

Enumerative Geometry of
Curves, Maps, and Sheaves

Part I: Cotangent lines

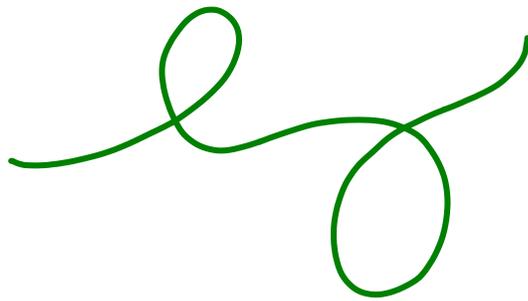
Rahul Pandharipande

ETH ZÜRICH

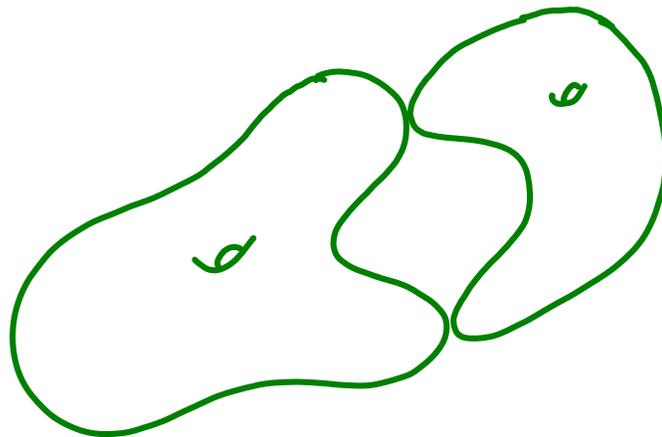
12 July 2021

(i) Deformations

Consider a nodal curve



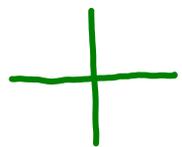
or perhaps



For us : nodal curves will be **Complex**

étale locally

algebraic varieties of dim 1



with at worst **nodal** singularities

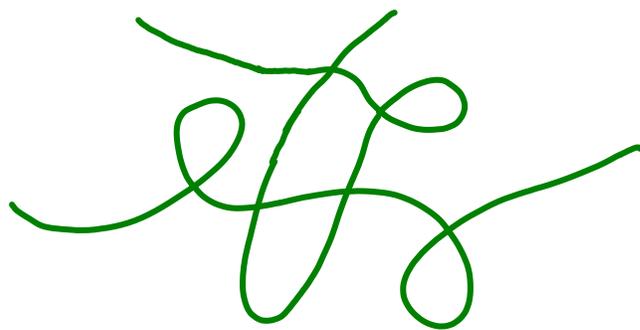
$$(xy=0) \subset \mathbb{A}^2$$

A nodal curve may be nonsingular

Usually a nodal curve will be assumed to be

- Connected
- Complete (= projective)

unless otherwise stated, but not irreducible.



We will start with some basics about the deformation theory of nodal curves.

Let C be a nodal curve.

If C is nonsingular of genus g , then

$$\text{Def}(C) = H^1(\text{Tan}_C)$$

$$H^2(\text{Tan}_C) = 0$$

$\dim = 3g - 3$
given by Riemann-Roch

so the deformations
are unobstructed

for $g \geq 2$

If C has nodes, the deformations are

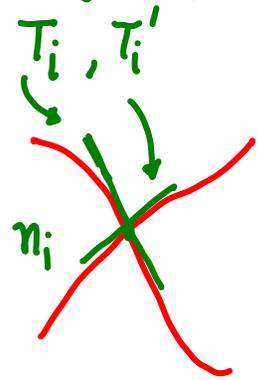
$$\text{Def}(C) = \text{Ext}^1(\Omega_C, \theta_C)$$

\nearrow
sheaf of Kähler differentials

again $\text{Ext}^2(\Omega_C, \theta_C) = 0$

so the deformations are again unobstructed

Tangent spaces



But nodal curves are more interesting

Let $n_1, n_2, \dots, n_g \in C$ be the nodes

By the local-to-global Ext sequence:

$$0 \rightarrow H^1(\mathcal{E}xt^0(\Omega_C, \mathcal{O}_C)) \rightarrow Ext^1(\Omega_C, \mathcal{O}_C) \rightarrow H^0(\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)) \rightarrow 0$$

Moreover $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)$ is a

sky scraper sheaf supported at the

nodes n_i . By a crucial local

calculation

smoothing
of the
nodes

$$\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C) = \bigoplus_{i=1}^g T_i \oplus T_i'$$

Tangent
Spaces
at n_i

So we have

$$\text{Def}(C) \xrightarrow{\mu} \bigoplus_{i=1}^{\delta} T_i \oplus T_i' \rightarrow 0$$

$\text{Ker}(\mu)$ = deformations which preserve the nodes.

(ii) Moduli

noncompact $\rightarrow M_g$ moduli space of genus g ($g \geq 2$)
nonsingular curves

compact $\rightarrow \bar{M}_g$ moduli space of genus g ($g \geq 2$)
Deligne-Mumford stable curves

C is nodal and connected, ω_C is ample

$\mathcal{M}_{g,n}$ moduli space of genus g
 n -pointed nonsingular curves

compact \rightarrow $\bar{\mathcal{M}}_{g,n}$ Deligne-Mumford stable curves
($2g-2+n > 0$)

C is nodal and connected,
 $p_1, p_2, \dots, p_n \in C$ distinct points in the nonsingular locus,
 $\omega_C(\sum p_i)$ is ample

$\bar{\mathcal{M}}_{g,n}$ is a nonsingular, irreducible

DM stack (or orbifold)

of $\dim_{\mathbb{C}} 3g-3+n$

Proven by Deligne-Mumford 1960's

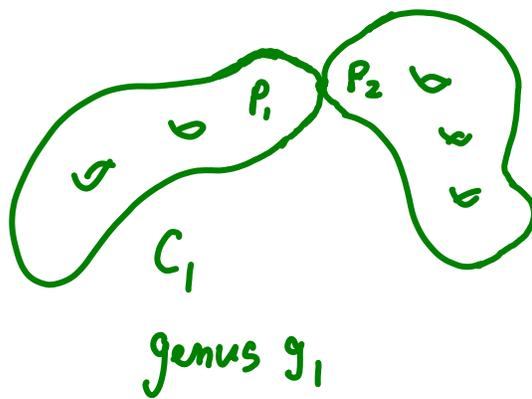
but goes back to Riemann 1860's

Tangent lines :

$$T_i \downarrow \bar{\mathcal{M}}_{g,n}$$

ϕ -line bundle determined by the tangent space at the i^{th} point

Example :



C_2 genus g_2

moduli of stable curves

$$\bar{\mathcal{M}}_{g_1,1} \times \bar{\mathcal{M}}_{g_2,1} \xrightarrow{\delta} \bar{\mathcal{M}}_{g_1+g_2}$$

