TSA Part 2: The Revenge
A HJB-POD approach for the control of nonlinear PDEs on a tree structure

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joint work with A. Alla and M. Falcone

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Outline

1. Extension to high-order
   - High-order TSA
   - Numerical test

2. Control of nonlinear PDEs by TSA
   - Model Order Reduction Methods
   - HJB-POD on a tree structure
   - Numerical tests

Outline

Extension to high-order (Falcone, Ferretti, '94)

We introduce a convergent one-step approximation

$$\begin{cases} y^{n+1} = y^n + \Delta t \, \Phi(y^n, \mathbf{U_n}, t_n, \Delta t), \\ y^0 = x, \end{cases}$$

where the admissible control matrix
$\mathbf{U_n} \in U_{\Delta t} \subset U \times U \ldots \times U \in \mathbb{R}^{M \times (q+1)}$, with $U \subset \mathbb{R}^M$.
We assume that the function Φ is consistent

$$\lim_{\Delta t \to 0} \Phi(x, \overline{\mathbf{u}}, t, \Delta t) = f(x, \overline{\mathbf{u}}, t),$$

where $\overline{\mathbf{u}} = (\bar{u}, \ldots, \overline{u}) \in \mathbf{U}$ for $\overline{u} \in U$ and Lipschitz continuous:

$$|\Phi(x, \mathbf{U}, t, \Delta t) - \Phi(y, \mathbf{U}, t, \Delta t)| \leq L_\Phi |x - y|.$$

Under these assumptions the scheme is convergent.

Extension to high-order schemes

Then, we consider the approximation of the cost functional

$$J_{x,t_n}^{\Delta t}(\{\mathbf{U_m}\}) = \Delta t \sum_{m=n}^{N-1} \sum_{i=0}^{q} w_i L(y^{m+\tau_i}, u_i^m, t_m) + g(y^N),$$

where τ_i and w_i are the nodes and weights of the quadrature formula. Finally we define the numerical value function as

$$V(t,x) = \inf_{\{\mathbf{U_n}\}} J_{x,t}^{\Delta t}(\{\mathbf{U_n}\})$$

Proposition (Discrete DPP)

$$V(t,x) = \inf_{\{\mathbf{U_m}\}} \left\{ \Delta t \sum_{i=0}^{q} w_i L(y^{n+\tau_i}, u_i^n, t_{n+\tau_i}) + V(t_{n+1}, y^{n+1}) \right\}$$

Pruning for high-order scheme

We can again define the pruned trajectory

$$\eta_j^{n+1} = \eta^n + \Delta t\, \Phi(\eta^n, \mathbf{U}_n, t_n, \Delta t) + \mathcal{E}_{\varepsilon_{\mathcal{T}}}(\eta^n + \Delta t\, \Phi(\eta^n, \mathbf{U}_n, t_n, \Delta t), \{\eta_i^{n+1}\}_i)$$

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Proposition

Given a one-step approximation $\{y^n\}_n$ and its perturbation $\{\eta^n\}_n$, then

$$|y^n - \eta^n| \leq \varepsilon_{\mathcal{T}} \frac{t_n - t}{\Delta t} e^{L_\Phi (t_n - t)}.$$

To guarantee p-th order convergence, the tolerance must be chosen such that

$$\varepsilon_{\mathcal{T}} \leq C(\Delta t)^{p+1}.$$

Test 1: Bilinear control for Advection Equation

$$\begin{cases} y_t + cy_x = yu(t) & (x,t) \in \Omega \times [0,T], \\ y(x,t) = 0 & (x,t) \in \partial\Omega \times [0,T], \\ y(x,0) = y_0(x) & x \in \Omega. \end{cases}$$

$$J_{y_0,t}(u) = \int_t^T \left(\|y(s) - \tilde{y}(s)\|_2^2 \, dx + 0.01|u(s)|^2 \right) \, ds + \|y(T) - \tilde{y}(T)\|_2^2.$$

