Taylor Expansions of the Value Function Associated with Stabilization Problems

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ICODE Workshop on numerical solutions of HJB equations, January 8, 2020

Introduction

We consider the following **bilinear optimal control problem**:

$$\inf_{u \in L^2(0,\infty)} \mathcal{J}(u, y_0) := \int_0^\infty \frac{1}{2}\|y(t)\|_Y^2 + \frac{\beta}{2}|u(t)|^2 \mathrm{d}t, \qquad (P(y_0))$$

$$\text{where: } \begin{cases} \dot{y}(t) = Ay(t) + Ny(t)u(t) + Bu(t), \\ y(0) = y_0 \in Y, \end{cases}$$

with associated **value function**: $\mathcal{V}(y_0) := \inf_{u \in L^2(0,\infty)} \mathcal{J}(u, y_0)$.

Key ideas:

- The derivatives $D^j\mathcal{V}(0)$ are characterized by a sequence of equations.
- This allows for the numerical approximation of $\mathcal{V}$ and the optimal feedback law (locally, around 0).

Assumptions

Functional framework:

- $V \subset Y \subset V^*$ is a Gelfand triple of real Hilbert spaces, where the embedding of V into Y is dense and compact
- $W(0,\infty) = \{y \in L^2(0,\infty;V) \,|\, \dot{y} \in L^2(0,\infty;V^*)\}$.

Assumptions:

(A1) The operator $-A$ can be associated with a V-Y coercive bilinear form $a\colon V \times V \to \mathbb{R}$ such that $\exists \lambda \in \mathbb{R}$ and $\delta > 0$ satisfying $a(v,v) \geq \delta \|v\|_V^2 - \lambda \|v\|_Y^2$, for all $v \in V$.

(A2) The operator N is such that $N \in \mathcal{L}(V,Y)$ and $N^* \in \mathcal{L}(V,Y)$.

(A3) **[Stabilizability]** There exists an operator $F \in \mathcal{L}(Y,\mathbb{R})$ such that the semigroup $e^{(A+BF)t}$ is exponentially stable on Y.

Another technical assumption is also needed.

Roadmap

The **Taylor expansion** of order k, denoted $\mathcal{V}_k$ is of the form:

$$\mathcal{V}_k(y_0) = \frac{1}{2}\mathcal{T}_2(y_0, y_0) + \frac{1}{3!}\mathcal{T}_3(y_0, y_0, y_0) + ... + \frac{1}{k!}\mathcal{T}_k(y_0, ..., y_0),$$

where $\mathcal{T}_j = D^j\mathcal{V}(0)$ is a **bounded multilinear form** from Y^j to $\mathbb{R}$.

Remark: $\mathcal{V}(0) = 0$, $D\mathcal{V}(0) = 0$.

We formally show that

- $\mathcal{T}_2$ is the unique solution to an **algebraic Riccati equation** (ARE)
- $\mathcal{T}_3$, $\mathcal{T}_4$,... are the unique solutions to (linear) **generalized Lyapunov equations** (GLE).

HJB equation

Proposition

Assume that there exists a neighborhood Y_0 of 0 such that

1. *Problem $P(y_0)$ has a continuous solution u, $\forall y_0 \in \mathcal{D}(A) \cap Y_0$*
2. *The value function is continuously differentiable on Y_0.*

Then, for all $y_0 \in \mathcal{D}(A) \cap Y_0$,

$$D\mathcal{V}(y_0)Ay_0 + \frac{1}{2}\|y_0\|_Y^2 - \frac{1}{2\beta}\big(D\mathcal{V}(y_0)(Ny_0 + B)\big)^2 = 0. \qquad \text{(HJB)}$$

Moreover, for all continuous solutions $\bar{u}$ to problem $P(y_0)$,

$$\bar{u}(t) = \underbrace{-\frac{1}{\beta}D\mathcal{V}(\bar{y}(t))(N\bar{y}(t) + B),}_{\textit{Control in feedback form!}} \quad \textit{for a.e. } t.$$

Taylor expansion

The equations characterizing $(\mathcal{T}_j)_{j=2,3,\dots}$ are then obtained by successive **differentiation** of the HJB equation.