$\uparrow P_1$ $\uparrow P_2$

Normal bundle to δ is $T_{P_1} \otimes T_{P_2}$

$$T_{P_1} \otimes T_{P_2} \rightarrow \bar{\mathcal{M}}_{g_1,1} \times \bar{\mathcal{M}}_{g_2,2}$$

(iii) Cotangent lines

$$\begin{array}{c} T_i^* \\ \downarrow \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

\mathcal{L} -line bundle
determined by
the cotangent space
at the i^{th} point

We will use the notation

$$\begin{array}{c} \mathcal{L}_i = T_i^* \\ \downarrow \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

First geometric result:

Proposition: \mathcal{L}_i is nef on $\overline{\mathcal{M}}_{g,n}$

Proof: Let \mathcal{C}

$$\begin{array}{c} \mathcal{C} \\ \pi \downarrow \nearrow \Delta_1, \dots, \Delta_n \\ B \end{array}$$

be a family of stable curves of genus g and n marked points over a 1-dimensional base B

for k large,

bundle of degree ≥ 0

$R^0 \pi_* (\omega_\pi(\Sigma_{\Delta_j})^k)$ is semipositive \nearrow

See "Projectivity of Complete moduli" by J. Kollár §4.7

By GRR $\Rightarrow \int c_1(\omega_\pi(\Sigma_{\Delta_j}))^2 \geq 0$

algebraic surface $\rightarrow \mathcal{C}$

Now

$$\int_{\mathcal{E}} c_1(\omega_{\pi}(\sum s_j)) \cdot [s_i] = \text{degree } \omega_{\pi}|_{s_i} + \text{degree } \mathcal{O}_{\mathcal{E}}(s_i)|_{s_i}$$

But on s_i , $\omega_{\pi} \cong \mathcal{O}_{\mathcal{E}}(s_i)^*$

so

$$\int_{\mathcal{E}} c_1(\omega_{\pi}(\sum s_j)) \cdot [s_i] = 0$$

Hence, by the Hodge index Theorem

for surfaces $\Rightarrow \int_{\mathcal{E}} [s_i]^2 \leq 0$

We use here

$$\mathcal{L}_i|_B \cong s_i^*(\mathcal{O}_{\mathcal{E}}(s_i)^*)$$



Why is 0 possible?
Hint: reducible curves.

$$\int_{\bar{\mathcal{M}}_{g,n}} [\beta] \cdot c_1(\mathcal{L}_i) \geq 0 \quad \square$$

Standard notation:

$$\psi_i = c_1(L_i) \in H^2(\bar{M}_{g,n})$$

We have the forgetful map

$$\pi: \bar{M}_{g,n+1} \rightarrow \bar{M}_{g,n} \quad \text{forget } (n+1)^{\text{st}} \text{ marking}$$

We define

$$\kappa_r = \pi_* \psi_{n+1}^{r+1} \in H^{2r}(\bar{M}_{g,n})$$

→
Kappa notation
goes back to
Mumford.

←
Convention of Arbarello-Cornalba

Another basic geometric fact:

$$H^2(M_{g,n}, \mathbb{Q}) \cong \text{Pic}(M_{g,n}, \mathbb{Q}) \quad \text{is generated by } \kappa_1, \psi_1, \dots, \psi_n$$

By nefness $\Rightarrow \int_{\bar{M}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \geq 0$

and the integral vanishes unless $\sum_{i=1}^n k_i = 3g-3+n$

Question: Can $\int_{\bar{M}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n}$ vanish if $\sum_{i=1}^n k_i = 3g-3+n$?

(iv) Witten's Conjecture

$$\langle \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n} = \int_{\bar{M}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n}$$

\uparrow
n is redundant

dim constraint: $\sum k_i = 3g-3+n$

$\langle \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n} = 0$ if constraint fails

Also 0
if
 $2g-2+n \leq 0$

by stability

To state Witten's Conjecture,
 we form a generating series

$$F_g(t_0, t_1, t_2, \dots)$$

$g \geq 0$
 genus

definition

$$\sum_{\{n_i\}} \prod_{i=1}^{\infty} \frac{t_i^{n_i}}{n_i!} \langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \dots \rangle_g$$

$$\langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \dots \rangle_g$$

Sum over all
 sequences of non-neg
 integers with only
 finitely many nonzero
 terms

$$\bar{M}_{g, \sum n_i} \left(\tau_1^0 \dots \tau_{n_0}^0 \tau_{n_0+1}^1 \dots \tau_{n_0+n_1}^1 \dots \right)$$

A more compact way :

$$\mathcal{F}_g = \sum_{n=0}^{\infty} \frac{\langle \phi^n \rangle_g}{n!}$$

Where $\phi = \sum_{i=0}^{\infty} t_i \tau_i$

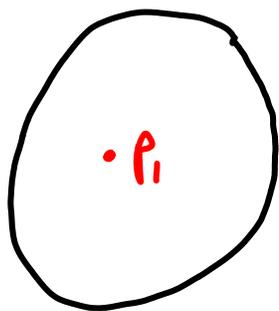
\mathcal{F}_g contains the data of all integrals of τ -classes over all moduli spaces $\bar{\mathcal{M}}_{g,n}$.

Called descendent integrals

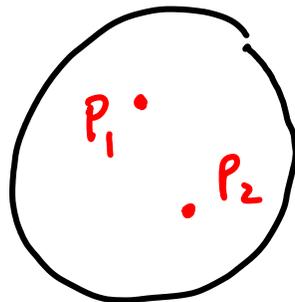
We can put all of the data for all genera together

$$F(\lambda, t) = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g$$

For F_0 , 1- and 2-point integrals are 0, since



and



are unstable

Notation for partial derivatives:

$$\langle\langle \tau_{k_1} \dots \tau_{k_n} \rangle\rangle = \frac{\partial}{\partial t_{k_1}} \frac{\partial}{\partial t_{k_2}} \dots \frac{\partial}{\partial t_{k_n}} \mathcal{F}$$

Witten's Conjecture / Kontsevich's Theorem (KdV):

For $n \geq 1$, we have:

$$(2n+1) \lambda^{-2} \langle\langle \tau_n \tau_0^2 \rangle\rangle = \\ \langle\langle \tau_{n-1} \tau_0 \rangle\rangle \langle\langle \tau_0^3 \rangle\rangle + 2 \langle\langle \tau_{n-1} \tau_0^2 \rangle\rangle \langle\langle \tau_0^2 \rangle\rangle \\ + \frac{1}{4} \langle\langle \tau_{n-1} \tau_0^4 \rangle\rangle$$

$$\text{Let } u = \frac{\partial^2 F}{\partial t_0^2}$$

Then the $n=1$ equation \Rightarrow

$$\frac{\partial u}{\partial t_1} = u \frac{\partial u}{\partial t_0} + \frac{1}{12} \frac{\partial^3 u}{\partial t_0^3} \quad (\text{set } \lambda=1)$$

