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- Joint spectral radius of general matrices: "quantum" dynamic programming operator acting on the space of positive semidefinite matrices, tropical analogue of completely positive maps. "states" = bunchs of positive semidefinite matrices. yields a piecewise quadratic approximate extremal norm.
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Thank you!