First differentiation of (HJB) w.r.t. y in some direction $z_1 \in \mathcal{D}(A)$:

$$D^2\mathcal{V}(y)(Ay, z_1) + D\mathcal{V}(y)Az_1 + \langle y, z_1 \rangle_Y \\ - \frac{1}{\beta}\big(D^2\mathcal{V}(y)(Ny + B, z_1) + D\mathcal{V}(y)Nz_1\big)\big(D\mathcal{V}(y)(Ny + B)\big) = 0.$$

Note: $y_0 \to y$.

Taylor expansion

Second differentiation of (HJB):

$$
\begin{aligned}
& D^3\mathcal{V}(y)(Ay, z_1, z_2) + D^2\mathcal{V}(y)(Az_2, z_1) + D^2\mathcal{V}(y)(Az_1, z_2) + \langle z_1, z_2\rangle_Y \\
& - \frac{1}{\beta}\big(D^2\mathcal{V}(y)(Ny+B, z_1) + D\mathcal{V}(y)Nz_1\big)\big(D^2\mathcal{V}(y)(Ny+B, z_2) + D\mathcal{V}(y)Nz_2\big) \\
& - \frac{1}{\beta}\big(D^3\mathcal{V}(y)(Ny+B, z_1, z_2)\big)\big(D\mathcal{V}(y)(Ny+B)\big) \\
& - \frac{1}{\beta}\big(D^2\mathcal{V}(y)(Nz_2, z_1) + D^2\mathcal{V}(y)(Nz_1, z_2)\big)\big(D\mathcal{V}(y)(Ny+B)\big) = 0.
\end{aligned}
$$

For $y = 0$, using the representation $D^2\mathcal{V}(0)(z_1, z_2) = \langle z_1, \Pi z_2\rangle$, where $\Pi\colon Y \to Y$, we obtain an algebraic Riccati equation:

$$A^*\Pi + \Pi A + \mathrm{Id} - \tfrac{1}{\beta}\Pi BB^*\Pi = 0. \qquad \text{(ARE)}$$

It has a unique self-adjoint and non-negative solution.

Taylor expansion

Third differentiation of (HJB), at $y = 0$:

$$\begin{aligned}
&D^3\mathcal{V}(0)(Az_3, z_1, z_2) + D^3\mathcal{V}(0)(Az_2, z_1, z_3) + D^3\mathcal{V}(0)(Az_1, z_2, z_3)\\
&- \frac{1}{\beta}\big(D^3V(0)(B, z_1, z_3) + D^2\mathcal{V}(0)(Nz_3, z_1) + D^2\mathcal{V}(0)(Nz_1, z_3)\big)D^2\mathcal{V}(0)(B, z_2)\\
&- \frac{1}{\beta}\big(D^3\mathcal{V}(0)(B, z_2, z_3) + D^2\mathcal{V}(0)(Nz_3, z_2) + D^2\mathcal{V}(0)(Nz_2, z_3)\big)D^2\mathcal{V}(0)(B, z_1)\\
&- \frac{1}{\beta}\big(D^3\mathcal{V}(0)(B, z_1, z_2) + D^2\mathcal{V}(0)(Nz_2, z_1) + D^2\mathcal{V}(0)(Nz_1, z_2)\big)D^2\mathcal{V}(0)(B, z_3) = 0.
\end{aligned}$$

We set: $A_\Pi = A - \frac{1}{\beta}BB^*\Pi$, we obtain:

$$\begin{aligned}
&\mathcal{T}_3(A_\Pi z_1, z_2, z_3) + \mathcal{T}_3(z_1, A_\Pi z_2, z_3) + \mathcal{T}_3(z_1, z_2, A_\Pi z_3)\\
&= \frac{1}{2\beta}\mathcal{R}_3(z_1, z_2, z_3), \quad \forall (z_1, z_2, z_3) \in \mathcal{D}(A)^3,
\end{aligned}$$

where the trilinear form $\mathcal{R}_3 \colon Y^3 \to \mathbb{R}$ is determined by Π, N, and B.