Korteweg-de Vries equation

$t_0 \rightsquigarrow x$ space

$t_1 \rightsquigarrow t$ time

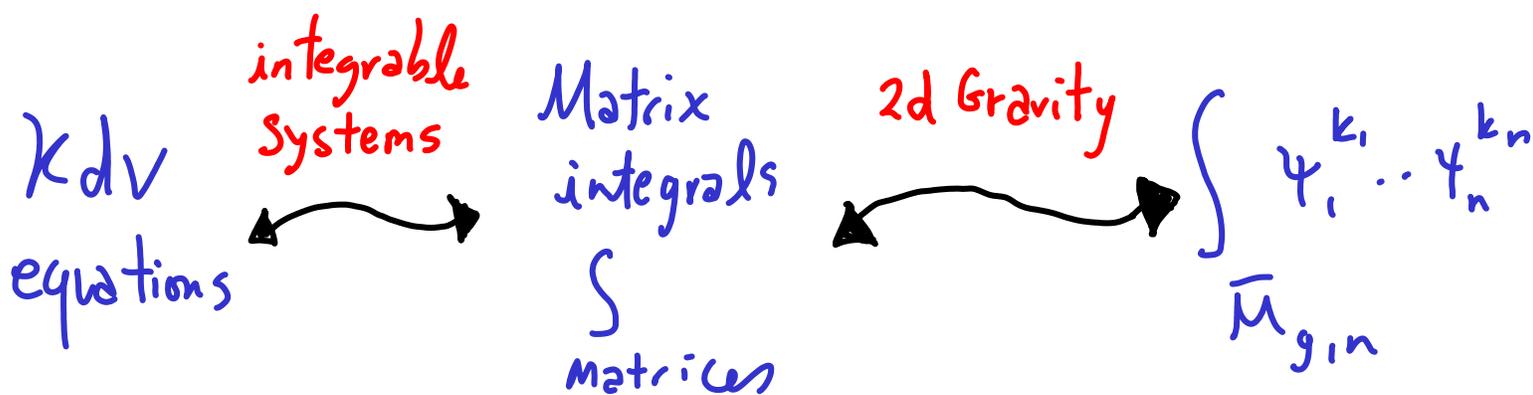
u height of wave

first written in the late 19th century
to model shallow water waves

What do water waves have to do with $\bar{M}_{g,n}$?

Long and interesting story

Witten, 2d quantum gravity and intersection theory on moduli space



Proven by Kontsevich 1992

Okounkov-P, GW theory, Hurwitz numbers and Matrix models 2001

2nd approach via Hurwitz by Kazarian-Lando 2006

Can we calculate $\langle \tau_1 \rangle_1 = \int_{\bar{M}_{1,1}} \tau_1$?

Take equation for $n=3$:

$$7 \langle \tau_3 \tau_0^2 \rangle_1 =$$

Set
 $\lambda=1$
and all
 $t_i=0$

$$\langle \tau_2 \tau_0 \rangle_1 \langle \tau_0^3 \rangle_0 + \frac{1}{4} \langle \tau_2 \tau_0^4 \rangle_0$$

After applying the String equation :

$$\langle \tau_3 \tau_0^2 \rangle_1 = \langle \tau_2 \tau_0 \rangle_1 = \langle \tau_1 \rangle_1$$

$$\langle \tau_2 \tau_0^4 \rangle_0 = \langle \tau_0^3 \rangle_0 = 1$$

$$\curvearrowright \int_{\bar{M}_{0,3}} 1 = 1$$

We find

$$7 \langle \tau_1 \rangle_1 = \langle \tau_1 \rangle_1 \cdot 1 + \frac{1}{4} \cdot 1$$

$$\text{So } 6 \langle \tau_1 \rangle_1 = \frac{1}{4}$$

$$\text{Finally } \langle \tau_1 \rangle_1 = \frac{1}{24}$$

We have used the String equation

$$\left\langle \tau_0 \prod_{i=1}^n \tau_{k_i} \right\rangle_{g, n+1} = \sum_{j=1}^n \left\langle \tau_{k_{j-1}} \prod_{i \neq j} \tau_{k_i} \right\rangle_{g, n}$$

Convention: $\tau_k = 0$ for $k < 0$.

Proof (String equation):

Consider the map

$$\pi: \bar{M}_{g,n+1} \rightarrow \bar{M}_{g,n}$$

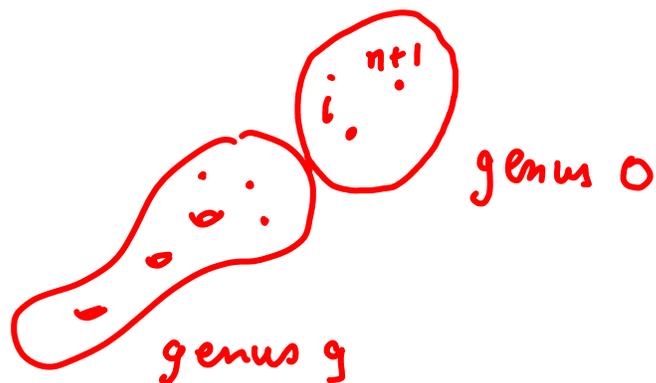
forgetting the last marking.

$$\left\langle \tau_0 \prod_{i=1}^n \tau_{k_i} \right\rangle_{g,n+1} = \int_{\bar{M}_{g,n+1}} \psi_1^{k_1} \cdots \psi_n^{k_n} \psi_{n+1}^0$$

We have the comparison equations

$$\psi_i = \pi^* \psi_i + \Delta_{i,n+1} \quad \text{for } 1 \leq i \leq n$$

Divisor of Curves



$$\int_{\bar{M}_{g,n+1}} \psi_1^{k_1} \cdots \psi_n^{k_n} \psi_{n+1}^0 = \int_{\bar{M}_{g,n+1}} \prod_{i=1}^n \psi_i \left(\pi^* \psi_i + \Delta_{i,n+1} \right)^{k_i-1}$$

$$= \int_{\bar{M}_{g,n+1}} \prod_{i=1}^n \psi_i \left(\pi^* \psi_i \right)^{k_i-1}$$

Since

$$\psi_i \Delta_{i,n+1} = 0$$

$$= \int_{\bar{M}_{g,n+1}} \prod_{i=1}^n \left(\pi^* \psi_i^{k_i} + \pi^* \psi_i^{k_i-1} \Delta_{i,n+1} \right)$$

$$\int_{\bar{M}_{g,n+1}} \prod_{i=1}^n \pi^* \psi_i^{k_i} = 0$$

$$= \sum_{j=1}^n \int_{\bar{M}_{g,n}} \psi_j^{k_j-1} \prod_{i \neq j} \psi_i^{k_i}$$

The last
expression
equals

$$\rightarrow \sum_{j=1}^n \langle \tau_{k_{j-1}} \prod_{i \neq j} \tau_{k_i} \rangle_{g,n} \quad \square$$

Exercise: Use the string equation

$$\text{to prove } \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{0,n} = \binom{n-3}{k_1, \dots, k_n}.$$

The initial condition $\langle \tau_0^3 \rangle_{0,3} = 1$
is required.

