Convergence rates for discretized optimal transport

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Based on joint work with F. Chazal and A. Delalande

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1. Motivations

Motivation 1: Monge-Kantorovich Quantiles

- Given $\mu \in \mathrm{Prob}(\mathbb{R})$, there exists a unique nondecreasing $T_\mu \in \mathrm{L}^1([0,1])$ satisfying $T_{\mu\#}\rho = \mu$, with $\rho =$ Lebesgue measure on $[0,1]$.

NB: $T_{\mu\#}\lambda = \mu \iff \forall B \subseteq \mathbb{R},\ \lambda(T_\mu^{-1}(B)) = \mu(B)$

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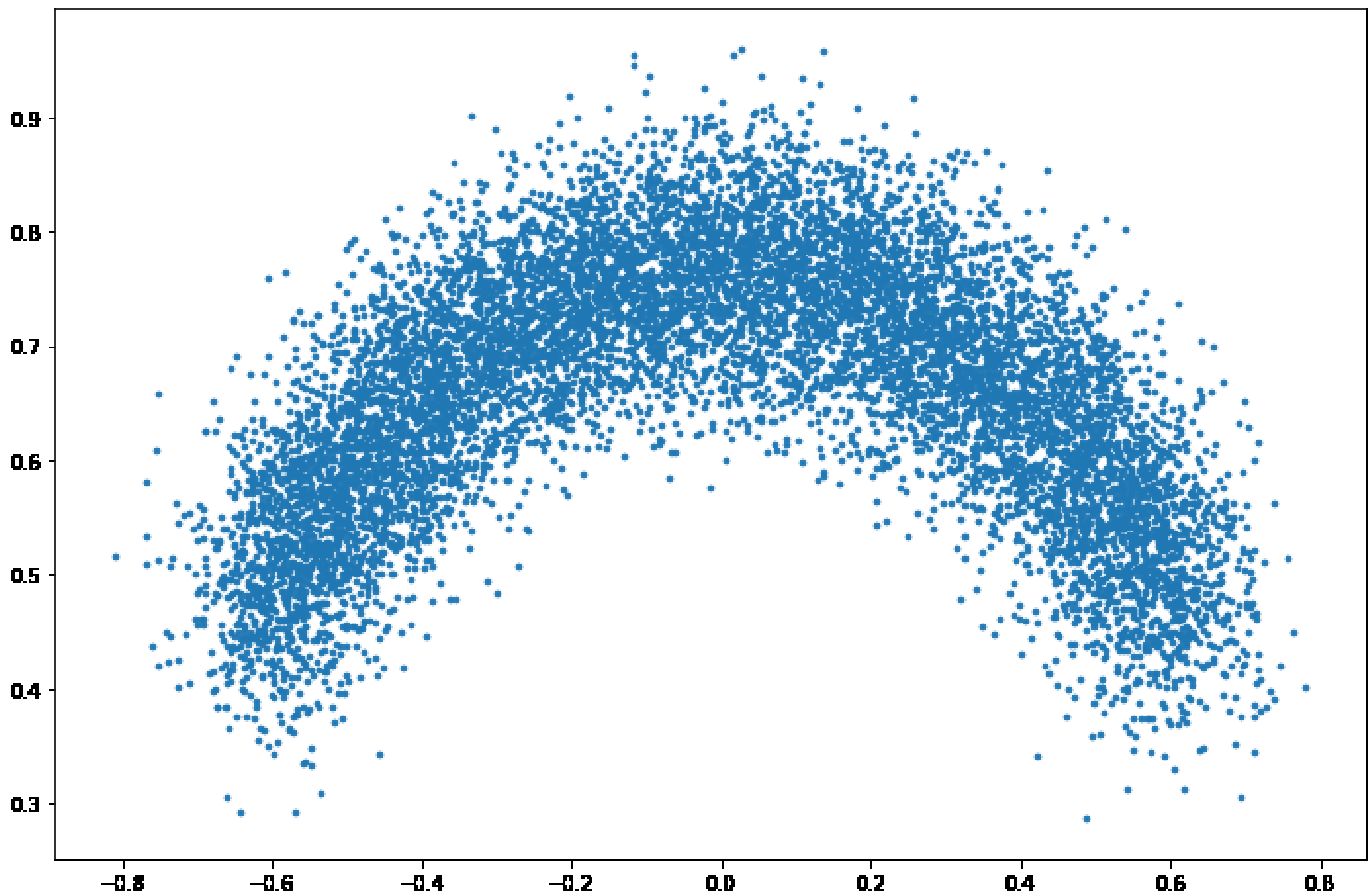
- $T_\mu : \mathrm{spt}(\rho) \to \mathbb{R}^d$ is *monotone*: $\langle T_\mu(x) - T_\mu(y) | x - y \rangle \geq 0$.

Numerical Example: Monge-Kantorovich Depth

Source: $\rho =$ uniform probability density on $B(0,1) \subseteq \mathbb{R}^2$

Target: $\mu = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{y_i}$ with $N = 10^4$ points



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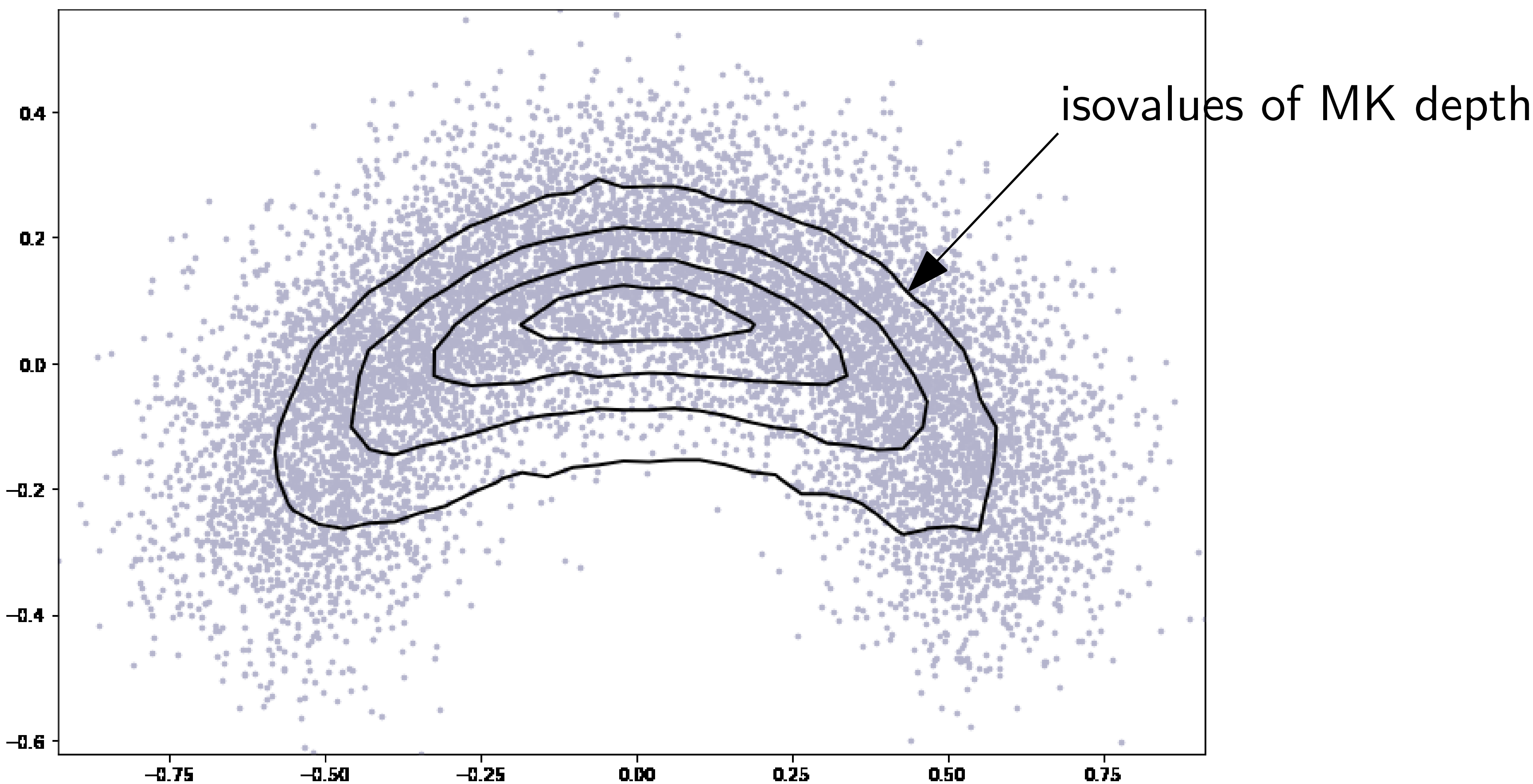