Taylor expansion

Differentiation of order j of (HJB), at $y = 0$:

$$\begin{aligned} & \mathcal{T}_j(A_\Pi z_1, z_2, ..., z_k) + ... + \mathcal{T}_j(z_1, ..., z_{k-1}, A_\Pi z_k) \\ & \quad = \frac{1}{2\beta}\mathcal{R}_j(z_1, ..., z_j), \quad \forall (z_1, ..., z_j) \in \mathcal{D}(A)^j. \qquad \text{(GLE}(j)\text{)} \end{aligned}$$

Properties of the derived **generalized Lyapunov equations**:

- linear equation
- computable right-hand side:
  the multilinear form $\mathcal{R}_j \colon Y^j \to \mathbb{R}$ is explicitely determined by Π, $D^3\mathcal{V}(0)$,...,$D^{j-1}\mathcal{V}(0)$, N, and B.

Theorem

There exists a unique sequence $(\mathcal{T}_j)_{j=3,4,\ldots}$ of symmetric bounded multilinear forms such that $\mathcal{T}_j \colon Y^j \to \mathbb{R}$ is a solution to GLE(j).

Proof. Representation formula:

$$\mathcal{T}_j(z_1, \ldots, z_k) = -\int_0^\infty \mathcal{R}_j\left(e^{A_\pi t} z_1, \ldots, e^{A_\pi} z_k\right) \mathrm{d}t.$$

Remark: the well-posedness of the GLEs can be established without knowledge regarding the differentiability of $\mathcal{V}$.

Feedback law

Polynomial $\mathcal{V}_k$ of degree k:

$$\mathcal{V}_k(y) = \sum_{k=2}^{k} \tfrac{1}{j!} \mathcal{T}_j(y, ..., y).$$

Feedback law $\mathbf{u}_k$ of order k:

$$\mathbf{u}_k : y \in Y \mapsto \mathbf{u}_k(y) = -\frac{1}{\beta} D\mathcal{V}_k(y)(Ny + B).$$

Closed-loop system of order k:

$$\dot{y}_k(t) = Ay_k(t) + (Ny_k(t) + B)\mathbf{u}_k(y_k(t)), \quad y_k(0) = y_0.$$

Open-loop control $\mathbf{U}_k(y_0)$ generated by the feedback $\mathbf{u}_k$ and y_0:

$$\mathbf{U}_k(y_0; t) = \mathbf{u}_k(y_k(t)).$$

Numerical approach

1. **Discretize** the operators A, N, and B in such a way that the bilinear structure is preserved (e.g. with finite differences)
2. Find a **reduced-order** model with a generalization of the balanced truncation method:

$$\inf_{u \in L^2(0,\infty)} J(u, y_0) := \int_0^\infty \frac{1}{2}\|C_r y_r(t)\|^2_{\mathbb{R}^n} + \frac{\beta}{2}|u(t)|^2 \mathrm{d}t,$$

$$\text{where: } \begin{cases} \dot{y}_r(t) = A_r y_r(t) + N_r y_r(t) u(t) + B_r u(t), \\ y_r(0) = y_{0,r} \in Y. \end{cases}$$

3. Solve the reduced GLE with a **tensor-calculus technique**.

Lyapunov equations

The associated reduced GLE of order k:

$$\begin{aligned} & \mathcal{T}_{k,r}(A_{\Pi,r}z_1, z_2, ..., z_k) + ... + \mathcal{T}_{k,r}(z_1, ..., z_{k-1}, A_{\Pi,r}z_k) \\ & \quad = \tfrac{1}{2\beta}\mathcal{R}_{k,r}(z_1, ..., z_k) \end{aligned}$$

is equivalent to a **linear system with** r^k variables. Solution:

$$\mathcal{T}_{k,r}(z_1, ..., z_k) = -\int_0^\infty \mathcal{R}_{k,r}(e^{A_{\Pi,r}t}z_1, ..., e^{A_{\Pi,r}t}z_k)\mathrm{d}t.$$

An approximation is given by:

$$\sum_{i=-\ell}^{\ell} w_i \mathcal{R}_{k,r}(e^{A_{\Pi,r}t_i}z_1, ..., e^{A_{\Pi,r}t_i}z_k),$$

for an appropriate choice of points t_i and weights w_i.

