TSA Part 2: The Revenge
A HJB-POD approach for the control of nonlinear PDEs on a tree structure

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joint work with A. Alla and M. Falcone

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Outline

1. Extension to high-order
   - High-order TSA
   - Numerical test

2. Control of nonlinear PDEs by TSA
   - Model Order Reduction Methods
   - HJB-POD on a tree structure
   - Numerical tests
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Extension to high-order (Falcone, Ferretti, ’94)

We introduce a convergent one-step approximation

\[
\begin{align*}
    y^{n+1} &= y^n + \Delta t \Phi(y^n, U_n, t_n, \Delta t), \\
    y^0 &= x,
\end{align*}
\]

where the admissible control matrix

\[U_n \in U_{\Delta t} \subset U \times U \ldots \times U \in \mathbb{R}^{M \times (q+1)},\text{ with } U \subset \mathbb{R}^M.\]

We assume that the function \( \Phi \) is consistent

\[
\lim_{\Delta t \to 0} \Phi(x, \overline{u}, t, \Delta t) = f(x, \overline{u}, t),
\]

where \( \overline{u} = (\overline{u}, \ldots, \overline{u}) \in U \) for \( \overline{u} \in U \) and Lipschitz continuous:

\[
|\Phi(x, U, t, \Delta t) - \Phi(y, U, t, \Delta t)| \leq L_\Phi |x - y|.
\]

Under these assumptions the scheme is convergent.
Extension to high-order schemes

Then, we consider the approximation of the cost functional

\[ J_{x,t_n}^{\Delta t}(\{U_m\}) = \Delta t \sum_{m=n}^{N-1} \sum_{i=0}^{q} w_i L(y^{m+\tau_i}, u_i^m, t_m) + g(y^N), \]

where \( \tau_i \) and \( w_i \) are the nodes and weights of the quadrature formula. Finally we define the numerical value function as

\[ V(t, x) = \inf_{\{U_n\}} J_{x,t}^{\Delta t}(\{U_n\}) \]

**Proposition (Discrete DPP)**

\[ V(t, x) = \inf_{\{U_m\}} \left\{ \Delta t \sum_{i=0}^{q} w_i L(y^{n+\tau_i}, u_i^n, t_{n+\tau_i}) + V(t_{n+1}, y^{n+1}) \right\} \]
Pruning for high-order scheme

We can again define the pruned trajectory

\[ \eta_{j}^{n+1} = \eta^{n} + \Delta t \Phi(\eta^{n}, U_{n}, t_{n}, \Delta t) + \mathcal{E}_{\epsilon_{T}}(\eta^{n} + \Delta t \Phi(\eta^{n}, U_{n}, t_{n}, \Delta t), \{\eta_{i}^{n+1}\}_{i}) \]
Pruning for high-order scheme

We can again define the pruned trajectory

\[
\eta_{n+1}^{j} = \eta^{n} + \Delta t \Phi(\eta^{n}, U_{n}, t_{n}, \Delta t) + \mathcal{E}_{\varepsilon_{T}}(\eta^{n} + \Delta t \Phi(\eta^{n}, U_{n}, t_{n}, \Delta t), \{\eta_{i}^{n+1}\}_{i})
\]

Proposition

Given a one-step approximation \(\{y_{n}^{n}\}_{n}\) and its perturbation \(\{\eta_{n}^{n}\}_{n}\), then

\[
|y^{n} - \eta^{n}| \leq \varepsilon_{T} \frac{t_{n} - t}{\Delta t} e^{L_{\Phi}(t_{n} - t)}.
\]

To guarantee \(p\)-th order convergence, the tolerance must be chosen such that

\[
\varepsilon_{T} \leq C(\Delta t)^{p+1}.
\]
**Test 1: Bilinear control for Advection Equation**

\[
\begin{aligned}
&\begin{cases}
y_t + cy_x = yu(t) \quad (x, t) \in \Omega \times [0, T], \\
y(x, t) = 0 \quad (x, t) \in \partial \Omega \times [0, T], \\
y(x, 0) = y_0(x) \quad x \in \Omega.
\end{cases}
\end{aligned}
\]

\[
J_{y_0, t}(u) = \int_t^T \left( \|y(s) - \tilde{y}(s)\|^2 + 0.01|u(s)|^2 \right) ds + \|y(T) - \tilde{y}(T)\|^2.
\]

