Numerical resolution of McKean-Vlasov Forward Backward Stochastic Differential Equations using neural networks

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McKean-Vlasov FBSDEs

We solve general McKean-Vlasov FBSDEs of the form

$$\begin{cases} X_t &= \xi + \int_0^t b(s, X_s, Y_s, Z_s, \mathcal{L}(X_s), \mathcal{L}(Y_s), \mathcal{L}(Z_s))\ \mathrm{d}s + \int_0^t \sigma(s, X_s)\ \mathrm{d}W_s \\ Y_t &= g(X_T, \mathcal{L}(X_T)) + \int_t^T f(s, X_s, Y_s, Z_s, \mathcal{L}(X_s), \mathcal{L}(Y_s), \mathcal{L}(Z_s))\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s \end{cases}$$

with **machine learning** techniques. W_s is a standard Brownian motion.

The processes dynamics depend on their laws $\mathcal{L}(\cdot)$. In practise we restrict to cases when the law dependence only concerns probability moments.

Main motivation: Mean Field Games (MFG)

Stochastic games introduced by (Lasry and Lions 2006), (Huang, Malhame, and Caines 2006) followed by (Carmona and Delarue 2012) dealing with a large number of interacting players. The empirical law of players states influences both the dynamics and cost.

Applications

- Population dynamics (crowd, traffic jam, bird flocking...).
- Market interactions (Cardaliaguet and Lehalle 2019).
- Energy storage (Matoussi, Alasseur, and Ben Taher 2018), electric cars management.
- Social networks.

N players stochastic game

Player i minimises a cost $J^i(\alpha^1, \cdots, \alpha^i, \cdots, \alpha^N)$ depending on the empirical distribution $\overline{\mu_t} = \frac{1}{N}\sum_{k=1}^N \delta_{X_t^k} \to$ mean field interaction.

$$\min_{\alpha^i \in \mathbb{A}} \quad \mathbb{E}\left[\int_0^T f(t, X_t^i, \overline{\mu}_t, \alpha_t^i)\, \mathrm{d}t + g(X_T^i, \overline{\mu}_T)\right]$$

$$\text{subject to} \quad \mathrm{d}X_t^i = b(t, X_t^i, \overline{\mu}_t, \alpha_t^i)\, \mathrm{d}t + \sigma(t, X_t^i, \overline{\mu}_t)\, \mathrm{d}W_t^i.$$

Difficult problem in general.

Asymptotic control problem when $N \to +\infty$

Given a family $(\mu_t)_{t\in[0,T]}$ of probability measures we solve for a representative player

$$\begin{aligned} \min_{\alpha} \quad & \mathbb{E}\left[\int_0^T f(t, X_t^\mu, \mu_t, \alpha_t)\,\mathrm{d}t + g(X_T^\mu, \mu_T)\right] \\ \text{subject to} \quad & \mathrm{d}X_t^\mu = b(t, X_t^\mu, \mu_t, \alpha_t)\,\mathrm{d}t + \sigma(t, X_t^\mu, \mu_t)\,\mathrm{d}W_t. \end{aligned}$$

Fixed point on probability measures

$$\mu_t = \mathcal{L}(X_t^\mu).$$

Simpler than the N players game $\to$ unique player in the limit!
Value function given by

$$v^\mu(t,x) = \inf_{\alpha\in\mathcal{A}_t} \mathbb{E}\left[\int_t^T f(s, X_s, \mu_s, \alpha_s)\,\mathrm{d}s + g(X_T, \mu_T)|X_t = x\right].$$

Optimality conditions

HJB equation

$$\begin{cases} \partial_t v + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top \partial^2_{xx} v) + \min_{\alpha\in\mathcal{A}} H^{\nu_t}(t,x,\partial_x v,\alpha) = 0 \\ v(T,x) = g(x,\nu_T) \end{cases}$$

coupled with Fokker-Planck equation:

$$\begin{cases} \partial_t \nu - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top \partial^2_{xx} \nu) - \mathrm{div}(b(t,x,\nu_t,\hat{\alpha}^{\nu_t}(t,x,\partial_x v))\nu) = 0 \\ \nu(0,\cdot) = \nu_0. \end{cases}$$

Solved in (Achdou and Capuzzo-Dolcetta 2010) with finite differences methods.