Fokker-Planck equation

Controlled **Fokker-Planck equation**:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nu \Delta \rho + \nabla \cdot (\rho \nabla G) + u \nabla \cdot (\rho \nabla \alpha_j) && \text{in } \Omega \times (0, \infty), \\ 0 &= (\nu \nabla \rho + \rho \nabla G) \cdot \vec{n} && \text{on } \Gamma \times (0, \infty), \\ \rho(x, 0) &= \rho_0(x) && \text{in } \Gamma, \end{aligned}$$

where $\Omega \in \mathbb{R}^d$ denotes a bounded domain with smooth boundary Γ.

For all t, $\rho(\cdot, t)$ is the probability density function of X_t, sol. to

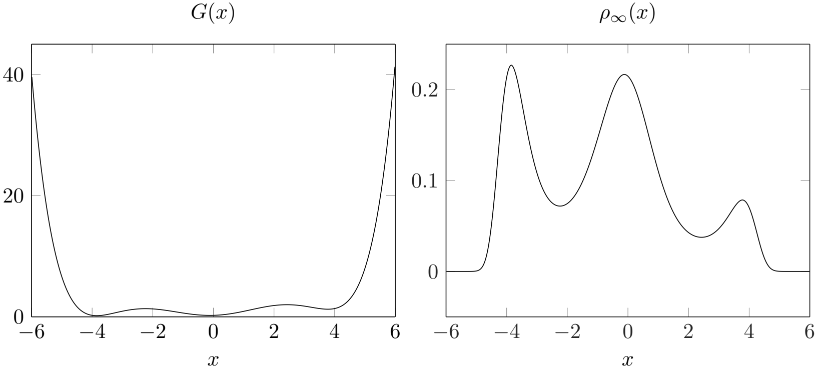
$$\mathrm{d}X(t) = -\nabla_x V(X(t), t)\mathrm{d}t + \sqrt{2\nu}\mathrm{d}W_t,$$

where the **potential** V is controlled by u:

$$V(x, t) = G(x) + u(t)\alpha(x), \quad \forall x \in \Omega, \ \forall t \geq 0.$$

Fokker-Planck equation

The uncontrolled Fokker-Planck equation is known to converge to its stationary distribution ρ_∞.



(a) Ground potential (b) Stationary distribution

Fokker-Planck equation

Optimal control problem:

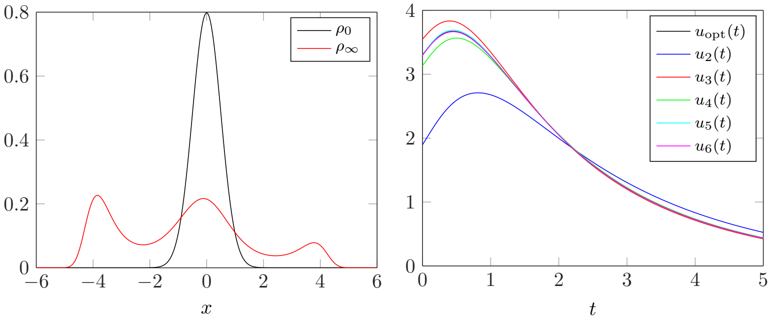
$$\inf_{u \in L^2(0,\infty)} \int_0^\infty \frac{1}{2} \|\rho(\cdot,t) - \rho_\infty(\cdot)\|^2_{L^2(\Omega)} + \beta |u(t)|^2 \mathrm{d}t,$$

where ρ satisfies the Fokker-Planck equation.

Under regularity assumptions on G and α, the problem can be reformulated, so that it falls in the abstract framework.