Semi-discrete problem (System dimension $= 10^2$)

$$\dot{y}(t) = Ay(t) + y(t)u(t),$$

$$\Delta x = 0.01, \Omega = [0,3] \text{ and } c = 1.5$$

Case 1: $\tilde{y} = 0$, $U = [-4, 0]$

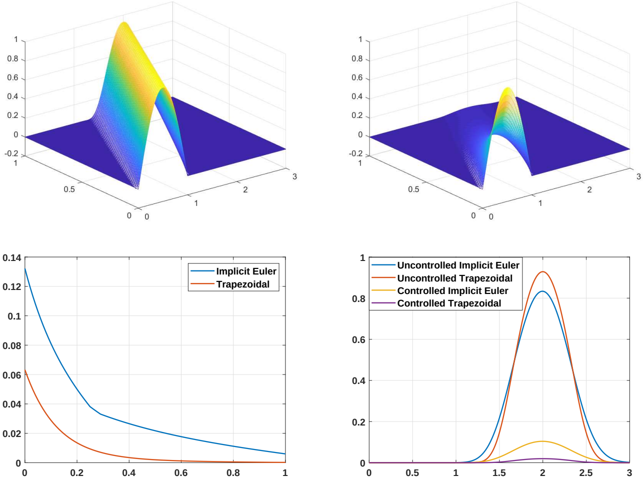

Figure: Top: Uncontrolled (left) and trapezoidal rule controlled solution (right). Bottom: cost functionals (left) and solutions at final time (right).

Case 2: $\tilde{y}(x,t) = y_0(x-ct)$, $U = [0,1]$

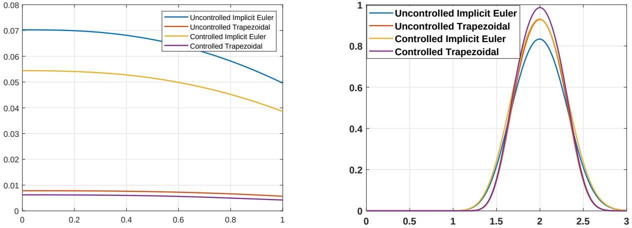

Figure: Comparison of the cost functionals (left) and the solutions at final time (right).

Δt	Nodes	CPU	$Error_2$	Order
0.1	506	0.11s	2.8e-2	
0.05	3311	0.7s	8e-3	1.84

Table: Trapezoidal rule with 2×2 discrete controls and $\varepsilon_{\mathcal{T}} = \Delta t^3$

How does the cardinality change?

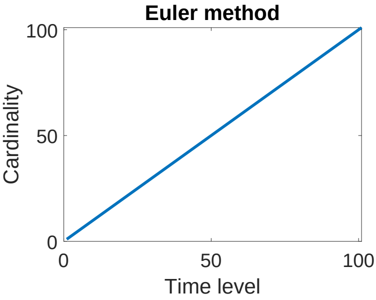

Figure: Implicit Euler: $|\mathcal{T}| = O(N^2)$, Trapezoidal rule: $|\mathcal{T}| = O(N^3)$

Method	Δt	Controls	Nodes	CPU	Error
Implicit Euler	2.5e-3	2	80982	9s	9e-3
Trapezoidal	5e-2	2×2	3311	0.7s	8e-3

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Problem Setting

Semidiscretized PDE

$$\begin{cases} \mathbf{M}\,\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{f}(t, \mathbf{y}(t)), \quad t \in (0, T], \\ \\ \mathbf{y}(0) = \mathbf{y_0}, \end{cases}$$

Assumptions

- $\mathbf{y_0} \in \mathbb{R}^n$ is a given initial data,
- $\mathbf{M}, \mathbf{A} \in \mathbb{R}^{n \times n}$ given matrices,
- $\mathbf{f} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ a continuous function in both arguments and locally Lipschitz-type with respect to the second variable

WARNING: High dimensional problems are computationally expensive.