Probabilistic point of view: McKean-Vlasov FBSDEs

$Y_t = \partial_x v(t, X_t) \rightarrow$ stochastic Pontryagin principle

$$\begin{cases} \mathrm{d}X_t &= b(t, X_t, \mathcal{L}(X_t), \hat{\alpha}^{\mathcal{L}(X_t)}(t, X_t, Y_t))\ \mathrm{d}t + \sigma\ \mathrm{d}W_t \\ X_0 &= \xi \\ \mathrm{d}Y_t &= -\partial_x H^{\mathcal{L}(X_t)}(t, X_t, Y_t, \hat{\alpha}^{\mathcal{L}(X_t)}(t, X_t, Y_t))\ \mathrm{d}t + Z_t\ \mathrm{d}W_t \\ Y_T &= \partial_x g\left(X_T, \mu_T\right). \end{cases}$$

In the form

$$\begin{cases} X_t &= \xi + \int_0^t b(s, X_s, Y_s, Z_s, \mathcal{L}(X_s), \mathcal{L}(Y_s), \mathcal{L}(Z_s))\ \mathrm{d}s + \int_0^t \sigma(s, X_s)\ \mathrm{d}W_s \\ Y_t &= g(X_T, \mathcal{L}(X_T)) + \int_t^T f(s, X_s, Y_s, Z_s, \mathcal{L}(X_s), \mathcal{L}(Y_s), \mathcal{L}(Z_s))\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s \end{cases}$$

Advantages of machine learning techniques

Curse of dimensionnality with finite differences methods if the state dimension is high (≥ 3 or 4).

$\longrightarrow$ Some machine learning schemes solve nonlinear PDE in high dimension (10, 50 or even 100): - Deep BSDE method of (Han, Jentzen, and E 2017)
- Deep Galerkin method of (Sirignano and Spiliopoulos 2017)
- Deep Backward Dynamic Programming of (Huré, Pham, and Warin 2020).

$\longrightarrow$ Open source libraries such as Tensorflow or Pytorch.

$\longrightarrow$ Efficient computation on GPU nodes.

A global method

Based on the Deep BSDE method of (Han, Jentzen, and E 2017)

$$\begin{cases} X_t &= \xi + \int_0^t b(s, X_s, Y_s, Z_s, \mathcal{L}(X_s), \mathcal{L}(Y_s), \mathcal{L}(Z_s))\ \mathrm{d}s + \int_0^t \sigma(s, X_s)\ \mathrm{d}W_s \\ Y_t &= g(X_T, \mathcal{L}(X_T)) + \int_t^T f(s, X_s, Y_s, Z_s, \mathcal{L}(X_s), \mathcal{L}(Y_s), \mathcal{L}(Z_s))\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s \end{cases}$$

Discretization:

$$\begin{cases} X_{t_{i+1}} &= X_{t_i} + b\left(t_i, X_{t_i}, Y_{t_i}, Z_{t_i}, \mu_{t_i}\right)\ \Delta t_i + \sigma\left(t_i, X_{t_i}\right) \Delta W_{t_i} \\ Y_{t_{i+1}} &= Y_{t_i} - f\left(t_i, X_{t_i}, Y_{t_i}, Z_{t_i}, \mu_{t_i}\right)\ \Delta t_i + Z^\theta(t_i, X_{t_i}) \Delta W_{t_i}. \end{cases}$$

Y_0 is a variable and Z_{t_i} is approached by a neural network $Z^\theta(t_i, \cdot)$ which minimizes the loss function $\mathbb{E}\left[\left(Y_T - g\left(X_T, \mu_T\right)\right)^2\right]$. The forward backward system is transformed into a **forward form** and an **optimization** problem. Also studied in (Fouque and Zhang 2019), (Carmona and Laurière 2019) in dimension 1 and with less generality.

Fixed point on probability measures

Estimation of state and value function moments:

- **Direct**: use of current particles empirical law

$$\mathbb{E}[X_{t_i}]^{(k+1)} \simeq \mu_{t_i}^{(k+1)} = \frac{1}{N_b} \sum_{j=1}^{N_b} X_{t_i}^j. \tag{1}$$

- **Dynamic**: keep in memory previously computed moments and average them with current particle moments

$$\mathbb{E}[X_{t_i}]^{(k+1)} \simeq \frac{N_m N_b \mu_{t_i}^{(k)} + \sum_{j=1}^{N_b} X_{t_i}^j}{N_b + N_m N_b}. \tag{2}$$

- **Expectation**: estimate $\mathbb{E}[X_t]$ by an additional neural network.

$$\mathbb{E}[X_t] \simeq \Psi_\kappa(t). \tag{3}$$

We add to the loss a term $\mathbb{E}\left[\lambda \sum_{i=0}^{N_t} \left(\Psi_\kappa(t_i) - \overline{X_{t_i}}\right)^2\right]$.