**Semi-discrete problem (System dimension = 10^2)**

\[
\dot{y}(t) = Ay(t) + y(t)u(t),
\]

\[
\Delta x = 0.01, \quad \Omega = [0, 3] \text{ and } c = 1.5.
\]
Case 1: $\tilde{y} = 0$, $U = [-4, 0]$
Case 2: $\tilde{y}(x, t) = y_0(x - ct)$, $U = [0, 1]$

![Graph showing comparison of cost functionals and solutions.](image)

**Figure:** Comparison of the cost functionals (left) and the solutions at final time (right).

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>Nodes</th>
<th>CPU</th>
<th>$\text{Error}_2$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>506</td>
<td>0.11s</td>
<td>2.8e-2</td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>3311</td>
<td>0.7s</td>
<td>8e-3</td>
<td>1.84</td>
</tr>
</tbody>
</table>

**Table:** Trapezoidal rule with $2 \times 2$ discrete controls and $\varepsilon_T = \Delta t^3$
How does the cardinality change?

**Figure:** Implicit Euler: $|T| = O(N^2)$, Trapezoidal rule: $|T| = O(N^3)$

<table>
<thead>
<tr>
<th>Method</th>
<th>$\Delta t$</th>
<th>Controls</th>
<th>Nodes</th>
<th>CPU</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implicit Euler</td>
<td>2.5e-3</td>
<td>2</td>
<td>80982</td>
<td>9s</td>
<td>9e-3</td>
</tr>
<tr>
<td>Trapezoidal</td>
<td>5e-2</td>
<td>2 × 2</td>
<td>3311</td>
<td>0.7s</td>
<td>8e-3</td>
</tr>
</tbody>
</table>
Outline

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Problem Setting

Semidiscretized PDE

\[ \begin{aligned}
M \dot{y}(t) &= A y(t) + f(t, y(t)), \quad t \in (0, T], \\
y(0) &= y_0,
\end{aligned} \]

Assumptions

- \( y_0 \in \mathbb{R}^n \) is a given initial data,
- \( M, A \in \mathbb{R}^{n \times n} \) given matrices,
- \( f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) a continuous function in both arguments and locally Lipschitz-type with respect to the second variable

WARNING: High dimensional problems are computationally expensive.
Proper Orthogonal Decomposition and SVD

Given snapshots \((y(t_0), \ldots, y(t_n)) \in \mathbb{R}^m\)

We look for an orthonormal basis \(\{\psi_i\}_{i=1}^\ell\) in \(\mathbb{R}^m\) with \(\ell \ll \min\{n, m\}\) s.t.

\[
J(\psi_1, \ldots, \psi_\ell) = \sum_{j=1}^n \alpha_j \|y_j - \sum_{i=1}^\ell \langle y_j, \psi_i \rangle \psi_i \|^2 = \sum_{i=\ell+1}^d \sigma_i^2
\]

reaches a minimum where \(\{\alpha_j\}_{j=1}^n \in \mathbb{R}^+\).

\[
\min J(\psi_1, \ldots, \psi_\ell) \quad \text{s.t.} \langle \psi_i, \psi_j \rangle = \delta_{ij}
\]

Singular Value Decomposition: \(Y = \Psi \Sigma V^T\).

For \(\ell \in \{1, \ldots, d = \text{rank}(Y)\}\), \(\{\psi_i\}_{i=1}^\ell\) are called POD basis of rank \(\ell\).

**ERROR INDICATOR:** \(E(\ell) = \frac{\sum_{i=1}^\ell \sigma_i^2}{\sum_{i=1}^d \sigma_i^2}\) with \(\sigma_i\) singular values of the SVD.
Reduced Order Modelling Control Problem

MOR ansatz

\[ y(t) \approx \Psi y^\ell(t) \quad \Psi^T \Psi = I, \quad \Psi \in \mathbb{R}^{n \times \ell} \]