Proper Orthogonal Decomposition and SVD

Given snapshots $(y(t_0), \ldots, y(t_n)) \in \mathbb{R}^m$

We look for an orthonormal basis $\{\psi_i\}_{i=1}^{\ell}$ in $\mathbb{R}^m$ with $\ell \ll \min\{n, m\}$ s.t.

$$J(\psi_1, \ldots, \psi_\ell) = \sum_{j=1}^{n} \alpha_j \left\| y_j - \sum_{i=1}^{\ell} \langle y_j, \psi_i \rangle \psi_i \right\|^2 = \sum_{i=\ell+1}^{d} \sigma_i^2$$

reaches a minimum where $\{\alpha_j\}_{j=1}^{n} \in \mathbb{R}^+$.

$$\min J(\psi_1, \ldots, \psi_\ell) \quad s.t. \langle \psi_i, \psi_j \rangle = \delta_{ij}$$

Singular Value Decomposition: $Y = \Psi \Sigma V^T$.

For $\ell \in \{1, \ldots, d = rank(Y)\}$, $\{\psi_i\}_{i=1}^{\ell}$ are called **POD basis** of rank ℓ.

ERROR INDICATOR: $\mathcal{E}(\ell) = \dfrac{\sum_{i=1}^{\ell} \sigma_i^2}{\sum_{i=1}^{d} \sigma_i^2}$ with σ_i singular values of the SVD.

Reduced Order Modelling Control Problem

MOR ansatz

$$y(t) \approx \Psi y^\ell(t) \quad \Psi^T \Psi = I, \qquad \Psi \in \mathbb{R}^{n\times \ell}$$

Compact Notations

$$x^\ell := \Psi^T x, \quad y^\ell(t) := \Psi^T y(t), \quad g^\ell(y^\ell(t)) := \Psi^T g(\Psi y^\ell(t)),$$
$$f^\ell(y^\ell(t), u(t), t) := \Psi^T f(\Psi y^\ell(t), u(t), t), L^\ell(y^\ell(t), u(t)) := L(\Psi y^\ell(t), u(t)).$$

$$\begin{cases} \dot{y}^\ell(t) = f^\ell(y^\ell(t), u(t)), \quad t \in [0, T], \\ y^\ell(0) = x^\ell \in \mathbb{R}^\ell. \end{cases}$$

The cost functional is:

$$J^\ell_{x^\ell}(u) = \int_0^T L^\ell(y^\ell(t), u(t), t) e^{-\lambda t}\, dt + g^\ell(y^\ell(T))$$

Reduced Order Modelling Control Problem

Reduced Value Function

$$v^\ell(x^\ell, t) = \inf_{u \in \mathcal{U}_{ad}} J^\ell_{x^\ell, t}(u)$$

Reduced HJB equation

$$-\frac{\partial v^\ell(x^\ell, t)}{\partial t} + \lambda v^\ell(x^\ell, t) + \sup_{u \in U}\{-\nabla_{x^\ell} v^\ell(x^\ell, t) \cdot f^\ell(x^\ell, u, t) - L^\ell(x^\ell, u, t)\} = 0$$

Feedback Control

$$u^{\ell,*}(x^\ell, t) = \arg\min_{u \in U}\{f^\ell(x^\ell, u, t) \cdot \nabla_{x^\ell} v^\ell(x^\ell, t) + L^\ell(x^\ell, u, t)\}$$

HJB-POD on a tree structure

Computation of the snapshots

- POD for optimal control problems presents a major bottleneck: the choice of the control inputs to compute the snapshots.
- We store the tree in the snapshots matrix $Y = \mathcal{T} = \cup_{n=0}^{N} \mathcal{T}^n$ for a chosen Δt and discrete control set U.