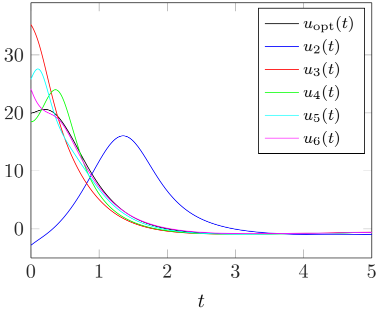
- Control shape function $\alpha(x) \approx x/12$.
- Discretization of $\Omega = (-6,6)$: $n = 100$.
- Reduction: $r = 21$ (selection of singular values above 10^{-6}).
- Results for two initial values (a close one/a further one), different values of β.

Numerical results (test case 1)

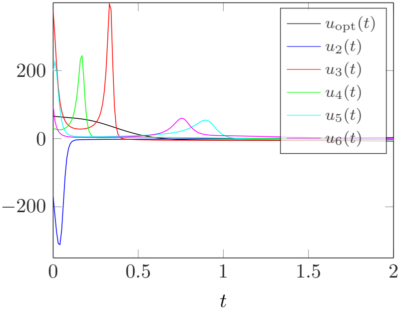


(a) Initial/stationary distributions (b) Controls for $\beta = 10^{-3}$

Numerical results (test case 1)



(a) Controls for $\beta = 10^{-4}$



(b) Controls for $\beta = 10^{-5}$

Numerical results (test case 1)

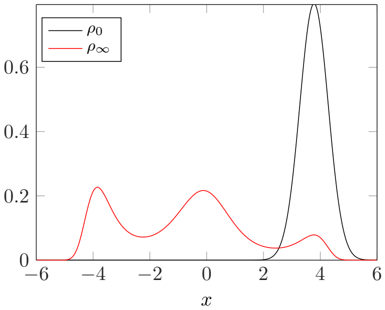
β	$J(u_2)$	$J(u_3)$	$J(u_4)$	$J(u_5)$	$J(u_6)$	$J(u_{\text{opt}})$
$1e^{-3}$	0.156	0.155	0.155	0.155	0.155	0.154
$1e^{-4}$	0.138	0.122	0.120	0.120	0.120	0.119
$1e^{-5}$	0.205	0.194	0.104	0.111	0.113	0.095

(a) Cost of the controls u_k

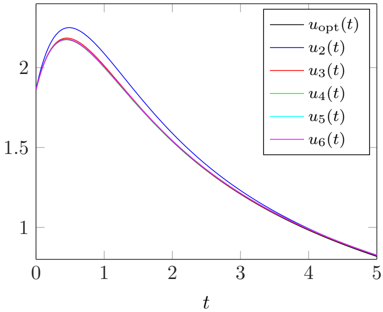
β	$\lVert u_k - u_{\text{opt}} \rVert_{L^2(0,T)}$				
	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
$1e^{-3}$	1.149	0.169	0.119	0.034	0.031
$1e^{-4}$	18.50	7.02	3.16	4.01	1.52
$1e^{-5}$	90.5	78.0	39.0	42.6	34.3

(b) L^2-distance between the controls u_k and the optimal control u_{opt}

Numerical results (test case 2)

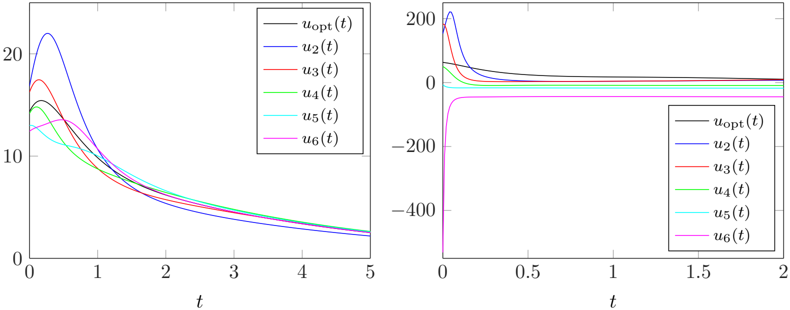


(a) Initial/stationary distributions



(b) Controls for $\beta = 10^{-2}$

Numerical results (test case 2)