A local approach

Inspired by (Huré, Pham, and Warin 2020).

Z and Y are approximated by neural networks $(Z^i_{\theta^i_m}(\cdot), Y^i_{\theta^i_m}(\cdot))_{\{i\in[\![0,N-1]\!]\}}$. At iteration m we simulate X_i with the previously computed parameters θ^i_m. The X_i's dynamics being frozen with parameters θ^i_m:

- First Y^N is set to the terminal condition $g(X_N, \mu_N)$.
- Then, we solve successively the local backward problems for i from $N-1$ to 0

$$\min_{\theta^i_m} \mathbb{E}\left[\left(Y^{i+1}_{\theta^{i+1}_{m+1}}(X_{i+1}) - Y^i_{\theta^i_m}(X_i) + f\left(t_i, X_i, Y^i_{\theta^i_m}(X_i), Z^i_{\theta^i_m}(X_i), \mu_{t_i}\right)\Delta t - Z^i_{\theta^i_m}(X_i)\Delta W_{t_i}\right)^2\right].$$

A linear quadratic test case: price impact

Large number of traders who want to sell a portfolio at the same time. Example from (Carmona and Delarue 2018).

$$\min_{\alpha \in \mathbb{A}} \quad \mathbb{E}\left[\int_0^T \left(\frac{c_\alpha}{2}\|\alpha_t\|^2 + \frac{c_X}{2}\|X_t\|^2 - \gamma X_t \cdot \mu_t\right) \mathrm{d}t + \frac{c_g}{2}\|X_T\|^2\right].$$

$$\text{subject to} \quad X_t = x_0 + \int_0^t \alpha_s \, \mathrm{d}s + \sigma \, W_t$$

and the fixed point $\mathbb{E}[\alpha_t] = \mu_t$. Optimality system:

$$\begin{cases} \mathrm{d}X_t &= -\frac{1}{c_\alpha} Y_t \, \mathrm{d}t + \sigma \, \mathrm{d}W_t \\ X_0 &= x_0 \\ \mathrm{d}Y_t &= -(c_X X_t + \frac{\gamma}{c_\alpha}\mathbb{E}[Y_t]) \, \mathrm{d}t + Z_t \, \mathrm{d}W_t \\ Y_T &= c_g X_T. \end{cases}$$

We take $c_X = 2, x_0 = 1, \sigma = 0.7, \gamma = 2, c_\alpha = 2/3, c_g = 0.3$. The simulations are conducted with $d = 10, \Delta t = 0.01$.