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Wasserstein space

- Let $\mathrm{Prob}_p(\mathbb{R}^d) = \{\mu \in \mathrm{Prob}(\mathbb{R}^d) \mid \int \|x\|^p \,\mathrm{d}\,\mu < +\infty\}$.

p-**Wasserstein distance** between $\mu, \nu \in \mathrm{Prob}_p(\mathbb{R}^d)$:

$$\mathrm{W}_p(\mu,\nu) = \left(\min_{\gamma\in\Gamma(\mu,\nu)} \|x-y\|^p \,\mathrm{d}\,\gamma(x,y)\right)^{1/p}.$$

where $\Gamma(\mu,\nu) =$ couplings between μ and $\nu \subseteq \mathrm{Prob}(\mathbb{R}^d \times \mathbb{R}^d)$.

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The previous embedding is false in higher dimension: $(\mathrm{Prob}_p, \mathrm{W}_p)$ is *curved*.

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tangent space	$\mathrm{T}_\rho M$	$\mathrm{T}_\rho\mathrm{Prob}_2(\mathbb{R}^d) \subseteq \mathrm{L}^2(\rho, X)$
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$\longrightarrow$ Representing family of probability measures by family of functions in $\mathrm{L}^2(\rho)$.

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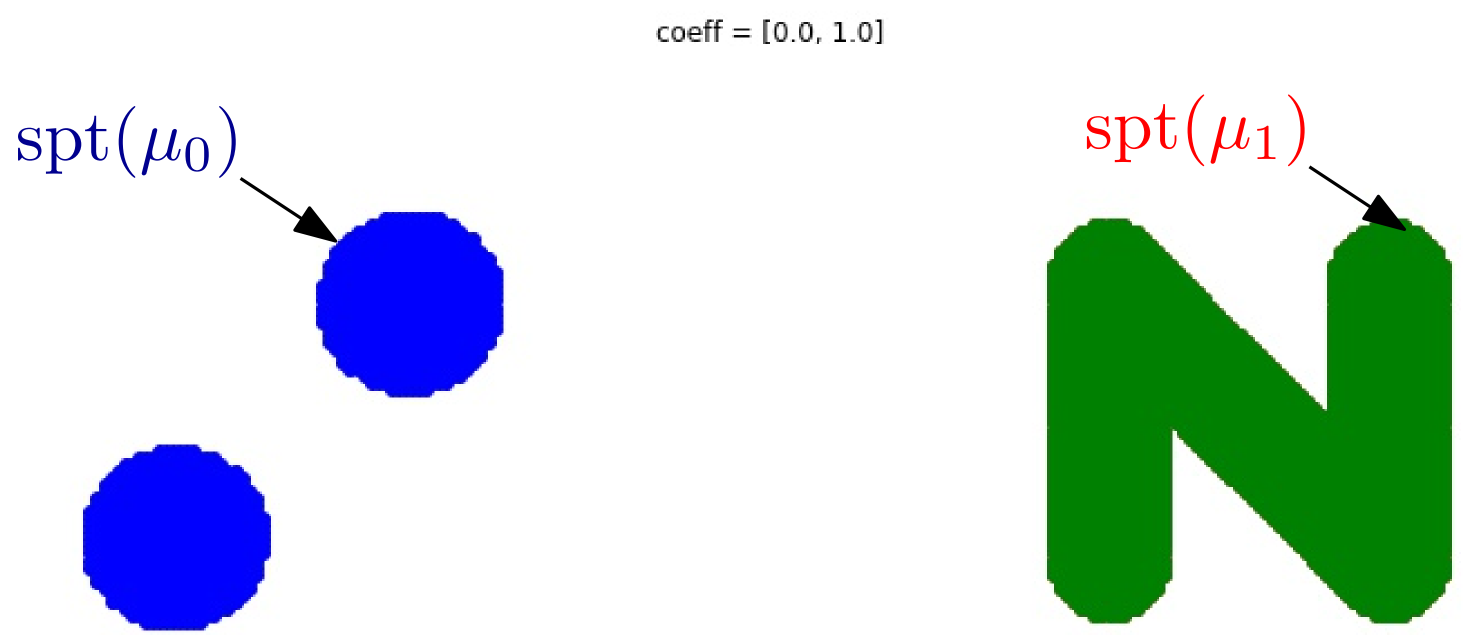
$\longrightarrow$ Simple expression once the transport maps $T_{\mu_i} : \rho \to \mu_i$ have been computed.



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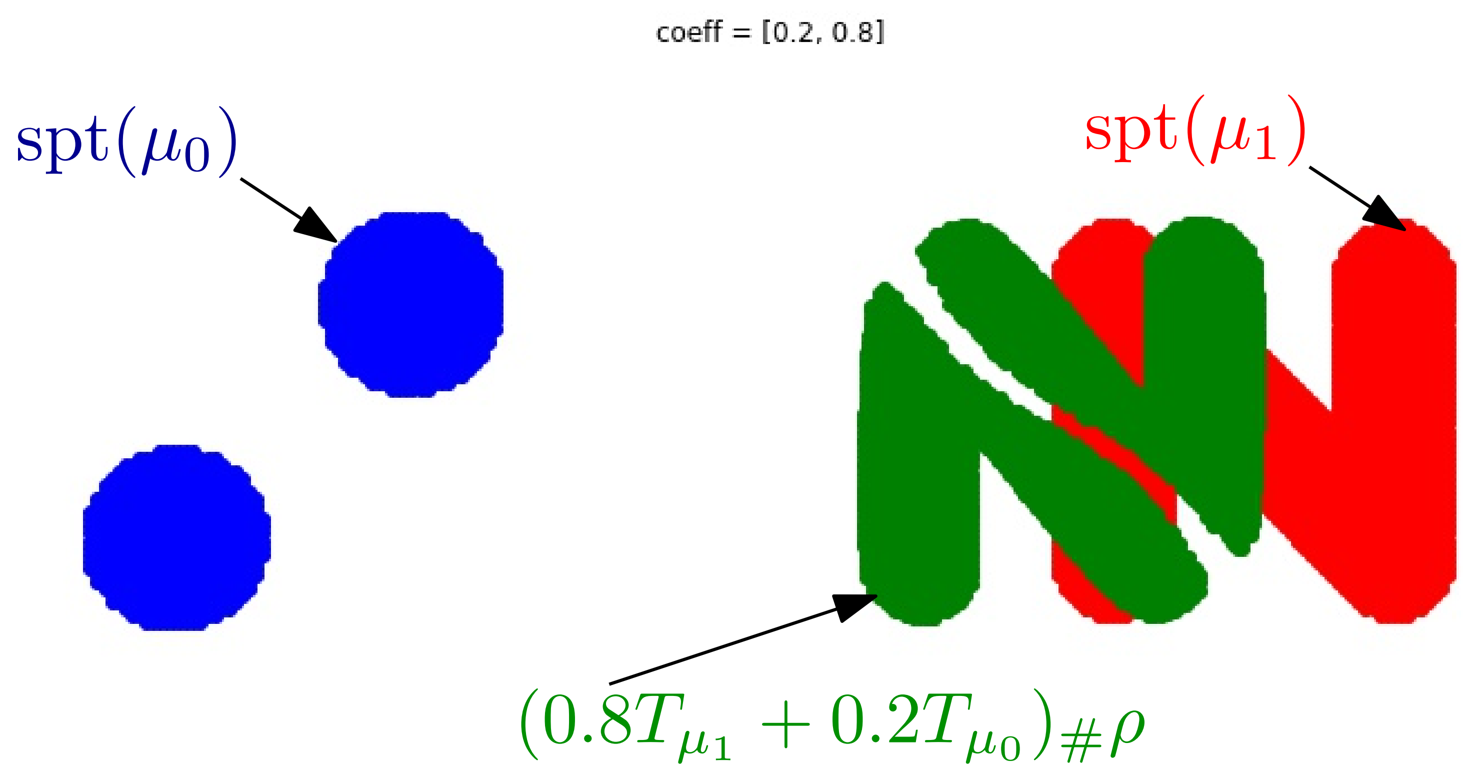