Exercise: Prove the dilaton equation

$$\langle \tau_1 \prod_{i=1}^n \tau_{k_i} \rangle_{g,n+1} = (2g-2+n) \langle \prod_{i=1}^n \tau_{k_i} \rangle_{g,n}.$$

The proof of the dilaton equation
uses the same geometry as the
proof of the string equation.

A Harder Exercise: Prove the genus 1

formula of Eftekhary - Setayesh:

elementary
symmetric
function

$$\langle T_{k_1} \dots T_{k_n} \rangle_{1,n} = \frac{1}{24} \binom{n}{k_1, \dots, k_n} \cdot \left[1 - \sum_{i=2}^n \frac{\sigma_i(k_1, \dots, k_n)}{i(i-1) \cdot \binom{n}{i}} \right]$$

A proof can be found in their paper "on the structure of the kappa ring"

What about removing T_k for $k > 1$?

It is possible using KdV + String

But easier with the Virasoro constraints.

(v) Virasoro Constraints

Recall
$$F(\lambda, t) = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g$$

Define
$$Z(\lambda, t) = \exp(F)$$

The string and dilaton equations
can be written using

$$L_{-1} = -\frac{\partial}{\partial t_0} + \frac{\lambda^{-2}}{2} t_0^2 + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i}$$

$$L_0 = -\frac{3}{2} \frac{\partial}{\partial t_1} + \sum_{i=0}^{\infty} \frac{2i+1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16}$$

as $L_{-1} \mathbb{Z} = 0$, $L_0 \mathbb{Z} = 0$

String

$$+ \langle \tau_0^3 \rangle_{0,3} = 1$$

Dilaton

$$+ \langle \tau_1 \rangle_{1,1} = \frac{1}{24}$$

The bracket is $[L_{-1}, L_0] = -L_{-1}$

Let $\mathcal{L}_n = -u^{n+1} \frac{\partial}{\partial u}$ $n \geq -1$

holomorphic differential operator
in the variable u .

Then, we have the Virasoro bracket

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m) \mathcal{L}_{n+m}$$

For $n > 0$, define

$$L_n = - \frac{(2n+3)!!}{2^{n+1}} \frac{\partial}{\partial t_{n+1}}$$

negative coefficient

$$+ \sum_{i=0}^{\infty} \frac{(2i+2n+1)!!}{(2i-1)!! 2^{n+1}} t_i \frac{\partial}{\partial t_{i+n}}$$

Positive

$$+ \frac{\lambda^2}{2} \sum_{i=0}^{n-1} \frac{(2i+1)!! (-2i+2n-1)!!}{2^{n+1}} \frac{\partial^2}{\partial t_i \partial t_{n-i}}$$

Then $[L_n, L_m] = (n-m) L_{n+m}$

Check algebraically

Virasoro Constraints: $L_n \mathbb{Z} = 0$ for $n \geq -1$

Corollary: $\langle T_{k_1} \dots T_{k_n} \rangle_{g,n} > 0$

whenever $\sum_{i=1}^n k_i = 3g - 3 + n$

Exercise: compute $\langle T_4 \rangle_{2,1} = \frac{1}{1152}$.

Proof (Virasoro Constraints): Two paths

Path I: KdV + String \Rightarrow Virasoro

Dijkgraaf - Verlinde - Verlinde 1991

Path II: Hyperbolic geometry
Mirzakhani's study of
volumes and geodesics 2007

How to use $L_n Z = 0$?

$$Z = \exp(F)$$

Can be interpreted as descendant series for moduli of disconnected curves

generating series of descendant integrals over $\overline{M}_{g,n}$, moduli of connected curves

However, we expand

$$L_n \frac{Z}{Z} = 0$$

$$\frac{\partial Z}{\partial t_i} / Z = \frac{\partial F}{\partial t_i}$$

So we obtain equations for F

$$\text{and } \frac{\partial}{\partial t_i} \frac{\partial Z}{\partial t_j} / Z = \frac{\partial F}{\partial t_i} \frac{\partial F}{\partial t_j} + \frac{\partial^2 F}{\partial t_i \partial t_j}$$

Enumerative Geometry of Curves, Maps, and Sheaves

Part II: Stable maps

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(i) Moduli of Stable maps

Let X be a **nonsingular** projective variety / \mathbb{C}

We will consider maps

$$f: C \rightarrow X$$

algebraic morphism \nearrow \nwarrow target

Complete connected nodal curve
of genus $g = 1 - \chi(\mathcal{O}_C)$

$$f_* [C] = \beta \in H_2(X, \mathbb{Z})$$

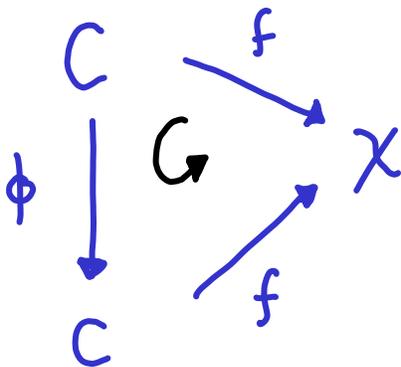
\nwarrow Curve class

$\bar{M}_g(X, \beta)$ is the moduli space of
Stable maps of genus g
curves to X representing β .

- $[f: C \rightarrow X] \in \bar{M}_g(X, \beta)$ is stable

if and only if $|\text{Aut}(f)| < \infty$.

- An automorphism of f is an automorphism of C which commutes with f :



$$\text{Aut}(f) \subset \text{Aut}(C)$$

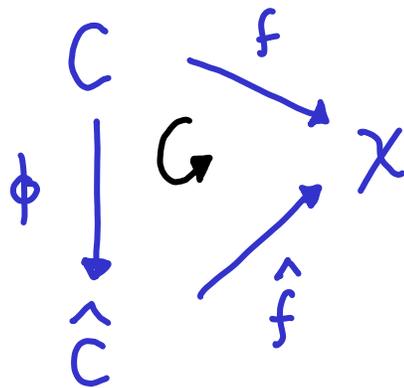
↖ if $|\text{Aut}(C)| < \infty$
 then $|\text{Aut}(f)| < \infty$
 and f is stable