Results 1/4: Local method/Price Impact model

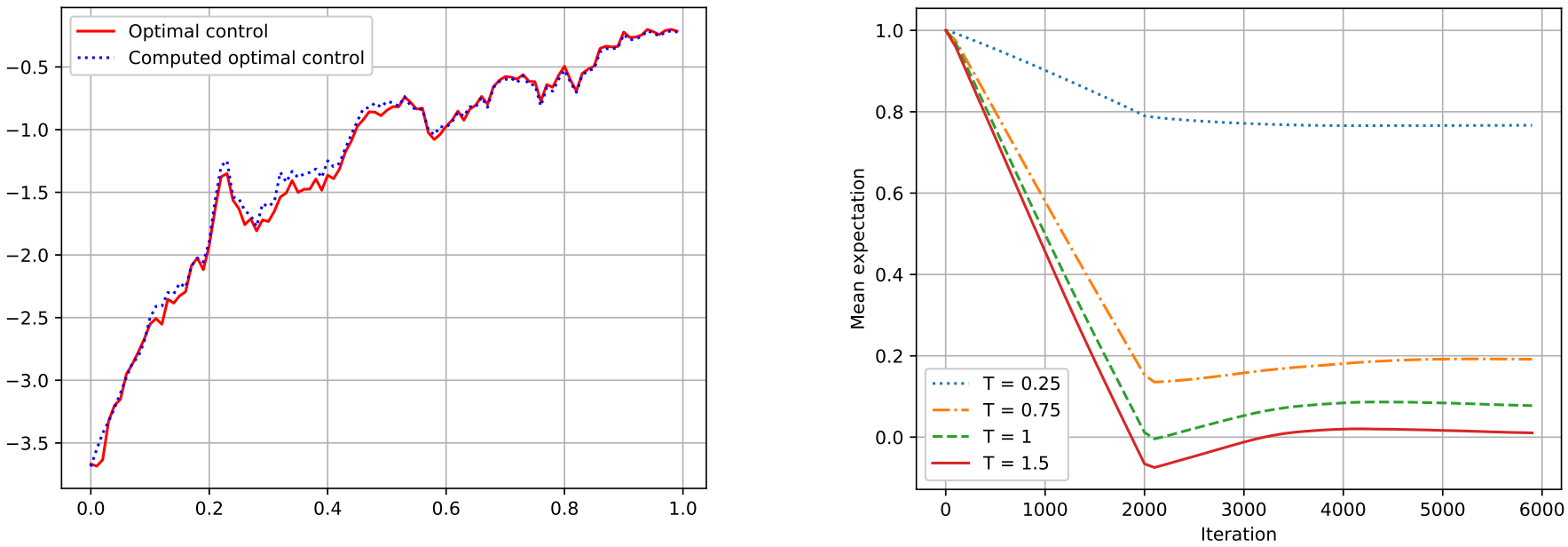

Figure: Computed optimal control after 6000 iterations and expectation of the state at terminal state for the local method.

Results 2/4: Global method/Price Impact model

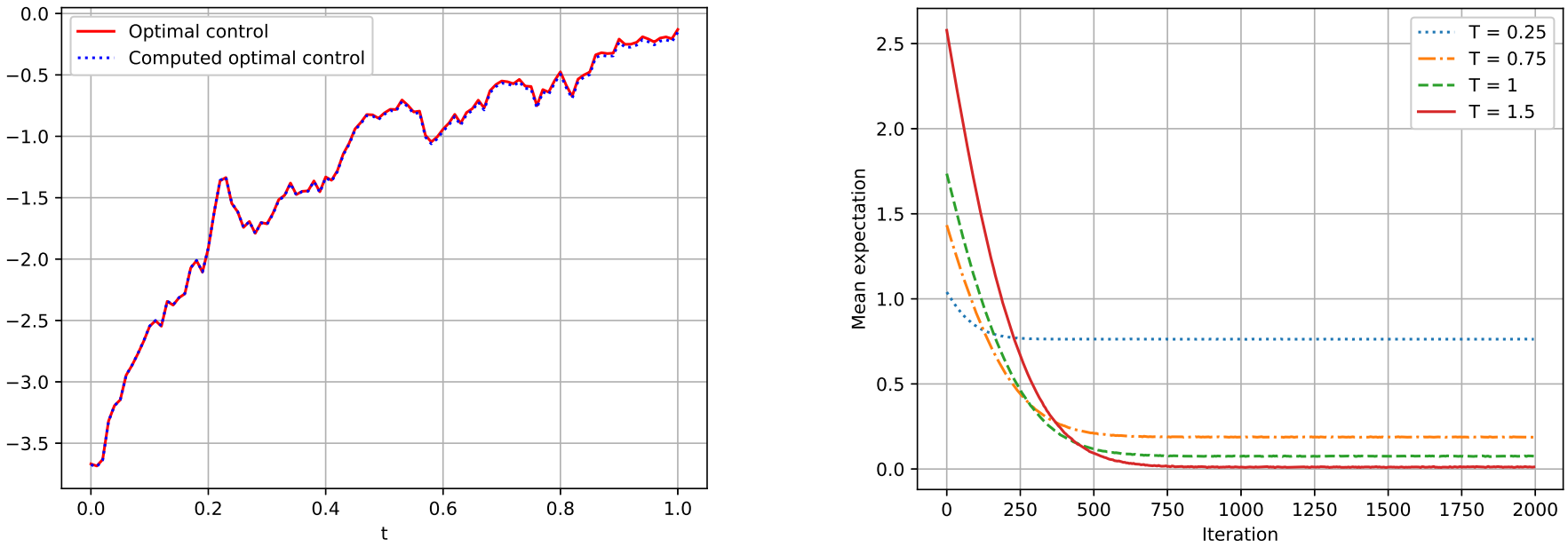

Figure: Computed optimal control after 2000 iterations and expectation of the state at terminal state for the global method.