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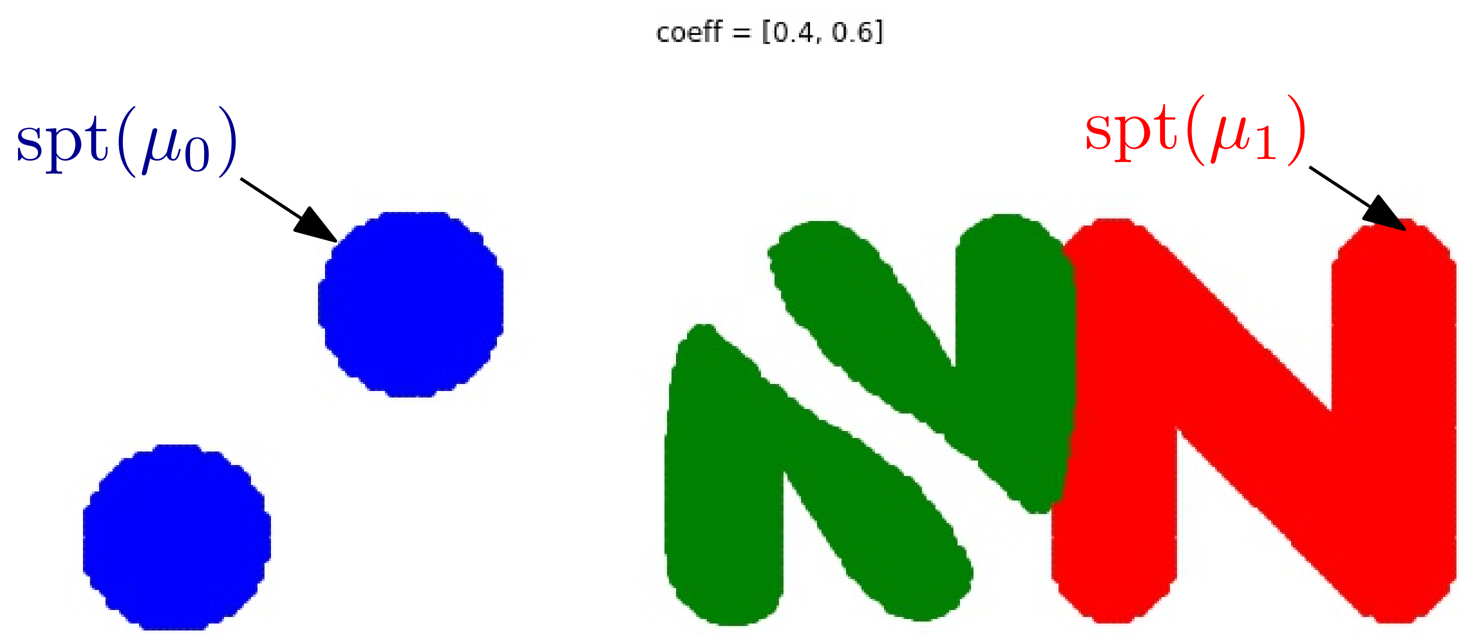

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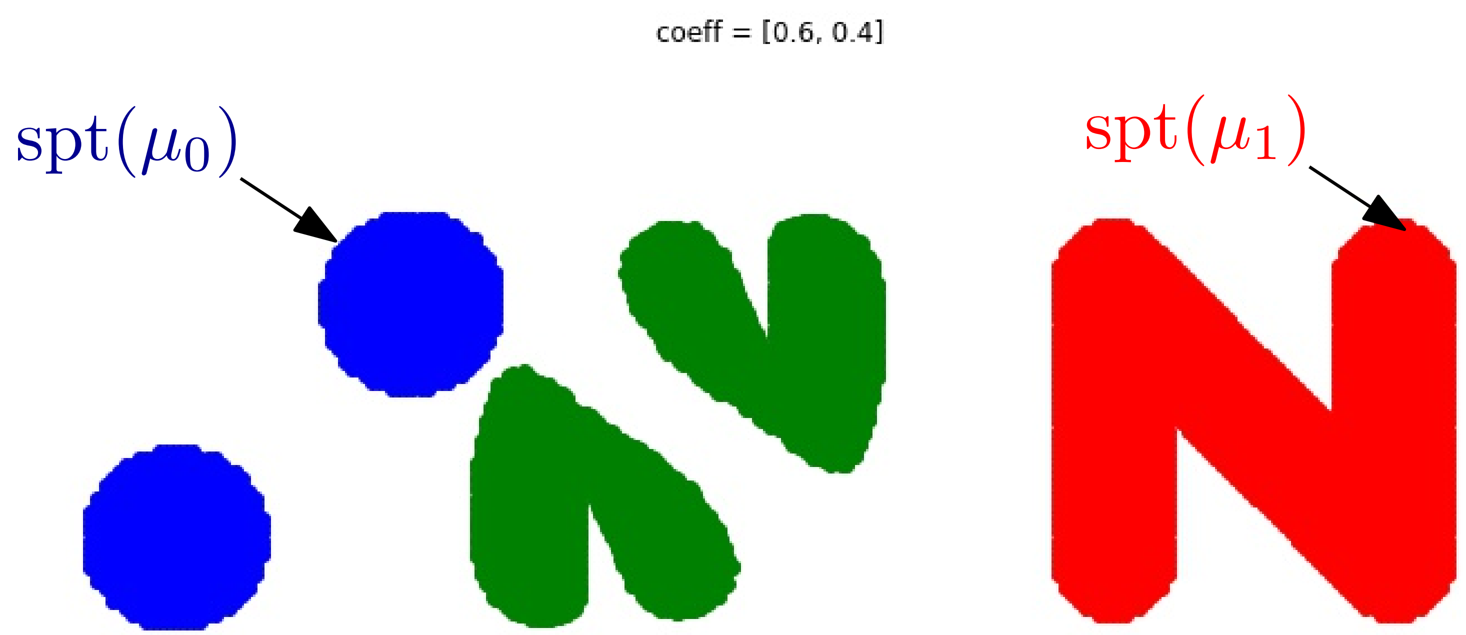

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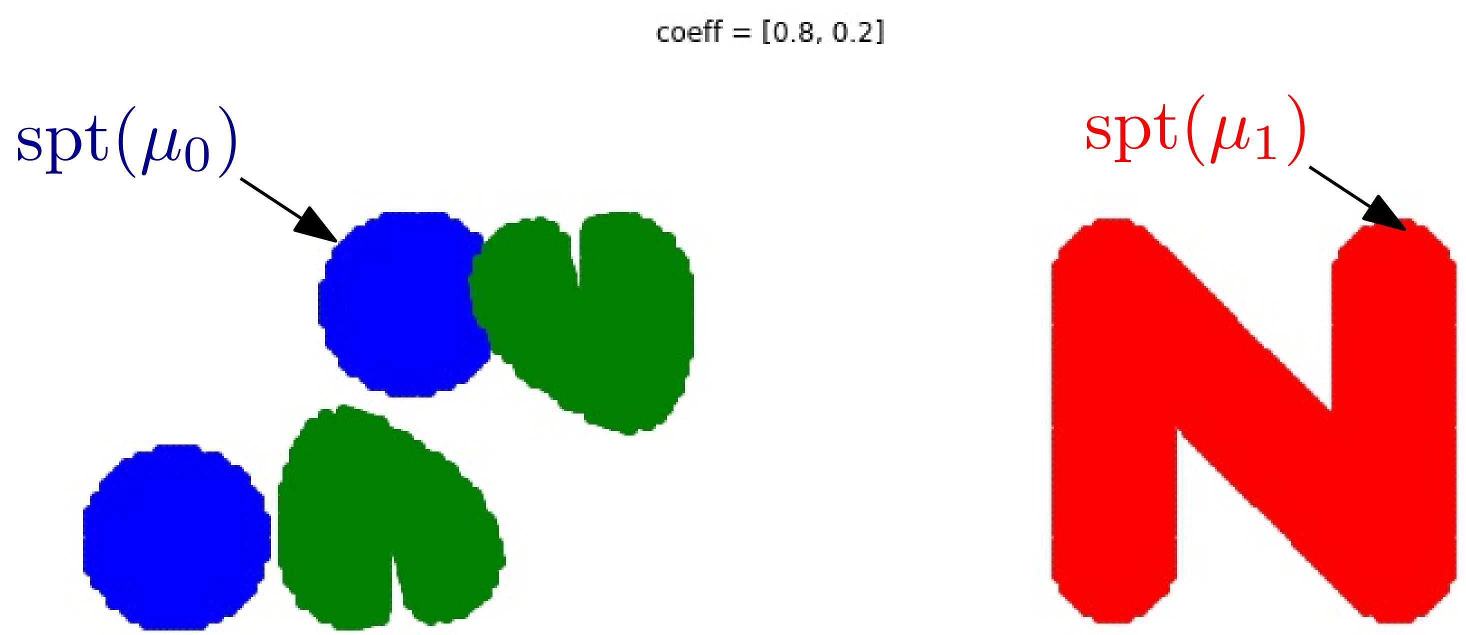