When are two stable maps

$$[f: C \rightarrow X], [\hat{f}: \hat{C} \rightarrow X]$$

isomorphic? If and only if

$$\exists \phi: C \xrightarrow[\sim]{\text{isom}} \hat{C} \quad \text{which commutes with } f, \hat{f} :$$



parallel definitions, Aut and isom must respect the markings

$$\bar{M}_g(x, \beta) \quad \text{and} \quad \bar{M}_{g,n}(x, \beta)$$

are Deligne-Mumford stack, but

may be **reducible, non-reduced, and very singular.**

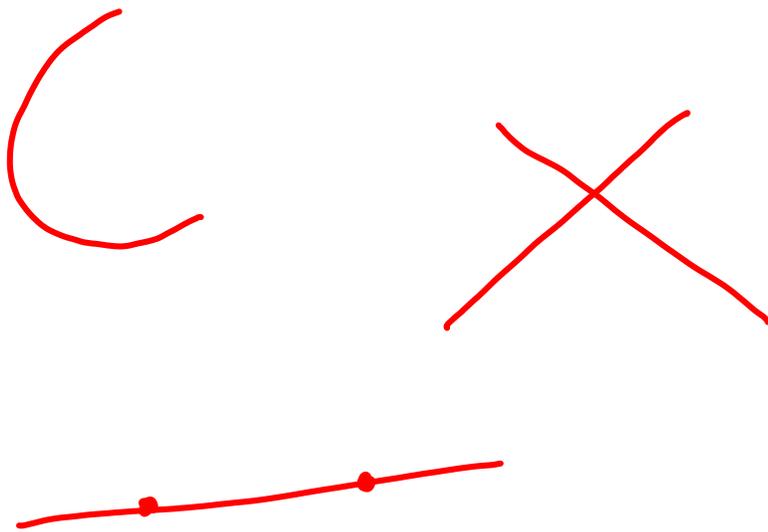
First examples:

• $\bar{M}_{g,n}(\chi, 0) = \bar{M}_{g,n}^{\chi} \chi$ for $2g-2+n > 0$

• $\bar{M}_{0,0}(\mathbb{P}^n, 1) = \text{Gr}(\mathbb{P}^1, \mathbb{P}^n)$

↑ class of
the line $L \in H_2(\mathbb{P}^n, \mathbb{Z})$

• $\bar{M}_{0,0}(\mathbb{P}^2, 2) =$ classical space of
complete conics



(ii) Obstruction theory

$\bar{M}_{g,n}(x, \beta)$ carries a Def-Obs theory with

$x(f^*T_x)$
↓

$$\text{vir dim } \bar{M}_{g,n}(x, \beta) = \int_{\beta} c_1(x) + \dim_{\mathbb{C}} x(1-g) + 3g - 3 + n$$

↑
dim of $\bar{M}_{g,n}$

The Def-Obs theory for a

fixed domain curve $f: C \rightarrow X$ is

Def $H^0(C, f^*T_x)$

Obs $H^1(C, f^*T_x)$

higher obstructions vanish

$\mathcal{M}_c(x, \beta)$ has Def-Obs theory

$$\begin{array}{ccc} \mathcal{M}_c \times C & & \\ \pi \downarrow & & \downarrow f \\ \mathcal{M}_c & & X \end{array}$$

↑
fixed domain

of vir dim $\chi(C, f^*T_X)$

← Artin Stack

$$\begin{array}{c} (R\pi_* f^*T_X)^\vee \\ \downarrow \\ \mathcal{L}_{\mathcal{M}_c} \end{array}$$

Then, since $\mathcal{M}_{g,n}$ is nonsingular,

we obtain a Def-Obs theory

for $\bar{\mathcal{M}}_{g,n}(x, \beta)$.

Behrend-Fantechi

Li-Tian

Gromov
witten
theory

$$[\bar{\mathcal{M}}_{g,n}(x, \beta)]^{\text{vir}} \in A_{\text{vir dim}}(\bar{\mathcal{M}}_{g,n}(x, \beta))$$

Exercise: The virtual class of $\bar{\mathcal{M}}_{g,n}(x, 0)$

is given by $c_{\text{top}}(\mathbb{E}_g^* \boxtimes T_X)$

on $\bar{\mathcal{M}}_{g,n}(x, 0) = \bar{\mathcal{M}}_{g,n} \times X$

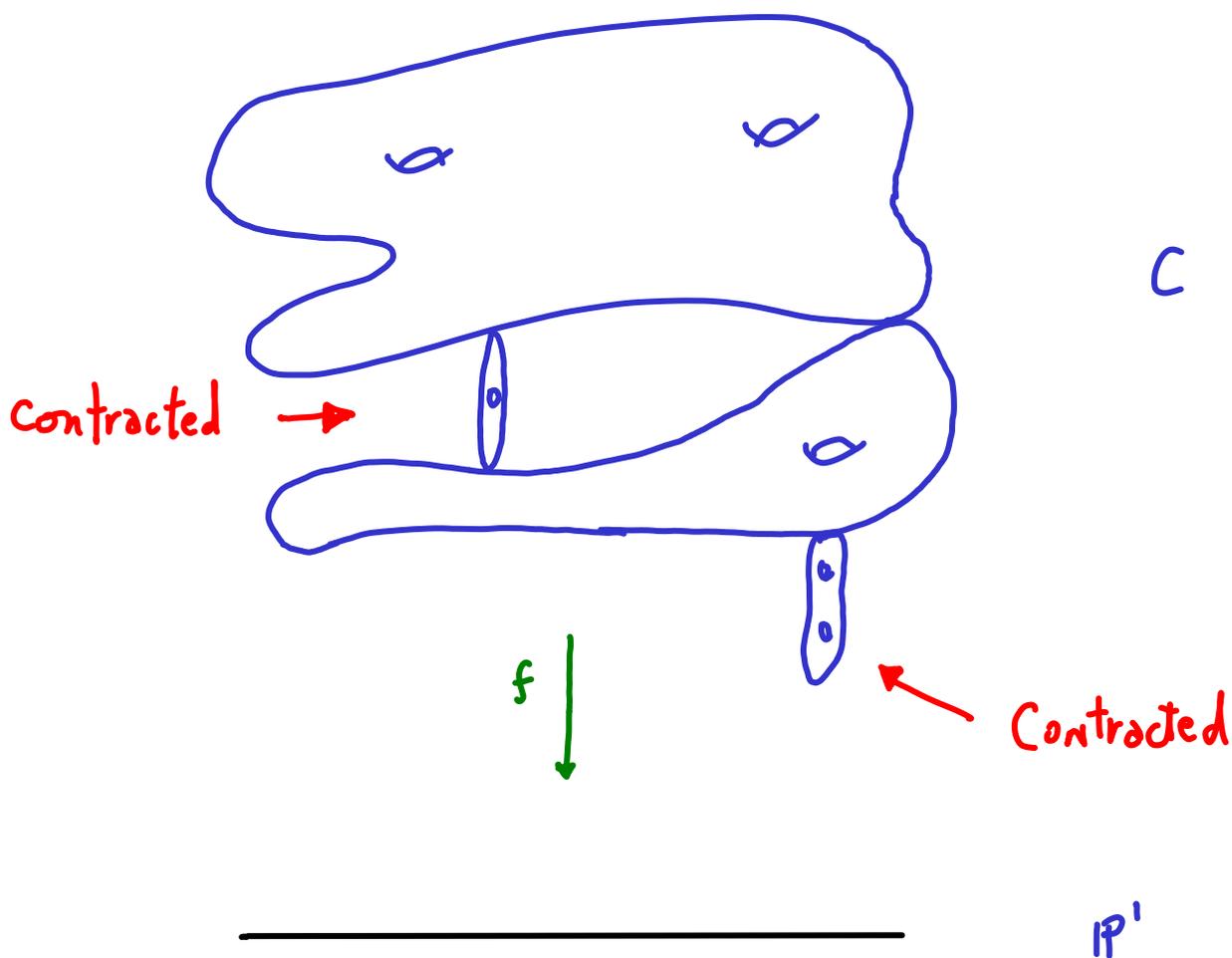
$\mathbb{E}_g \rightarrow \bar{\mathcal{M}}_{g,n}$
Hodge bundle
with fiber
 $H^0(C, \omega_C)$