A non linear quadratic test case

Take X_t log normal, $Y_t = \exp(\alpha t)\log(\prod_i X_t^i)$ and quadratic dynamics.

$$\begin{cases} \mathrm{d}X_t^i &= (a^i X_t^i + b(Y_t + Z_t^i + \mathbb{E}[X_t^i] + \mathbb{E}[Y_t] + \mathbb{E}[Z_t^i]) \\ & -b\left(e^{\alpha t}\log\left(\prod_{i=1}^d X_t^i\right) + \sigma_t^i e^{\alpha t} + g_t^i + c_t + e_t^i\right) \\ & +c\Bigg(Y_t^2 + (Z_t^i)^2 + \mathbb{E}[(X_t^i)^2] + \mathbb{E}[Y_t^2] + \mathbb{E}[(Z_t^i)^2]) \\ & -c\left(e^{2\alpha t}\log\left(\prod_{i=1}^d X_t^i\right)^2 + (\sigma_t^i)^2 e^{2\alpha t} + (g_t^i)^2 + c_t^2 + (e_t^i)^2\right)\Bigg)\ \mathrm{d}t + \sigma_t^i X_t^i\ \mathrm{d}W_t^i \\ X_0^i &= \xi^i \\ \mathrm{d}Y_t &= \Bigg(\phi(t, X_t) + b(Y_t + \tfrac{1}{d}\textstyle\sum_{i=1}^d Z_t^i + \tfrac{1}{d}\sum_{i=1}^d \mathbb{E}[X_t^i] + \mathbb{E}[Y_t] + \tfrac{1}{d}\sum_{i=1}^d \mathbb{E}[Z_t^i]) \\ & -b\left(e^{\alpha t}\log\left(\prod_{i=1}^d X_t^i\right) + \tfrac{1}{d}\textstyle\sum_{i=1}^d \sigma_t^i e^{\alpha t} + \tfrac{1}{d}\sum_{i=1}^d g_t^i + c_t + \tfrac{1}{d}\sum_{i=1}^d e_t^i\right) \\ & +c(Y_t^2 + \tfrac{1}{d}\textstyle\sum_{i=1}^d (Z_t^i)^2 + \tfrac{1}{d}\sum_{i=1}^d \mathbb{E}[(X_t^i)^2] + \mathbb{E}[Y_t^2] + \tfrac{1}{d}\sum_{i=1}^d \mathbb{E}[(Z_t^i)^2]) \\ & -c\left(e^{2\alpha t}\log\left(\prod_{i=1}^d X_t^i\right)^2 + c_t^2 + \tfrac{1}{d}\textstyle\sum_{i=1}^d (\sigma_t^i)^2 e^{2\alpha t} + (g_t^i)^2 + (e_t^i)^2\right)\Bigg)\ \mathrm{d}t + Z_t\ \mathrm{d}W_t \\ Y_T &= e^{\alpha T}\log\left(\prod_{i=1}^d X_T^i\right). \end{cases}$$

We take $a = b = c = 0.1,\ \alpha = 0.5, \sigma = 0.4, \xi = 1$. The simulations are conducted with $d = 10, \Delta t = 0.01$.