Compact Notations

\[ x^\ell := \Psi^T x, \quad y^\ell(t) := \Psi^T y(t), \quad g^\ell(y^\ell(t)) := \Psi^T g(\Psi y^\ell(t)), \]
\[ f^\ell(y^\ell(t), u(t), t) := \Psi^T f(\Psi y^\ell(t), u(t), t), \quad L^\ell(y^\ell(t), u(t)) := L(\Psi y^\ell(t), u(t)). \]

\[
\begin{cases} 
  \dot{y}^\ell(t) = f^\ell(y^\ell(t), u(t)), & t \in [0, T], \\
  y^\ell(0) = x^\ell \in \mathbb{R}^\ell.
\end{cases}
\]

The cost functional is:

\[ J^\ell_x(u) = \int_0^T L^\ell(y^\ell(t), u(t), t) e^{-\lambda t} \, dt + g^\ell(y^\ell(T)) \]
Reduced Order Modelling Control Problem

Reduced Value Function

\[ v^\ell(x^\ell, t) = \inf_{u \in \mathcal{U}_{ad}} J^\ell_{x^\ell, t}(u) \]

Reduced HJB equation

\[ -\frac{\partial v^\ell(x^\ell, t)}{\partial t} + \lambda v^\ell(x^\ell, t) + \sup_{u \in \mathcal{U}} \left\{ -\nabla_x v^\ell(x^\ell, t) \cdot f^\ell(x^\ell, u, t) - L^\ell(x^\ell, u, t) \right\} = 0 \]

Feedback Control

\[ u^{\ell,*}(x^\ell, t) = \arg \min_{u \in \mathcal{U}} \left\{ f^\ell(x^\ell, u, t) \cdot \nabla_x v^\ell(x^\ell, t) + L^\ell(x^\ell, u, t) \right\} \]
HJB-POD on a tree structure

**Computation of the snapshots**
- POD for optimal control problems presents a major bottleneck: the choice of the control inputs to compute the snapshots.
- We store the tree in the snapshots matrix \( Y = T = \bigcup_{n=0}^{N} T^n \) for a chosen \( \Delta t \) and discrete control set \( U \).

**Computation of the basis functions**
- We solve

\[
\min_{\psi_1, \ldots, \psi_\ell \in \mathbb{R}^d} \sum_{j=1}^{N} \sum_{u_j \subset U} \left| y(t_j, u_j) - \sum_{i=1}^{\ell} \langle y(t_j, u_j), \psi_i \rangle \psi_i \right|^2, \quad \langle \psi_i, \psi_j \rangle = \delta_{ij},
\]

- No restrictions on the choice of the number of basis \( \ell \), since we will solve the HJB equation on a tree structure.
- We choose \( \ell \) such that \( \mathcal{E}(\ell) \approx 0.999 \),
HJB-POD on a tree structure

Construction of the reduced tree

- Construction of a new (projected) tree $T^\ell$ with a smaller $\Delta t$ and/or a finer control space with respect to the snapshots set.
- The first level of the tree is contains the projection of the initial condition, i.e. $T^{0,\ell} = \Psi^T x$.
- Again we have

$$T^{n,\ell} = \left\{ \zeta_i^{n-1,\ell} + \Delta t f^\ell(\zeta_i^{n-1,\ell}, u_j, t_{n-1}) \right\}_{j=1}^M, \quad i = 1, \ldots, M^{n-1},$$

where the reduced nonlinear term $f^\ell$ can be done via POD or POD-DEIM.
- The procedure follows the full dimensional case, but with the projected dynamics.
HJB-POD on a tree structure

Approximation of the reduced value function

The numerical reduced value function $V^\ell(x^\ell, t)$ will be computed on the tree nodes in space as

$$V^\ell(x^\ell, t_n) = V^{n,\ell}(x^\ell), \quad \forall x^\ell \in T^{n,\ell}.$$ 

The computation of the reduced value function follows directly from the DPP:

$$V^{n,\ell}(\zeta^{n,\ell}_i) = \min_{u \in U} \{ V^{n+1,\ell}(\zeta^{n,\ell}_{i+1} + \Delta t f^{\ell}(\zeta^{n,\ell}_i, u, t_n)) + \Delta t L^{\ell}(\zeta^{n,\ell}_i, u, t_n) \},$$

$$\zeta^{n,\ell}_i \in T^{n,\ell}, n = N - 1, \ldots, 0,$$

$$V^{N,\ell}(\zeta^{N,\ell}_i) = g^{\ell}(\zeta^{N,\ell}_i), \quad \zeta^{N,\ell}_i \in T^{N,\ell}.$$
Nonlinear dynamics

Since $\Psi^T f(\Psi y^\ell, t)$ is computationally expensive ($\Psi y^\ell \in \mathbb{R}^d$), we apply Discrete Empirical Interpolation Method to obtain $f^{DEIM}(y^{DEIM}, t)$ which is independent of the original dimension. This method is based on a further SVD of the matrix $\{f(y(t_i), t_i)\}_i$.