(iii) Maps to \mathbb{P}^1

$\bar{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ has virtual dim $2g + 2d - 2 + n$

↑
degree d
maps

A general map looks like



We can connect Gromov-Witten theory to Hurwitz's older enumerative geometry of maps to \mathbb{P}^1 .

How does a ramification condition appear?

$$\bar{\mathcal{M}}_{g,1}(\mathbb{P}^1, d) \xrightarrow{\text{ev}_1} \mathbb{P}^1 \quad \text{evaluation map}$$

$$\begin{array}{c} \mathbb{L}_1 \\ \downarrow \\ \bar{\mathcal{M}}_{g,1}(\mathbb{P}^1, d) \end{array} \quad \begin{array}{l} \swarrow \text{cotangent line on} \\ \text{the domain} \\ \text{at the marking} \end{array}$$

$$\psi_1 \cdot \text{ev}_1^*(p) \quad \rightsquigarrow \quad \text{imposition of a ramification condition over } p \in \mathbb{P}^1$$

$$df: T_{C, x} \rightarrow T_{\mathbb{P}^1, f(x)}$$

differential of
 f at the marked
point $x \in C$

We can rewrite as a section

$$df \in H^0 \left(T_{C, 1}^* \otimes f^* T_{\mathbb{P}^1} \right)$$

"

$$H^0 \left(\mathcal{L}_1 \otimes f^* T_{\mathbb{P}^1} \right)$$

After we also impose $ev_1^{-1}(p)$,

$f^* T_{\mathbb{P}^1} \Big|_{ev_1^{-1}(p)}$ is trivial.

So the vanishing of df

restricted to $ev_1^{-1}(p)$ represents γ_1

and occurs at critical points of f .

Theorem: Basic GW/Hurwitz Correspondence P 1999

$$\int \prod_{i=1}^{2g+2d-2} \gamma_i \text{ev}_i^*(p) = \text{Hur}_{g,d}^0$$

\uparrow
Hurwitz Count
of Connected
Covers with
Simple ramifications

$[\bar{\mathcal{M}}_{g, 2g+2d-2}(\mathbb{P}^1, d)]^{\text{vir}}$

Proof: Show the intersections avoid
all pathologies of the moduli
of maps to \mathbb{P}^1

Branch morphism
Fantechi-P
is useful here

$$\text{br: } \bar{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \rightarrow \text{Sym}^{2g+2d-2}(\mathbb{P}^1)$$

(iv) Descendants

For $\gamma \in H^*(x, \mathbb{Q})$, notation

$$\tau_k(\gamma) \leftrightarrow \gamma^k \text{ev}^*(\gamma)$$

and for integrals

$$\left\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \right\rangle_{g, n, \beta}^x$$

redundant

=

$$\int \prod_{i=1}^n \gamma_i^{k_i} \text{ev}_i^*(\gamma_i)$$

$$[\bar{\mathcal{M}}_{g, n}(x, \beta)]^{\text{vir}}$$

Restatement: $\left\langle \tau_1(p) \right\rangle_{g, d}^{2g+2d-2, \mathbb{P}^1} = \text{Hur}_{g, d}^{\circ}$

Two immediate questions:

- (A) is there a such a statement for higher descendents?
- (B) is there a generalization of Witten's Conjecture which controls descendents for the target \mathcal{X} ?

The answers are Yes in both cases
with some qualifications

(A) for $\mathbb{P}^1 \Rightarrow$ full GW/Hurwitz
Correspondence Okounkov-P

for $\mathcal{X} \Rightarrow$ Relative GW theory,
Descendent / Relative Maulik-P

Theorem: GW/Hurwitz Correspondence Okounkov - P

$$\left\langle \prod_{i=1}^n \tau_{k_i}(p) \right\rangle_{g,d} = \text{disconnected domains}$$

$$\prod_{i=1}^n \frac{1}{k_i!} \text{Hur}_{g,d} \left(\prod_{i=1}^n \overline{(k_i+1)} \right)$$

$\overline{(k_i+1)}$ is the completed cycle,

an object in the theory of symmetric functions

Kerov, Olshanski, Okounkov (and others)

$$\overline{(k+1)} = (k+1) + \text{corrections}$$

↑ usual cycle

Examples (see "GW theory, Hurwitz theory and Completed cycles"
 Okounkov - P 2006)

degenerate
 constant
 contribution

$$(\bar{1}) = (1) - \frac{1}{24} (\)$$

First
 case

→ $(\bar{2}) = (2)$

$$(\bar{3}) = (3) + (1,1) + \frac{1}{12} \cdot (1) + \frac{7}{2880} (\)$$

There is a simple formula

for these correction coefficients

$$S(z) = \frac{\sinh(x/2)}{x/2} = \frac{e^{x/2} - e^{-x/2}}{x}$$

Then we have

$$\overline{(\kappa)} = \sum_{\mu} \rho_{\kappa, \mu} (\mu)$$

Partition
 $\mu = \{\mu_i\}$
of size $|\mu|$
and length $l(\mu)$

$$\rho_{\kappa, \mu} = (\kappa-1)! \frac{\prod \mu_i}{|\mu|!} [z^{\kappa+1-|\mu|-l(\mu)}] S(z)^{|\mu|-1} \cdot \prod S(\mu_i z)$$

(B) Virasoro Constraints for
arbitrary targets X

Eguchi
Hori
Xiong
S. Katz

For simplicity, we

assume X has only even cohomology
of type (p, p) .

The general case is important
even for $\dim_{\mathbb{C}} X = 1$!

Let X be a nonsingular projective variety

of $\dim_{\mathbb{C}} X = r$.