Computation of the basis functions

- We solve

$$\min_{\psi_1,\ldots,\psi_\ell \in \mathbb{R}^d} \sum_{j=1}^{N} \sum_{\underline{u}_j \subset U^j} \left| y(t_j, \underline{u}_j) - \sum_{i=1}^{\ell} \langle y(t_j, \underline{u}_j), \psi_i \rangle \psi_i \right|^2, \quad \langle \psi_i, \psi_j \rangle = \delta_{ij},$$

- No restrictions on the choice of the number of basis ℓ, since we will solve the HJB equation on a tree structure.
- We choose ℓ such that $\mathcal{E}(\ell) \approx 0.999$,

HJB-POD on a tree structure

Construction of the reduced tree

- Construction of a new (projected) tree $\mathcal{T}^\ell$ with a smaller Δt and/or a finer control space with respect to the snapshots set.
- The **first level** of the tree is contains the projection of the initial condition, i.e. $\mathcal{T}^{0,\ell} = \Psi^T x$.
- Again we have

$$\mathcal{T}^{n,\ell} = \{\zeta_i^{n-1,\ell} + \Delta t\, f^\ell(\zeta_i^{n-1,\ell}, u_j, t_{n-1})\}_{j=1}^M \quad i = 1, \ldots, M^{n-1},$$

  where the reduced nonlinear term f^ℓ can be done via POD or POD-DEIM.
- The procedure follows the full dimensional case, but with the projected dynamics.

HJB-POD on a tree structure

Approximation of the reduced value function

The numerical reduced value function $V^\ell(x^\ell, t)$ will be computed on the tree nodes in space as

$$V^\ell(x^\ell, t_n) = V^{n,\ell}(x^\ell), \quad \forall x^\ell \in \mathcal{T}^{n,\ell}.$$

The computation of the reduced value function follows directly from the DPP:

$$\begin{cases} V^{n,\ell}(\zeta_i^{n,\ell}) = \min\limits_{u\in U}\{V^{n+1,\ell}(\zeta_i^{n,\ell} + \Delta t\, f^\ell(\zeta_i^{n,\ell}, u, t_n)) + \Delta t\, L^\ell(\zeta_i^{n,\ell}, u, t_n)\}, \\ \qquad\qquad \zeta_i^{n,\ell} \in \mathcal{T}^{n,\ell}, n = \overline{N}-1, \dots, 0, \\ V^{\overline{N},\ell}(\zeta_i^{\overline{N},\ell}) = g^\ell(\zeta_i^{\overline{N},\ell}), \qquad \zeta_i^{\overline{N},\ell} \in \mathcal{T}^{\overline{N},\ell}. \end{cases}$$

HJB-POD on a tree structure

Nonlinear dynamics

Since $\Psi^T f(\Psi y^\ell, t)$ is computationally expensive ($\Psi y^\ell \in \mathbb{R}^d$), we apply Discrete Empirical Interpolation Method to obtain $f^{DEIM}(y^{DEIM}, t)$ which is independent of the original dimension.
This method is based on a further SVD of the matrix $\{f(y(t_i), t_i)\}_i$.

Computation of the feedback control

- When we compute the reduced value function, we store the control indices corresponding to the argmin of the hamiltonian and then we follow the path of tree,
- We can consider a postprocessing procedures with a control set $\tilde{U} \supset U$, involving interpolation on scattered data.
- If the dynamics is linear in $u \in \mathbb{R}$, we can consider 1D interpolation.

HJB-POD on a tree structure

Theorem (Alla, S., 2019)

Let f, L and g be Lipschitz continuous, bounded. Moreover let L and g be semiconcave and $f \in C^1$, then there exists a constant $C(T)$ such that

$$\sup_{s\in[0,T]} |v(x,s) - V^\ell(\Psi^T x, s)| \leq C(T) \left(\left(\sum_{i\geq l+1} \sigma_i^2 \right)^{1/2} + \Delta t \right),$$

where $\{\sigma_i\}_i$ are the singular values of the snapshots matrix.