Computation of the feedback control

- When we compute the reduced value function, we store the control indices corresponding to the argmin of the hamiltonian and then we follow the path of tree,

- We can consider a postprocessing procedures with a control set $\tilde{U} \supset U$, involving interpolation on scattered data.

- If the dynamics is linear in $u \in \mathbb{R}$, we can consider 1D interpolation.
Theorem (Alla, S., 2019)

Let $f$, $L$ and $g$ be Lipschitz continuous, bounded. Moreover let $L$ and $g$ be semiconcave and $f \in C^1$, then there exists a constant $C(T)$ such that

$$\sup_{s \in [0,T]} |v(x, s) - V^\ell(\Psi^T x, s)| \leq C(T) \left( \left( \sum_{i \geq l+1} \sigma_i^2 \right)^{1/2} + \Delta t \right),$$

where $\{\sigma_i\}_i$ are the singular values of the snapshots matrix.
Pros about TSA-POD

- We build the snapshots set upon all the trajectories that appear in the tree, avoiding the selection of a forecast for the control inputs which is always not trivial for model reduction.
- The application of POD also allows an efficient pruning since it reduces the dimension of the problem.
- We avoid to define the numerical domain for the projected problem, which is a difficult task since we lose the physical meaning of the reduced coordinates.
- We are not restricted to consider a very low dimensional reduced space.
Test 1: Heat equation

\[
\begin{aligned}
\partial_t y(x, t) &= \sigma y_{xx}(x, t) + y_0(x)u(t) \quad (x, t) \in \Omega \times [0, T], \\
y(x, t) &= 0 \quad (x, t) \in \partial\Omega \times [0, T], \\
y(x, 0) &= y_0(x) \quad x \in \Omega,
\end{aligned}
\]

\[
U = [-1, 0], \; \sigma = 0.15, \; T = 1 \text{ and } \Omega = [0, 1].
\]

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2 discrete controls and \(\Delta t = 0.1\).
We choose \(\ell = 2\) basis with projection error \(Err = 7 \cdot e^{-4}\).
## Test 1: Heat equation

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>Nodes</th>
<th>Pruned/Full</th>
<th>CPU</th>
<th>$Err_2$</th>
<th>$Err_\infty$</th>
<th>Order$_2$</th>
<th>Order$_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>134</td>
<td>4.3e-10</td>
<td>0.1s</td>
<td>0.244</td>
<td>0.220</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>825</td>
<td>1.0e-19</td>
<td>0.56s</td>
<td>0.102</td>
<td>9.4e-2</td>
<td>1.25</td>
<td>1.22</td>
</tr>
<tr>
<td>0.025</td>
<td>11524</td>
<td>2.1e-39</td>
<td>8.74s</td>
<td>3.1e-2</td>
<td>3.0e-2</td>
<td>1.73</td>
<td>1.67</td>
</tr>
<tr>
<td>0.0125</td>
<td>194426</td>
<td>7.8e-80</td>
<td><strong>151s</strong></td>
<td>1.0e-2</td>
<td>8.2e-3</td>
<td>1.60</td>
<td>1.85</td>
</tr>
</tbody>
</table>

### Table: Test 1: Error analysis for TSA-POD method with $\varepsilon_T = \Delta t^2$, 11 discrete controls and 2 POD basis.