$\gamma_0 = \text{Id class}$

Let $\{\gamma_a\}$ be a basis of X

with $\gamma_a \in H^{2p_a}(X, \mathbb{Q})$

By our assumptions,

$\gamma_a \in H^{p_a, p_a}(X)$

As before, we define

$$\langle \prod \tau_{k_i}(\gamma_{a_i}) \rangle_{g, \beta}^X = \int [\bar{M}_{g, n}(X, \beta)]^{\text{vir}} \prod \psi_i^{k_i} \text{ev}_i^*(\gamma_{a_i})$$

Let
$$\phi = \sum_{a, X} t_a^X \tau_k(\gamma_a)$$

and
$$F^X = \sum_{g \geq 0} \lambda^{2g-2} \sum_{\beta} q^{\beta} \sum_{n \geq 0} \frac{1}{n!} \langle \phi^n \rangle_{g, \beta}^X$$

Finally, let $\mathcal{Z}^x = \exp(\mathcal{F}^x)$

Virasoro Conjecture: $L_k \mathcal{Z}^x = 0$

for all $k \geq -1$

Where

$$L_k = \sum_{m=0}^{\infty} \sum_{i=0}^{k+1} \left([b_a + m]_i^k (C^i)_a^b \tilde{t}_m^a \partial_{b, m+k-i} \right.$$

$$+ \frac{\lambda^2}{2} (-1)^{m+1} [-b_a - m]_i^k (C^i)^{ab} \partial_{a, m} \partial_{b, k-m-i-1} \left. \right)$$

$$+ \frac{\lambda^{-2}}{2} (C^{k+1})_{ab} \tilde{t}_0^a \tilde{t}_0^b$$

$$+ \frac{\delta_{k0}}{48} \int_x ((3-r) C_r(x) - 2 C_1(x) C_{r-1}(x))$$

Repeated
indices
summed

Various terms require definition

- $g_{ab} = \int_{\mathcal{X}} \gamma_a \gamma_b$ intersection pairing

used to raise and lower indices

- matrix C_a^b is defined by

$$C_a^b \gamma_b = c_1(x) \vee \gamma_a$$

- Combinatorial coefficients

$$b_a = p_a + (1-r)/2$$

$$[x]_i^k = e_{k+1-i}(\lambda, \lambda+1, \dots, \lambda+k)$$

 elementary symmetric function

• Variables with dilation shift

$$\tilde{t}_m^a = t_m^a - \delta_{a0} \delta_{m1}$$

$$\partial_{a,m} = \frac{\partial}{\partial t_m^a}$$

Cases known

• X is a point Witten's Conjecture

• $\dim_{\mathbb{C}} X = 1$ Okounkov-P

• $QH^*(X)$ is semisimple Teimann,
Givental-Teimann classification
 $X = \mathbb{P}^n$ or G/P

• for $X = C43$, trivial

• for all X in genus 0 X.Liu-Tian
Getzler

Unknown for most varieties: hypersurfaces,
surfaces of general type, Fano's, etc.

(v) Projective plane \mathbb{P}^2

The basic Gromov-Witten invariants
Count Severi degrees.

$$N_{g,d} = \left\langle \tau_0(p)^{3d-1+g} \right\rangle_{g,d}^{\mathbb{P}^2}$$

$$= \int \prod_{i=1}^{3d-1+g} \text{ev}_i^*(p)$$

$$[\bar{M}_{g,3d-1+g}(\mathbb{P}^2, d)]^{\text{vir}}$$

Classical dimension bounds imply
that $N_{g,d}$ is the actual count of
genus g , degree d curves through
 $3d-1+g$ general points in \mathbb{P}^2

genus 0 : The fundamental equation comes from quantum cohomology (associativity of the quantum product \star)

$$N_{0,d} = \sum_{\substack{d_1+d_2=d \\ d_i > 0}} N_{d_1} N_{d_2} \left[\binom{3d-4}{3d_1-2}^{d_1, d_2} - \binom{3d-4}{3d_1-1}^{d_1, d_2} \right]$$

Kontsevich
WDVV

$$N_{0,1} = 1, N_{0,2} = 1, N_{0,3} = 12, N_{0,4} = 620, \dots$$

initial condition

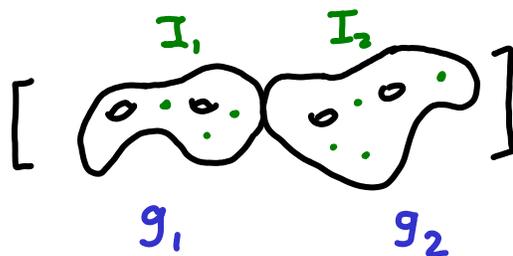
There are many good expositions.

$$\bar{M}_{g,n}(x, \beta) \leftarrow \bar{M}_{g,n}(x, \beta) \times \bar{M}_{g,n}^D$$

$$\varepsilon \downarrow$$

$$\downarrow$$

$$\bar{M}_{g,n} \leftarrow \delta^* D = \bar{M}_{g_1, I_1+\bullet} \times \bar{M}_{g_2, I_2+\star}$$



The rule for computing the pull-back:

$$\delta^* [\bar{M}_{g,n}(x, \beta)]^{\text{vir}} =$$

$$\sum_{\beta_1 + \beta_2 = \beta} [\bar{M}_{g_1, I_1+\bullet}(x, \beta_1)]^{\text{vir}} \times [\bar{M}_{g_2, I_2+\star}(x, \beta_2)]^{\text{vir}}$$

$$\beta_1 + \beta_2 = \beta$$

$$\cdot \text{ev}_{\bullet, \star}^*(\Delta)$$

class of diagonal in $X \times X$

Splitting Axioms in Gromov-Witten theory
Behrend, Behrend-Fantechi

WDVV equations are obtained from the splitting axioms and the relation

$$\left[\begin{array}{c} 1 \\ \circ \quad \circ \\ \vdots \quad \vdots \\ 2 \quad 4 \end{array} \right] = \left[\begin{array}{c} 1 \\ \circ \\ \vdots \\ \circ \\ \vdots \\ 2 \quad 4 \end{array} \right] \text{ in } H^2(\bar{M}_{0,4})$$

Exercise: Derive the recursion for $N_{0,d}$

Requires some geometric ideas.

We see that the cohomology of $\bar{M}_{g,n}$

constrains the Gromov-Witten invariants.

Also the opposite is true. The geometry of the moduli space of stable maps

constrains the cohomology of $\bar{M}_{g,n}$.

Example:
Proof of
Pixton's relations

genus 1: Much more subtle,
and less well-known.

$$\mathcal{N}_{1,d} = \frac{1}{12} \binom{d}{3} \mathcal{N}_{0,d}$$

Eguchi
Hori
Xiong

$$+ \sum_{\substack{d_1+d_2=d \\ d_i > 0}} \frac{3d_1^2 d_2 - 2d_1 d_2}{9} \binom{3d-1}{3d_1-1} \mathcal{N}_{0,d_1} \mathcal{N}_{1,d_2}$$

$$\mathcal{N}_{1,1} = 0, \quad \mathcal{N}_{1,2} = 0, \quad \mathcal{N}_{1,3} = 1, \quad \mathcal{N}_{1,4} = 225$$

Consequence of the Virasoro constraints
for \mathbb{P}^2 (also can be derived
from Getzler's relation in $\bar{\mathcal{M}}_{1,4}$).