Pros about TSA-POD

- We build the snapshots set upon all the trajectories that appear in the tree, avoiding the selection of a forecast for the control inputs which is always not trivial for model reduction.
- The application of POD also allows an efficient pruning since it reduces the dimension of the problem.
- We avoid to define the numerical domain for the projected problem, which is a difficult task since we lose the physical meaning of the reduced coordinates.
- We are not restricted to consider a very low dimensional reduced space.

Test 1: Heat equation

$$\begin{cases} \partial_t y(x,t) = \sigma y_{xx}(x,t) + y_0(x)u(t) & (x,t) \in \Omega \times [0,T], \\ y(x,t) = 0 & (x,t) \in \partial\Omega \times [0,T], \\ y(x,0) = y_0(x) & x \in \Omega, \end{cases}$$

$U = [-1, 0]$, $\sigma = 0.15$, $T = 1$ and $\Omega = [0, 1]$.

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2 discrete controls and $\Delta t = 0.1$.
We choose $\ell = 2$ basis with projection error $Err = 7.e - 4$.

Test 1: Heat equation

Δt	Nodes	Pruned/Full	CPU	Err_2	Err_∞	$Order_2$	$Order_\infty$
0.1	134	4.3e-10	0.1s	0.244	0.220		
0.05	825	1.0e-19	0.56s	0.102	9.4e-2	1.25	1.22
0.025	11524	2.1e-39	8.74s	3.1e-2	3.0e-2	1.73	1.67
0.0125	194426	7.8e-80	151s	1.0e-2	8.2e-3	1.60	1.85

Table: Test 1: Error analysis for TSA-POD method with $\varepsilon_{\mathcal{T}} = \Delta t^2$, 11 discrete controls and 2 POD basis.

Δt	Nodes	Pruned/Full	CPU	Err_2	Err_∞	$Order_2$	$Order_\infty$
0.1	134	4.7e-09	0.14s	0.279	0.241		
0.05	863	1.2e-18	0.65s	0.144	0.118	0.95	1.03
0.025	15453	3.1e-38	12.88s	5.5e-2	5.3e-2	1.40	1.17
0.0125	849717	3.8e-78	1.1e3s	1.6e-2	1.6e-2	1.77	1.42

Table: Test 1: Error analysis for TSA with $\varepsilon_{\mathcal{T}} = \Delta t^2$ and 11 discrete controls.

Test 1: Feedback reconstruction

- First, we apply TSA-POD with 2 basis e 3 discrete controls.
- Then, we consider the feedback law

$$u_*^{n,\ell} := \arg\min_{u\in U} \left\{ V^{n+1,\ell}(\zeta_*^{n,\ell} + \Delta t\, f^\ell(\zeta_*^{n,\ell}, u, t_n)) + \Delta t\, L^\ell(\zeta_*^{n,\ell}, u, t_n) \right\},$$

Scattered Interpolation

We fix $\widetilde{U}$ with 100 controls and we apply scattered interpolation in dimension ℓ.

1D Interpolation

Since the dynamics is linear in $u \in \mathbb{R}$, the sons of a node lie on a segment and we consider 1D interpolation (*e.g.* quadratic).

Test 1: Feedback reconstruction

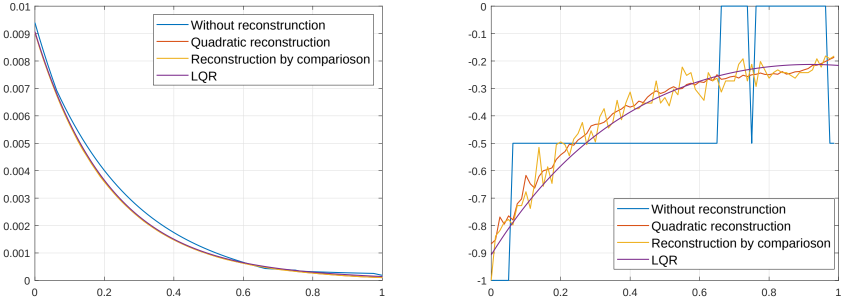

Figure: Test 1: Cost functional (top) and optimal control (bottom) with different techniques for the feedback reconstruction.