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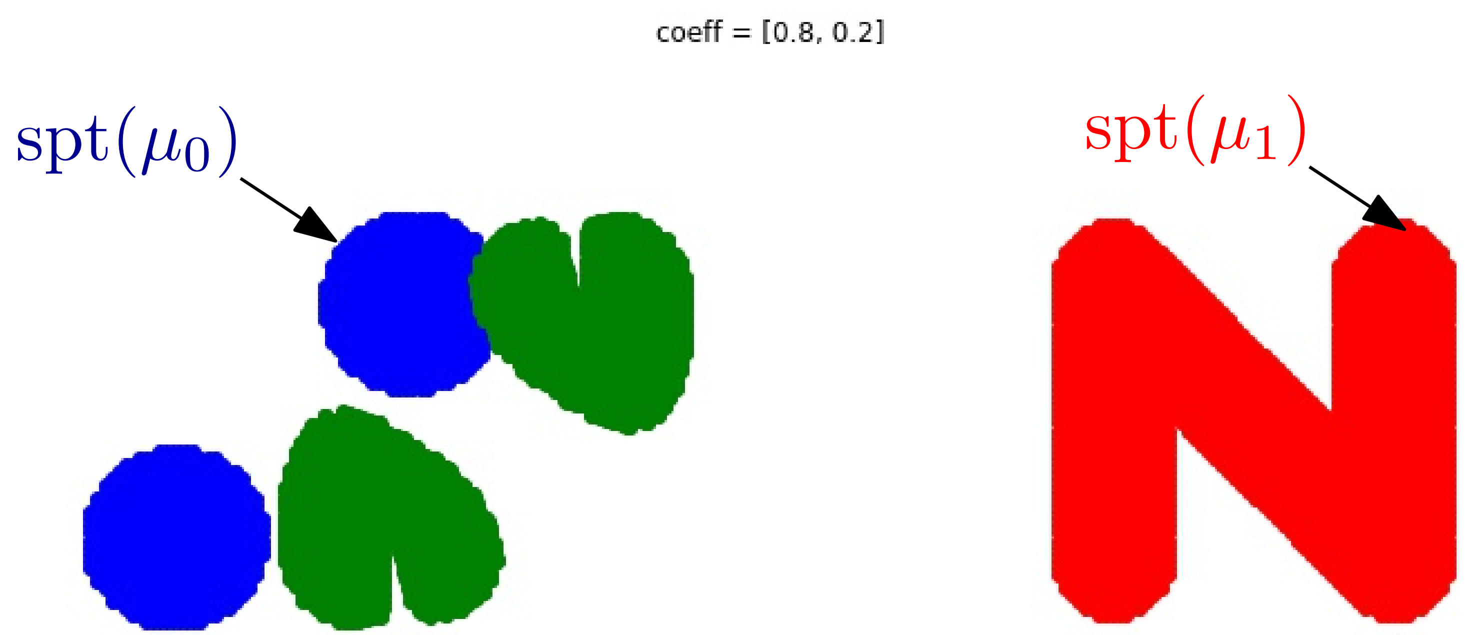

$\boxed{\text{What amount of the Wasserstein geometry is preserved by the embedding } \mu \mapsto T_\mu\text{?}}$

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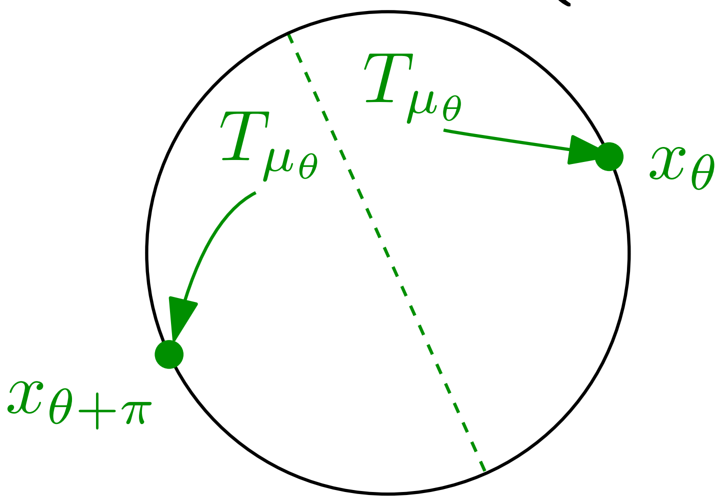

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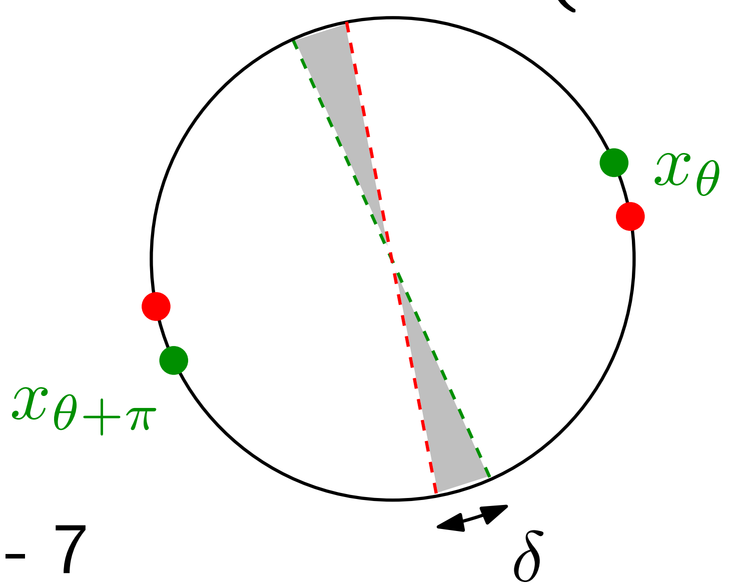

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(convexity: $\psi_\nu(y) - \psi_\nu(x) \geq \langle y - x|\nabla\psi_\nu(x)\rangle$) $\quad \geq \int\langle\nabla\psi_\mu - \nabla\psi_\nu|\nabla\psi_\nu(\nabla\phi_\nu)\rangle\,\mathrm{d}\,\rho$

$$= \int\langle\nabla\psi_\mu - \nabla\psi_\nu|\mathrm{id}\rangle\,\mathrm{d}\,\rho$$

- $\int \psi_\mu\,\mathrm{d}(\nu-\mu) \geq \int\langle\nabla\psi_\nu - \nabla\psi_\mu|\mathrm{id}\rangle\,\mathrm{d}\,\rho + \frac{L}{2}\|\nabla\phi_\mu - \nabla\phi_\nu\|_{L^2(\rho)}$
  ($T_\mu = \nabla\phi_\mu$ L-Lipschitz $\iff \psi_\mu = \phi_\mu^*$ is L-strongly convex)

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Thm (Berman, '18): Let $\rho \in \mathrm{Prob}^{\mathrm{ac}}(X)$ and $\mu, \nu \in \mathrm{Prob}(Y)$ with X, Y compact.
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- Proof of Berman's theorem relies on techniques from complex geometry.