(a) Controls for $\beta = 10^{-3}$ (b) Controls for $\beta = 10^{-4}$

Numerical results

β	$J(u_2)$	$J(u_3)$	$J(u_4)$	$J(u_5)$	$J(u_6)$	$J(u_{\text{opt}})$
$1e^{-2}$	0.788	0.788	0.788	0.788	0.788	0.787
$1e^{-3}$	0.525	0.511	0.511	0.512	0.510	0.507
$1e^{-4}$	0.381	0.368	2.689	∞	∞	0.246

(a) Cost of the controls u_k

β	$\|u_k - u_{\text{opt}}\|_{L^2(0,T)}$				
	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$1e^{-2}$	0.19	0.15	0.15	0.15	0.15
$1e^{-3}$	4.88	1.50	1.77	2.31	1.52
$1e^{-4}$	46.34	35.36	57.08	∞	∞

(b) L^2-distance between the controls u_k and the optimal control u_{opt}

Elements of analysis

Theorem

There exists $\delta > 0$ such that

- *for all $y_0 \in B(\delta)$, problem $P(y_0)$ has a* **unique solution** $\bar{u}$,
- *the value function $\mathcal{V}$ is* **infinitely differentiable** *on $B(\delta)$.*

For all $k \geq 2$, there exist $\delta > 0$ and $C > 0$ such that:

- *The* **closed-loop system** *(of order k) is well-posed and generates an open-loop control in $L^2(0, \infty)$.*
- *The following estimates hold true:*

$$\mathcal{J}(\mathbf{U}_k(y_0), y_0) \leq \mathcal{V}(y_0) + C\|y_0\|_Y^{2k}$$
$$\|\bar{u} - \mathbf{U}_k(y_0)\|_{L^2(0,\infty)} \leq C\|y_0\|_Y^k.$$

Remark: **local result**, δ and C depend on k.

Elements of analysis

Result 1 (optimality conditions for the original problem). For all solutions $\bar{u}$ with trajectory $\bar{y}$, there exists $\bar{p} \in W(0,\infty)$ such that

$$\dot{\bar{p}} + (A + \bar{u}N)^* \bar{p} + \bar{y} = 0, \quad \beta \bar{u} + (Ny + B)^* \bar{p} = 0.$$

Result 2 (optimality conditions for the closed loop system). For the control u_k and the trajectory y_k generated by the feedback of order k, there exists $p_k \in L^2(0,\infty;V)$ such that

$$\dot{p}_k + (A + u_k N)^* p_k + y_k = w_k, \quad \beta u_k + (Ny_k + B)^* p_k = 0,$$

where $\|w_k\| \leq C \|y_0\|_Y^k$.

Elements of analysis

Result 3 (sensitivity analysis).
The mapping $\Phi\colon (y,u,p) \in W(0,\infty) \times L^2(0,\infty) \times L^2(0,\infty;V) \mapsto$

$$\Phi(y,u,p) = \begin{pmatrix} y(0) \\ \dot{y} - (Ay + Nyu + Bu) \\ -\dot{p} - (A + uN)^* p - y \\ \beta u + (Ny + B)^* p \end{pmatrix}$$

is **locally invertible** around $(0,0,0)$, with a C^∞ inverse.

Proof: application of the inverse mapping theorem.

$$D\Phi(0,0,0)(\delta y, \delta u, \delta p) = (\omega_1, \omega_2, \omega_3, \omega_4)$$

$$\Longleftrightarrow \begin{pmatrix} \delta y(0) = & \omega_1 \\ \delta \dot{y} = & A\delta y + B\delta u + \omega_2 \\ -\delta \dot{p} = & A^* \delta p + \delta y + \omega_3 \\ \beta \delta u + B^* \delta p = & \omega_4 \end{pmatrix}$$

$\Longleftrightarrow (\delta y, \delta u)$ unique sol. of a LQ problem.

Elements of analysis

Conclusion (for $\|y_0\|$ small enough).