Results 3/4: Local method/Non linear quadratic equation

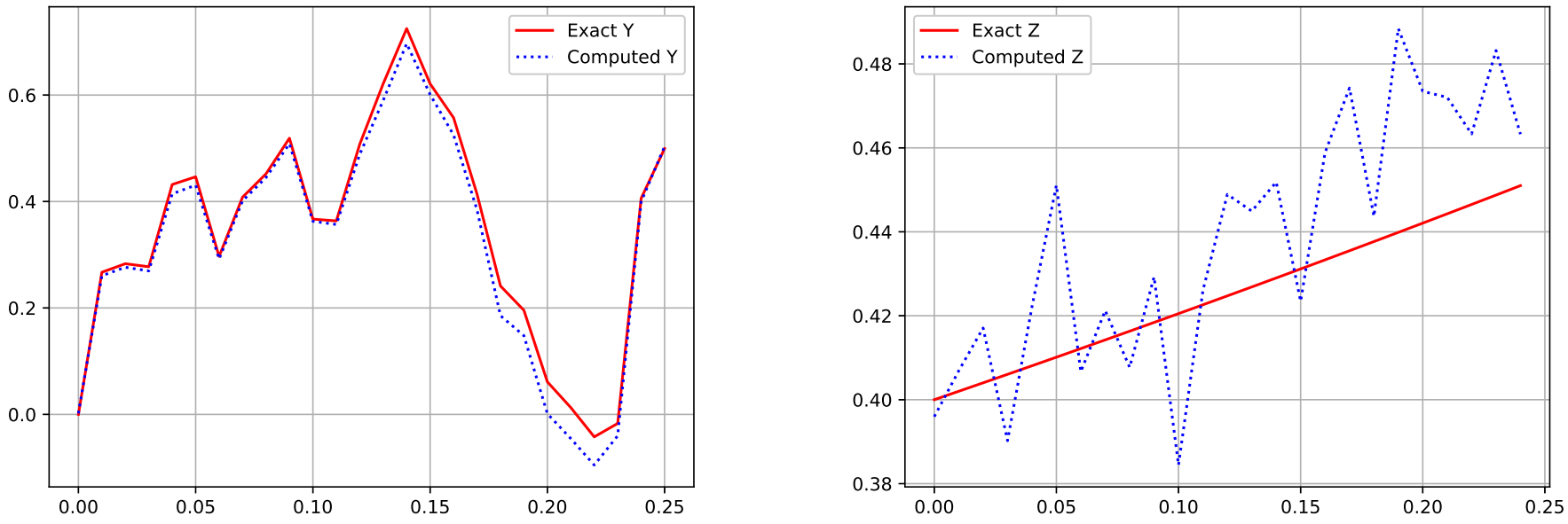

Figure: Y and Z for the local method after 6000 iterations

Results 4/4: Global method/Non linear quadratic equation

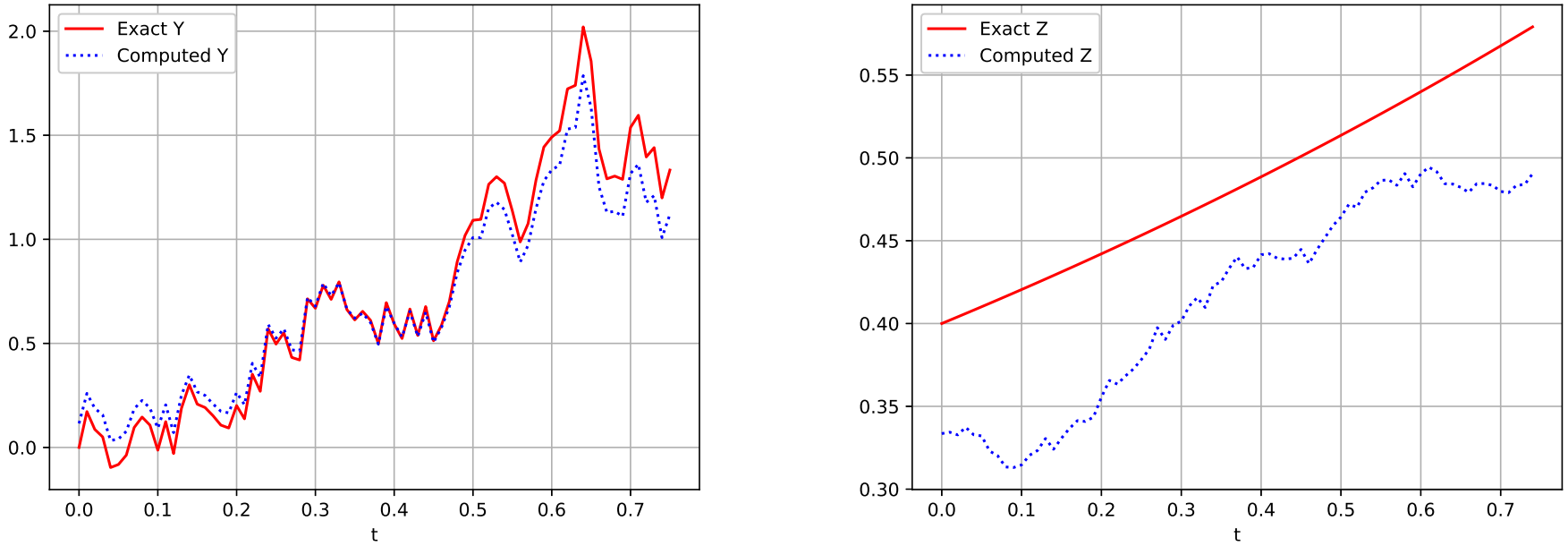

Figure: Y and Z for the global method after 2000 iterations

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