2. Global, dimension-independent, Hölder-continuity of $\mu \mapsto T_\mu$.

Main theorem

Thm (M., Delalande, Chazal '19): Let X convex compact with $|X| = 1$ and $\rho = \mathrm{Leb}_X$, and let Y be compact. Then, there exists C s.t. for all $\mu, \nu \in \mathrm{Prob}(Y)$,

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- The constant C depend polynomially on $\mathrm{diam}(X), \mathrm{diam}(Y)$.
- Proof relies on the semidiscrete setting, i.e. the bound is established in the case

  $$\mu = \sum_i \mu_i \delta_{y_i},\ \nu = \sum_i \nu_i \delta_{y_i}.$$

  and one concludes using a density argument.

Semidiscrete OT for $c(x,y) = -\langle x|y\rangle$

- Let $\rho, \nu \in \mathrm{Prob}_1^{\mathrm{ac}}(\mathbb{R}^d)$ and $\Gamma(\rho,\mu) =$ couplings between ρ, μ,

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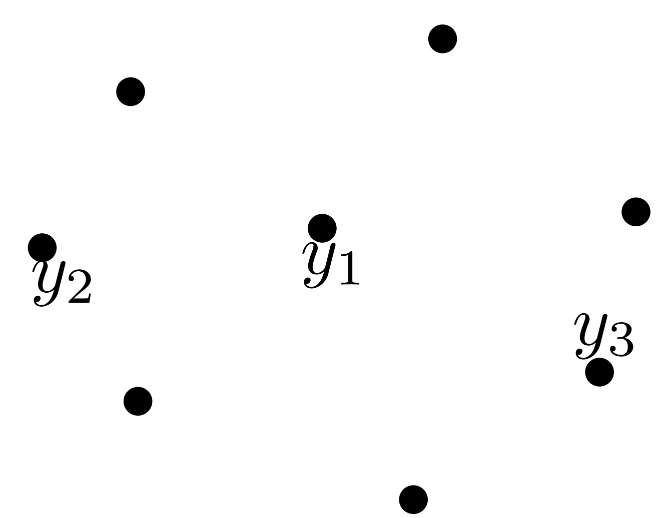

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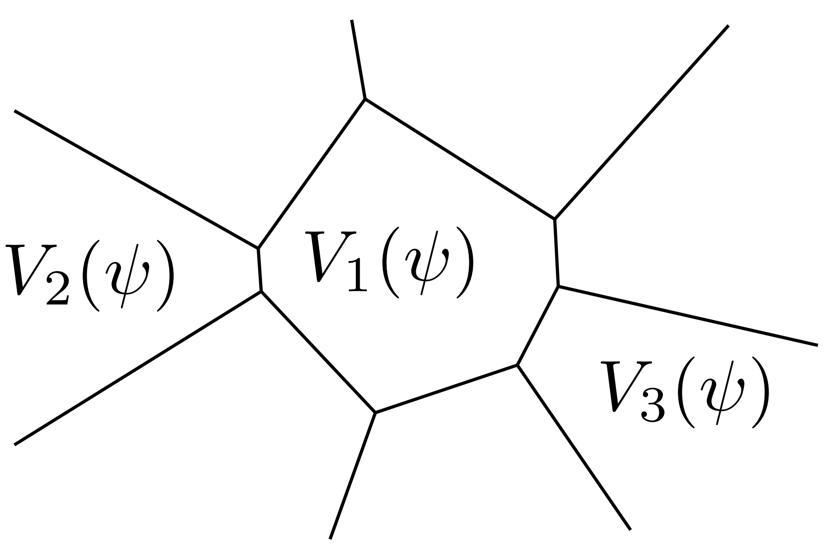

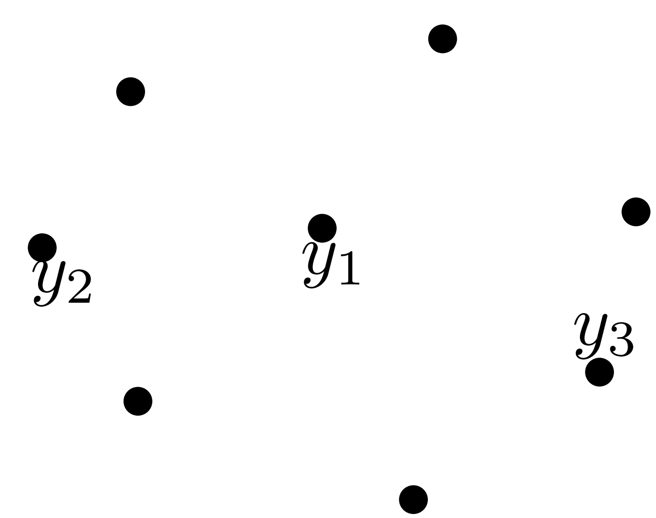

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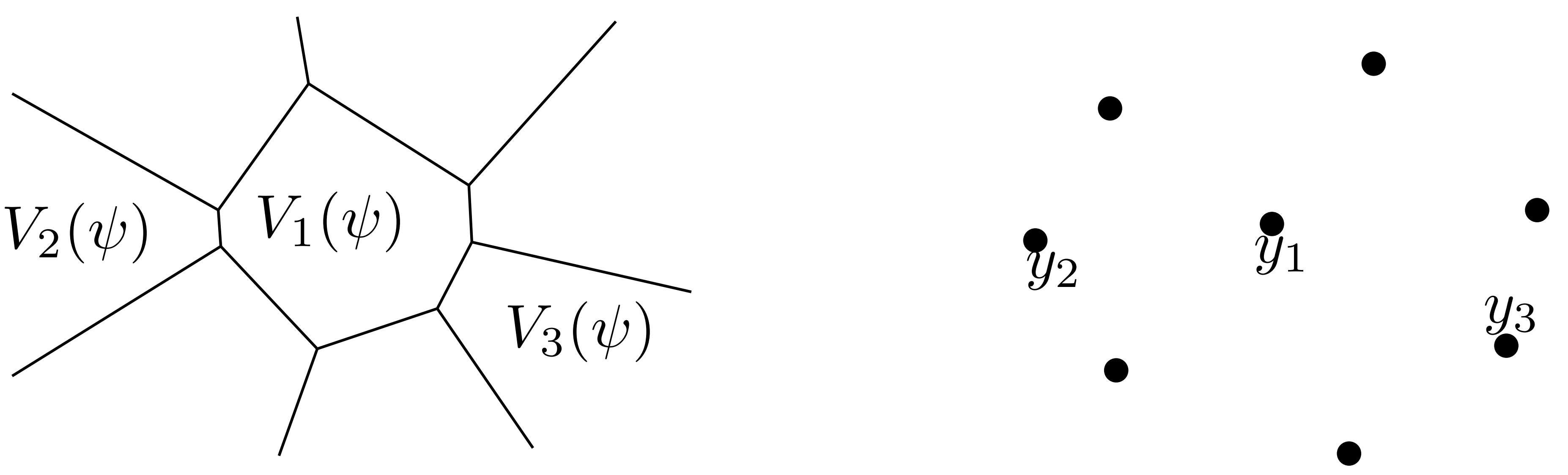

$$\boxed{\text{Thus, } \mathcal{T}(\rho,\mu) = \min_{\psi\in\mathbb{R}^N} \sum_i \int_{V_i(\psi)} \langle x|y_i\rangle - \psi_i \,\mathrm{d}\,\rho(x) + \sum_i \mu_i\psi_i}$$

Optimality condition and economic interpretation

$\mathcal{T}(\rho, \mu) = \min_{\psi \in \mathbb{R}^N} \Phi(\psi) - \sum_i \mu_i \psi_i$, where: $\boxed{\Phi(\psi) := \sum_i \int_{V_i(\psi)} \langle x | y_i \rangle - \psi_i \,\mathrm{d}\,\rho(x)}$

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- **Algorithm (Oliker–Prussner):** coordinate-wise increment. Complexity: $O(N^3)$.