Test 2: 2D Reaction diffusion equation

$$\begin{cases} \partial_t y(x,t) = \sigma \Delta y(x,t) + \mu \left(y^2(x,t) - y^3(x,t) \right) + y_0(x) u(t) \\ \partial_n y(x,t) = 0 \\ y(x,0) = y_0(x) \end{cases}$$

$$J_{y_0,t}(u) = \int_t^T \left(\int_\Omega |y(x,s)|^2 dx + \frac{1}{100} |u(s)^2| \right) ds + \int_\Omega |y(x,T)|^2 dx$$

POD-DEIM resolution

$T = 1$, $\sigma = 0.1$, $\mu = 5$, and $N_x = 961$.
6 POD basis to obtain a projection ratio equal to 0.9999.

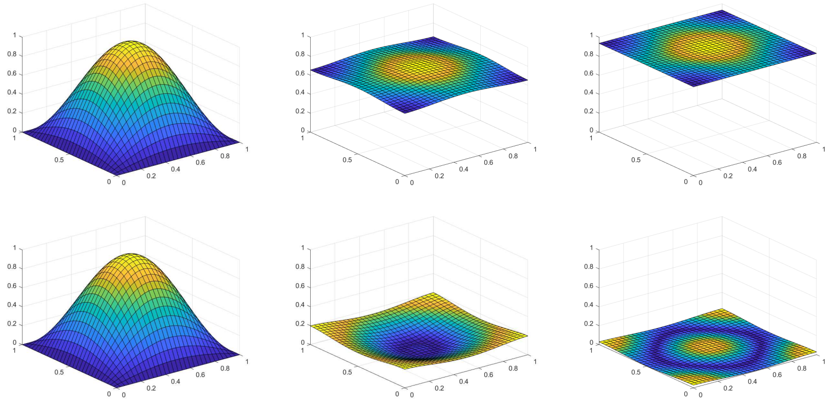

Figure: Uncontrolled solution (top) and controlled solution with full tree (bottom) for time $t = \{0, 0.5, 1\}$

Test 2: 2D Reaction diffusion equation

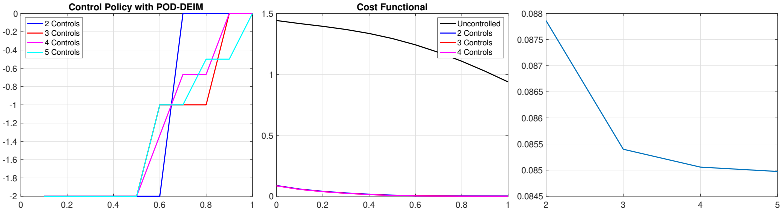

Figure: Test 1: Optimal policy (left), cost functional (middle) and $J_{y_0,0}$ (right) for U_n with $n = \{2, 3, 4, 5\}$.

	U_2	U_3	U_4	U_5
TSA-Full	6s	241s	3845s	$>$ 4 days
TSA-POD	0.5s	20s	432s	1e4s

Table: CPU time of the TSA and the TSA-POD with a different number of controls

Thank you for your attention

1. A. Alla, M. Falcone, L. Saluzzi, *An efficient DP algorithm on a tree-structure for finite horizon optimal control problems*, SISC, 2019
2. A. Alla, M. Falcone, L. Saluzzi, *High-order Approximation of the Finite Horizon Control Problem via a Tree Structure Algorithm*, IFAC CPDE 2019
3. A. Alla, L. Saluzzi, *A HJB-POD approach for the control of nonlinear PDEs on a tree structure*, APNUM, 2019
4. M. Falcone, R. Ferretti, *Discrete time high-order schemes for viscosity solutions of Hamilton-Jacobi-Bellman equations*, Numerische Mathematik, 1994
5. L. Saluzzi, A. Alla, M. Falcone, *Error estimates for a tree structure algorithm on dynamic programming equations*, submitted, 2019