- $(\bar{y}, \bar{u})$ is a solution to $P(y_0)$ with costate $\bar{p}$ implies

$$\Phi(\bar{y}, \bar{u}, \bar{p}) = (y_0, 0, 0, 0) \iff (\bar{y}, \bar{u}, \bar{p}) = \Phi^{-1}(y_0, 0, 0, 0).$$

  Uniqueness and smoothness of $\mathcal{V}$ follow.

- (y_k, u_k, p_k) is as in Step 2 implies

$$\Phi(y_k, u_k, p_k) = (y_0, 0, w_k, 0) \iff (y_k, u_k, p_k) = \Phi^{-1}(y_0, 0, w_k, 0).$$

- Error estimate:

$$\begin{aligned}\|(y_k, u_k, p_k) - (\bar{y}, \bar{u}, \bar{p})\| &= \|\Phi^{-1}(y_0, 0, w_k, 0) - \Phi^{-1}(y_0, 0, 0, 0)\| \\ &\leq C\|w_k\| \leq C\|y_0\|_Y^k.\end{aligned}$$

Introduction

Main result: an upper bound of

$$\|y_{RH} - \bar{y}\|_{W(0,\infty)} + \|u_{RH} - \bar{u}\|_{L^2(0,\infty)}, \quad \text{where:}$$

- $(\bar{y}, \bar{u})$ is the solution to $P(y_0)$
- (y_{RH}, u_{RH}) is an approximate solution obtained with the **Receding-Horizon** method (= **Model Predictive Control**).

We aim at analyzing the effect of

- the sampling time τ
- the prediction horizon T
- the penalty function ϕ.

Algorithm

Main idea of the RHC method: replace $P(y_0)$ by a sequence of (tractable) finite-horizon problems.

For a given **terminal cost function** $\phi\colon Y \to \mathbb{R}$, consider the **truncated problem**

$$\inf_{u \in L^2(0,\infty)} \int_0^T \frac{1}{2}\|y(t)\|_Y^2 + \frac{\beta}{2}|u(t)|^2 \mathrm{d}t + \phi(y(T)),$$

$$\text{where: } \begin{cases} \dot{y}(t) = Ay(t) + Ny(t)u(t) + Bu(t), \\ y(0) = y_{\text{init}} \in Y, \end{cases}$$

$$(P_{T,\phi}(y_{\text{init}}))$$

Algorithm

Method.

1. Set $n = 0$.
2. Compute a solution (y, u) to $P_{T,\phi}(y_n)$.
3. Set $u_{RH}(t) = u(n\tau + t)$, $y_{RH}(t) = y(n\tau + t)$ for $t \in (0, \tau)$.
4. Set $y_{n+1} = y_{RH}((n+1)\tau)$, $n = n + 1$, and go back to Step 2.

Remark

- *If $\mathcal{V}$ is used as a terminal cost, then by the* **dynamic programming principle***, the RH-algorithm generates the exact solution to the problem.*
- *Limit case when $(\tau, T) \to 0$: Feedback control.*
  Limit case when $(\tau, T) \to \infty$: Open-loop control.

Result

Theorem

For all $k \geq 1$, there exist $\tau_0 > 0$, $\delta > 0$, and $M > 0$ such that for all $\tau \geq \tau_0$, for all $T \geq \tau$, and all $y_0 \in B_Y(\delta)$, the RHC method with

$$\phi = \mathcal{V}_k$$

is well-posed. Moreover,

$$\|y_{RH} - \bar{y}\|_{W_\infty} + \|u_{RH} - \bar{u}\|_{L^2(0,\infty)} \leq M e^{-\lambda(T-\tau)-\lambda k T} \|y_0\|_Y^k$$

where $\bar{u}$ is the unique solution to the problem with trajectory $\bar{y}$.

Proof: based on a sensitivity analysis.

Conclusion

Summary:

- General method for deriving polynomial feedback laws
- Implementation for an infinite-dimensional problem thanks to model reduction
- Good results, but only locally.
- Theoretical result for the RHC method.

Extensions:

- Other systems, with different non-linearities.
- Analysis of other kind of feedback mechanisms (e.g. SDRE).
- Analysis of other kind of problems (e.g. problems with turnpike property).

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Thank you for your attention!