Hessian on Φ and Newton's Algorithm

(Recall that $G_i(\psi) = \rho(V_i(\psi))$ and $\nabla\Phi = -(G_1, \ldots, G_N)$)

Proposition: ▶ If $\rho \in \mathcal{C}^0(X)$ and $(y_i)_{1\leq i\leq N}$ is generic, then $\Phi \in \mathcal{C}^2(\mathbb{R}^N)$ and

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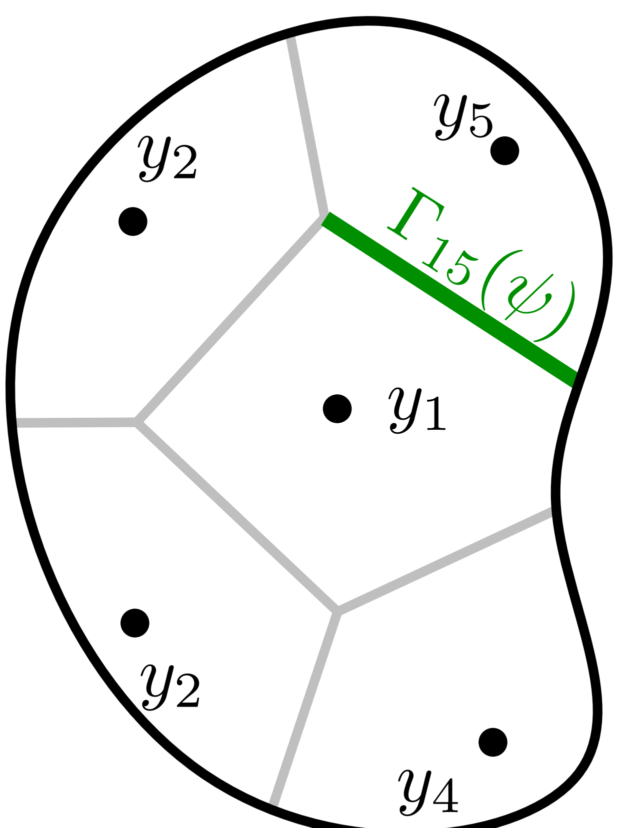

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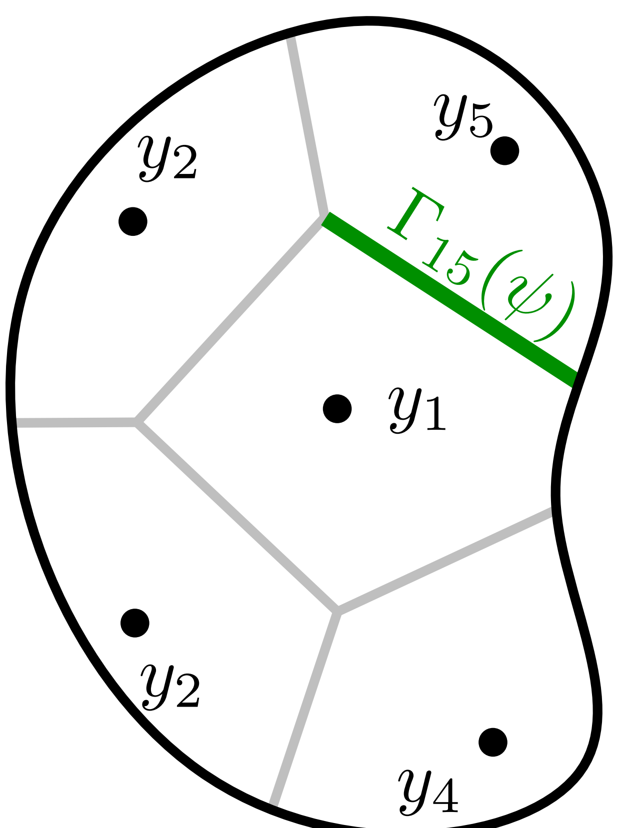
Let $E = \{\psi \in \mathbb{R}^N \mid \forall i, G_i(\psi) > 0\}$

▶ If $\Omega = \{\rho > 0\}$ is connected and $\psi \in E$, then $\mathrm{Ker} \mathrm{D}^2 \Phi(\psi) = \mathbb{R}(1, \ldots, 1)$.



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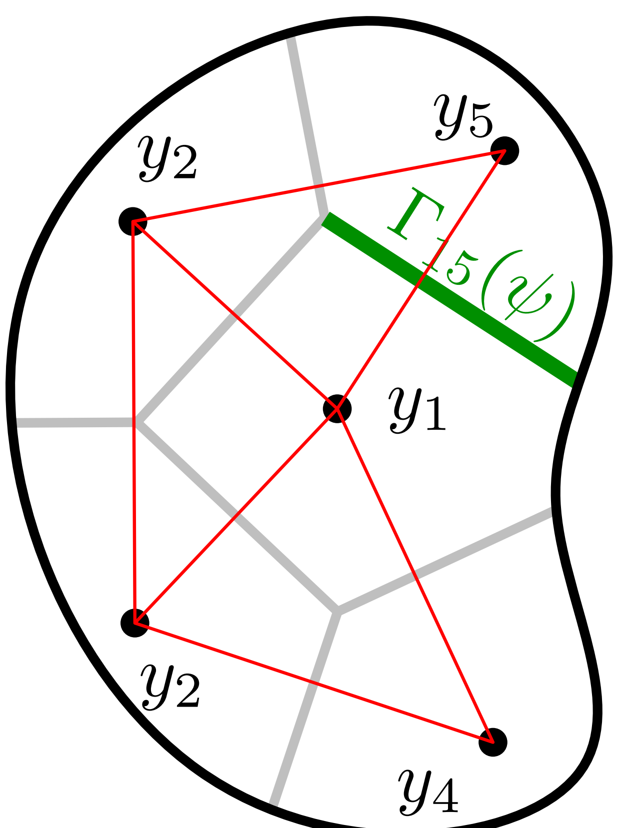

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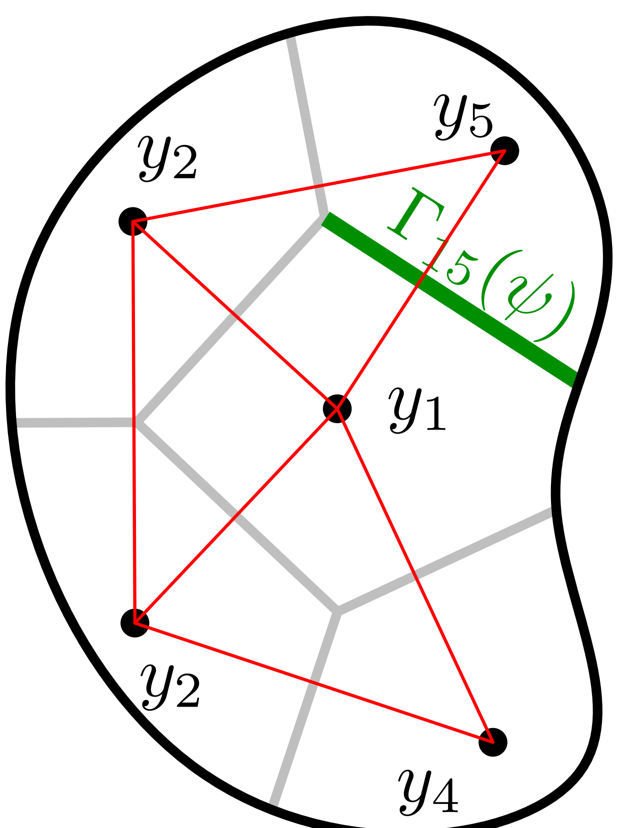