---

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>Nodes</th>
<th>Pruned/Full</th>
<th>CPU</th>
<th>$Err_2$</th>
<th>$Err_\infty$</th>
<th>Order$_2$</th>
<th>Order$_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>134</td>
<td>4.7e-09</td>
<td>0.14s</td>
<td>0.279</td>
<td>0.241</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>863</td>
<td>1.2e-18</td>
<td>0.65s</td>
<td>0.144</td>
<td>0.118</td>
<td>0.95</td>
<td>1.03</td>
</tr>
<tr>
<td>0.025</td>
<td>15453</td>
<td>3.1e-38</td>
<td>12.88s</td>
<td>5.5e-2</td>
<td>5.3e-2</td>
<td>1.40</td>
<td>1.17</td>
</tr>
<tr>
<td>0.0125</td>
<td>849717</td>
<td>3.8e-78</td>
<td><strong>1.1e3s</strong></td>
<td>1.6e-2</td>
<td>1.6e-2</td>
<td>1.77</td>
<td>1.42</td>
</tr>
</tbody>
</table>

### Table: Test 1: Error analysis for TSA with $\varepsilon_T = \Delta t^2$ and 11 discrete controls.
First, we apply TSA-POD with 2 basis and 3 discrete controls.

Then, we consider the feedback law

$$u_{n,\ell}^* := \arg \min_{u \in U} \left\{ V^{n+1, \ell}(z_{n,\ell}^* + \Delta t f^\ell(z_{n,\ell}^*, u, t_n)) + \Delta t L^\ell(z_{n,\ell}^*, u, t_n) \right\},$$

Scattered Interpolation

We fix $\tilde{U}$ with 100 controls and we apply scattered interpolation in dimension $\ell$.

1D Interpolation

Since the dynamics is linear in $u \in \mathbb{R}$, the sons of a node lie on a segment and we consider 1D interpolation (e.g. quadratic).
Test 1: Feedback reconstruction

**Figure:** Test 1: Cost functional (top) and optimal control (bottom) with different techniques for the feedback reconstruction.
Test 2: 2D Reaction diffusion equation

\[
\begin{aligned}
\begin{cases}
\partial_t y(x, t) &= \sigma \Delta y(x, t) + \mu (y^2(x, t) - y^3(x, t)) + y_0(x)u(t) \\
\partial_n y(x, t) &= 0 \\
y(x, 0) &= y_0(x)
\end{cases}
\end{aligned}
\]

\[
J_{y_0, t}(u) = \int_t^T \left( \int_\Omega |y(x, s)|^2 dx + \frac{1}{100} |u(s)|^2 \right) ds + \int _\Omega |y(x, T)|^2 dx
\]

**POD-DEIM resolution**

\(T = 1, \sigma = 0.1, \mu = 5, \) and \(N_x = 961.\)

6 POD basis to obtain a projection ratio equal to 0.9999.
Figure: Uncontrolled solution (top) and controlled solution with full tree (bottom) for time $t = \{0, 0.5, 1\}$
Figure: Test 1: Optimal policy (left), cost functional (middle) and $J_{y_0,0}$ (right) for $U_n$ with $n = \{2, 3, 4, 5\}$.

<table>
<thead>
<tr>
<th></th>
<th>$U_2$</th>
<th>$U_3$</th>
<th>$U_4$</th>
<th>$U_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>TSA-Full</td>
<td>6s</td>
<td>241s</td>
<td>3845s</td>
<td>&gt; 4 days</td>
</tr>
<tr>
<td>TSA-POD</td>
<td>0.5s</td>
<td>20s</td>
<td>432s</td>
<td>1e4s</td>
</tr>
</tbody>
</table>

Table: CPU time of the TSA and the TSA-POD with a different number of controls
A. Alla, M. Falcone, L. Saluzzi, *An efficient DP algorithm on a tree-structure for finite horizon optimal control problems*, SISC, 2019

A. Alla, M. Falcone, L. Saluzzi, *High-order Approximation of the Finite Horizon Control Problem via a Tree Structure Algorithm*, IFAC CPDE 2019

A. Alla, L. Saluzzi, *A HJB-POD approach for the control of nonlinear PDEs on a tree structure*, APNUM, 2019

M. Falcone, R. Ferretti, *Discrete time high-order schemes for viscosity solutions of Hamilton-Jacobi-Bellman equations*, Numerische Mathematik, 1994

L. Saluzzi, A. Alla, M. Falcone, *Error estimates for a tree structure algorithm on dynamic programming equations*, submitted, 2019