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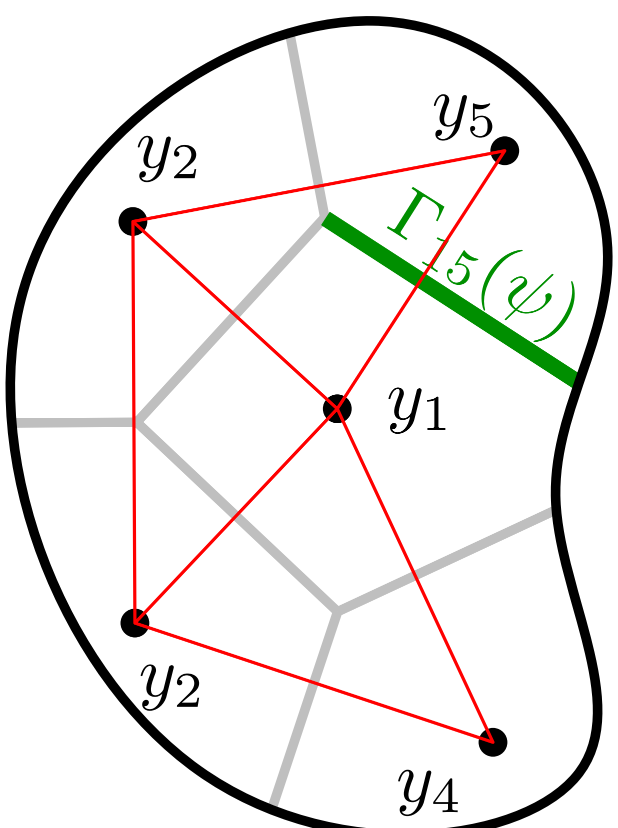

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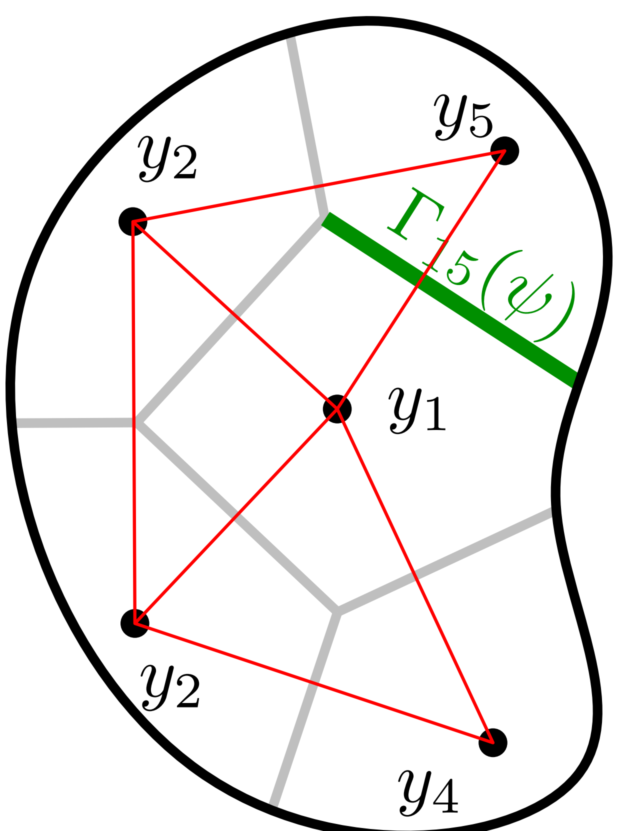

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Corollary: Global convergence of a damped Newton algorithm.

[Kitagawa, M., Thibert 16]

Numerical example

Source: $\rho =$ uniform on $[0,1]^2$,

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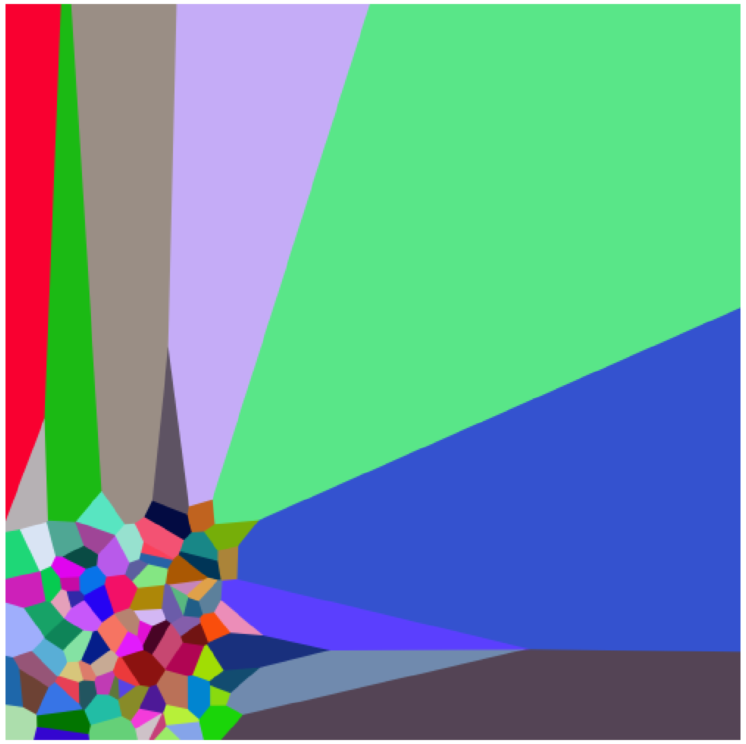

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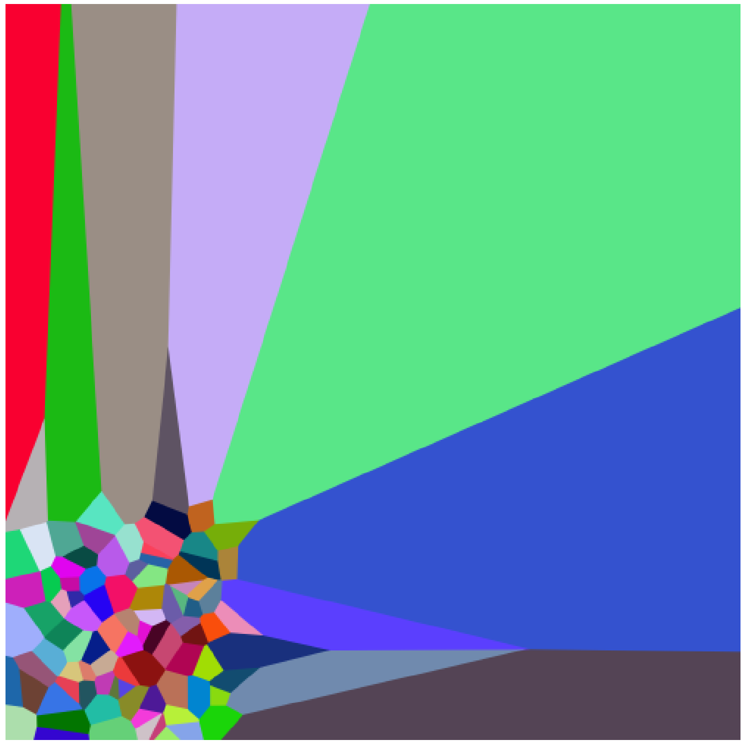

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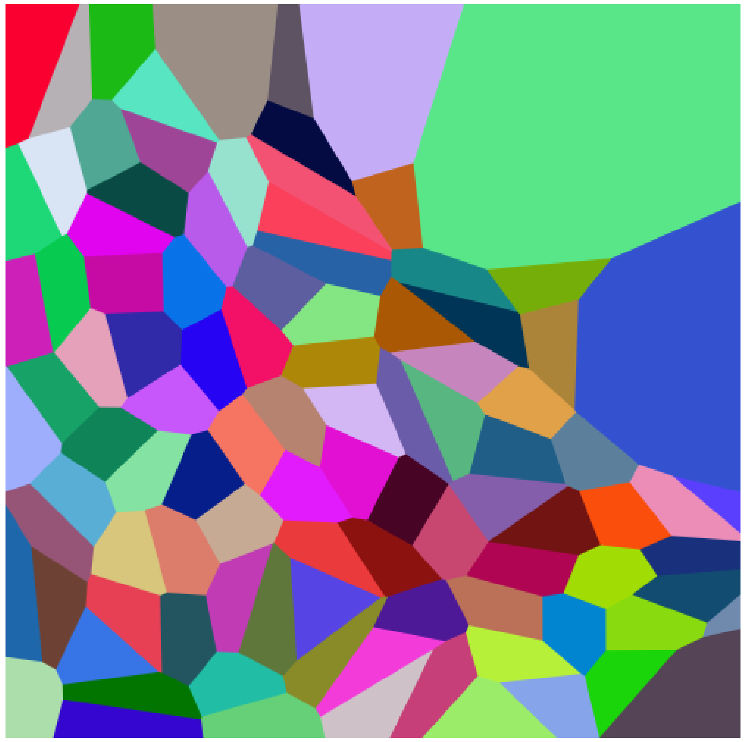

$\psi_1 = \mathrm{Newt}(\psi_0)$

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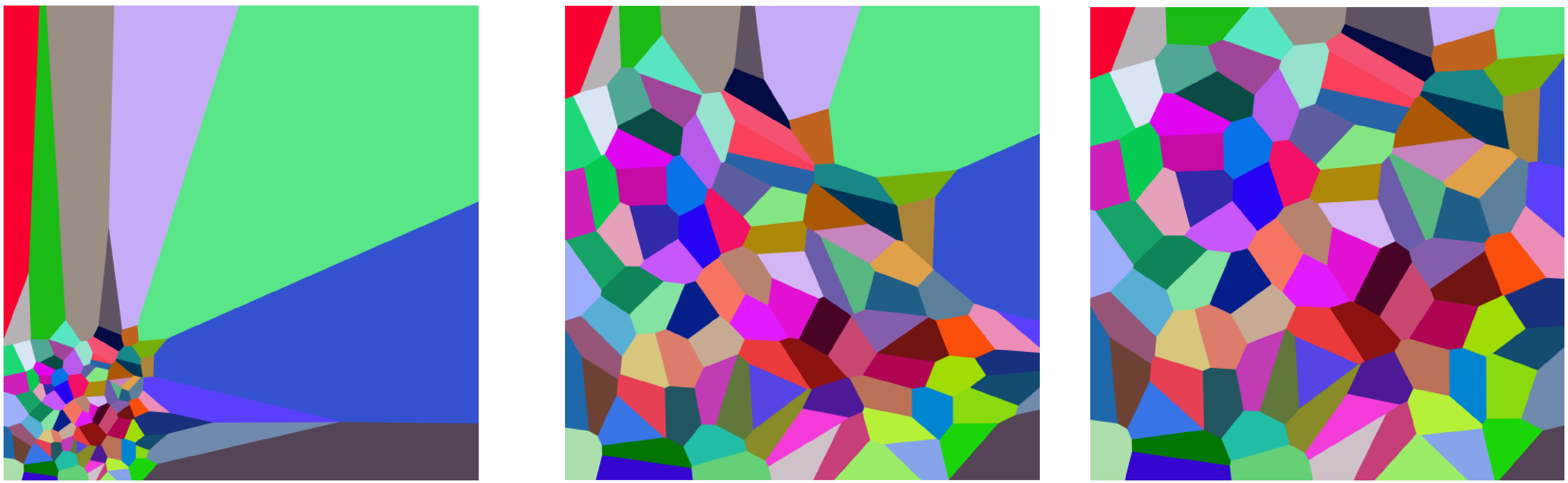

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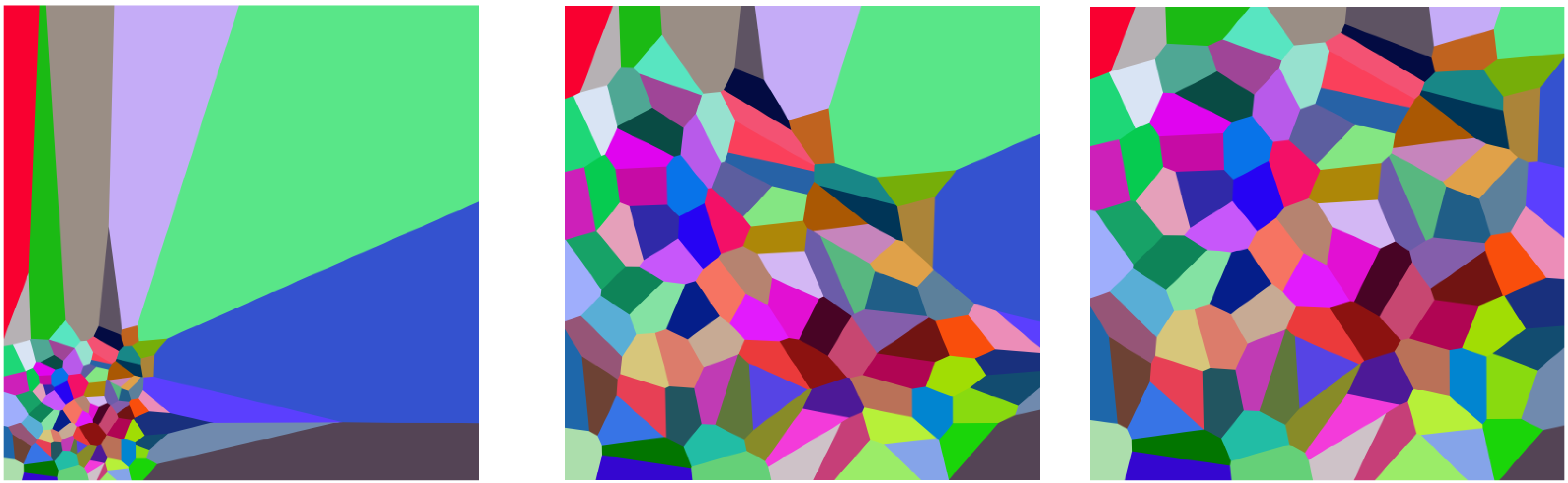

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Convergence is *very* fast when $\mathrm{spt}(\rho)$ convex: 17 Newton iterations for $N \geq 10^7$ in 3D.

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- We lose a little in the exponent to control the difference between OT maps...

A toy application

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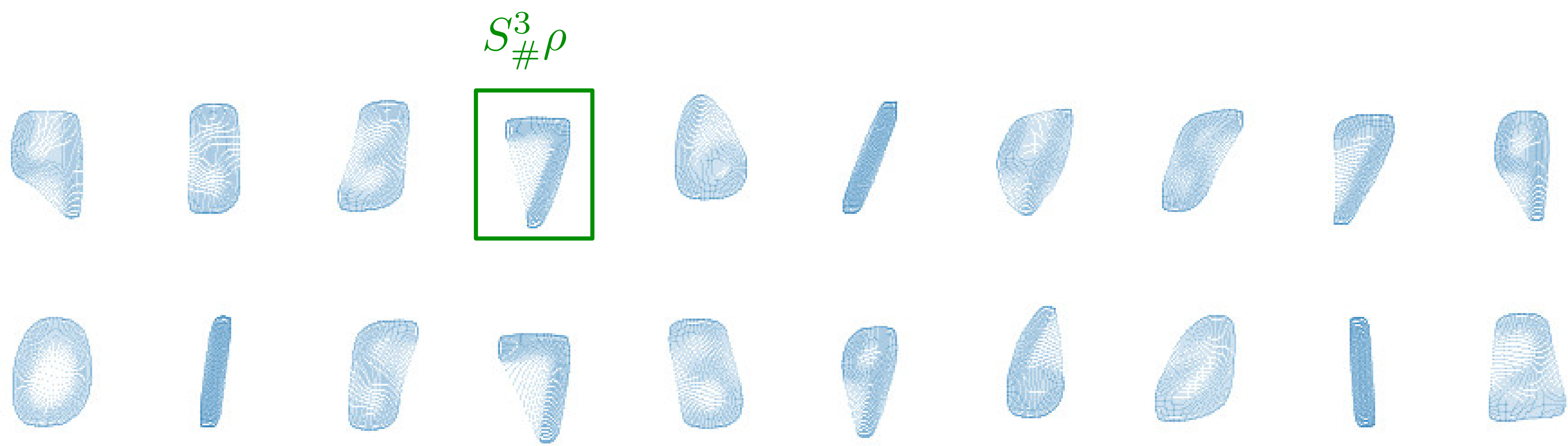

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Thank you for your attention!