How to prove the $g=1$ recursion?

Step 1. Write L_1 explicitly
for \mathbb{P}^2

Step 2. Extract the coefficient

of $\lambda^0 (t_0^2)^{3d-1}$ in $\frac{L_1 \mathbb{Z}^{\mathbb{P}^2}}{\mathbb{Z}^{\mathbb{P}^2}}$.

You will find

$$-\frac{g}{(3d-1)!} \left\langle \tau_0(p)^{3d} \right\rangle_{1,d} + \dots$$

which vanishes by the
Virasoro constraints.

Step 3. Terms with the insertions

$T_1(H)$ and $T_2(1)$

require applications of

TRR in genus 1:

$$\langle T_r(\sigma) \cdot T_0(p)^m \rangle_{1,d}^{\mathbb{P}^2}$$

\equiv

$$\frac{d^2}{24} \langle T_0(\sigma) \cdot T_0(p)^m \rangle_{0,d}^{\mathbb{P}^2}$$

[and the String, divisor equation]

$$+ \sum_{\substack{d_1+d_2=d \\ m_1+m_2=m}} \binom{m}{m_1} \langle T_{r-1}(\sigma) T_0(p)^{m_1} T_0(\gamma^a) \rangle_{0,d_1}^{\mathbb{P}^2} g^{ab} \langle T_0(\gamma^b) T_0(p)^{m_2} \rangle_{1,d_2}^{\mathbb{P}^2}$$

$$d_1 + d_2 = d$$

$$m_1 + m_2 = m$$

genus $g \geq 2$ Are there
higher genus recursions?

Answer is Yes, but
much more complicated
forms involving additional
recursions for descendants

- Using Virasoro \Rightarrow Gathmann
+ TRR

- There are also recursions using degenerations
Z. Ran
Caporaso-Harris

For other surface, many

open questions:

- Virasoro constraints

for the Enriques surface

not known even in genus 1

- Can not be formulated

in symplectic geometry

in general (requires Hodge decomposition)



Very strange state of affairs which

led to some doubts about the constraints.

Derivation of the Eguchi-Hori-Xiong

$g=1$ recursion from L_1 for \mathbb{P}^2 .

Notes written by Longting Wu:

Let us set $p^0=1$, $p^1=H$, $p^2=H^2$. Then variables

$$t_m^i \rightsquigarrow \tau_m(H^i)$$

By taking the coefficient $(t_0^2)^{3d-1}$ of

$$L_1 \exp(F^{\mathbb{P}^2}) = 0$$

we get

$$\frac{15}{4} (3d-1) \langle \tau_1(H^2) \underbrace{\tau_0(H^2) \cdots \tau_0(H^2)}_{3d-2} \rangle_{1,d} - 9N_{1,d}$$

$$- \frac{3}{4} \langle \tau_2(1) \underbrace{\tau_0(H^2) \cdots \tau_0(H^2)}_{3d-1} \rangle_{1,d} - 6 \langle \tau_1(H) \underbrace{\tau_0(H^2) \cdots \tau_0(H^2)}_{3d-1} \rangle_{1,d}$$

$$+ \frac{d^2}{8} N_{0,d} + \frac{1}{4} \sum_{\substack{d_1+d_2=d \\ d_i > 0}} (d_1 d_2) \binom{3d-1}{3d-1} N_{0,d_1} N_{1,d_2} - \frac{d}{32} N_{0,d} = 0$$

*

We set

$$N_{1,d} = \langle \underbrace{\tau_0(H^2) \cdot \dots \cdot \tau_0(H^2)}_{3d} \rangle_{1,d}$$

pull back relation
 $\Psi_1 = \frac{1}{12}[\rho]$

$$N_{0,d} = \langle \tau_0(H^2) \cdot \dots \cdot \tau_0(H^2) \rangle_{0,d}$$

in $\overline{M}_{1,1}$

Using the topology recursion for genus 1, we get

$$\langle \tau_1(H^2) \underbrace{\tau_0(H^2) \cdot \dots \cdot \tau_0(H^2)}_{3d-2} \rangle_{1,d} = \frac{1}{24} d^2 N_{0,d} - \frac{d}{8} N_{0,d} + \sum_{\substack{d_1+d_2=d \\ d_i > 0}} \binom{3d-2}{3d_1-2} d_1 d_2 N_{0,d_1} N_{1,d_2}$$

$$\langle \tau_1(H) \underbrace{\tau_0(H^2) \cdot \dots \cdot \tau_0(H^2)}_{3d-1} \rangle_{1,d} = \frac{1}{24} d^3 N_{0,d} - \frac{d^2}{8} N_{0,d} + \sum_{\substack{d_1+d_2=d \\ d_i > 0}} \binom{3d-1}{3d_1-1} d_1^2 d_2 N_{0,d_1} N_{1,d_2}$$

$$\langle \tau_2 \underbrace{\tau_0(H^2) \cdot \dots \cdot \tau_0(H^2)}_{3d-1} \rangle_{1,d} = \frac{d^2(3d-1)}{24} N_{0,d} - \frac{(3d-2)d}{8} N_{0,d} + \sum_{\substack{d_1+d_2=d \\ d_i > 0}} \binom{3d-1}{3d_1-1} (3d_1-2) d_1 d_2 N_{0,d_1} N_{1,d_2}$$

Plug these into previous relation *

We get recursion

$$N_{1,d} = \frac{1}{12} \binom{d}{3} N_{0,d} + \sum_{\substack{d_1+d_2=d \\ d_i > 0}} \frac{3d_1^2 d_2 - 2d_1 d_2}{9} \binom{3d-1}{3d_1} N_{0,d_1} N_{1,d_2}$$

Enumerative Geometry of
Curves, Maps, and Sheaves

Part III : Sheaf Counting

Rahul Pandharipande

ETH ZÜRICH

14 July 2021

A fundamental property of

Gromov-Witten theory is the

uniform definition for all

targets X .

[nonsingular,
projective of
any dimension]

Sheaf counting is more delicate:

the standard theories are for

sheaves on X with $\dim_{\mathbb{C}} X \leq 3$

Recent work by

R. Thomas and J. Oh

on CY4-folds

What is the reason for the difference?

- Def-Obs theory for a stable map $f: C \rightarrow X$

$$\text{inf Aut} = 0 \quad [\text{map stability}]$$

$$\text{Def} = H^0(C, f^* T_X)$$

$$\text{Obs} = H^1(C, f^* T_X)$$

is *always* 2-term \Rightarrow virtual fundamental class.

- For a sheaf $\mathcal{F} \rightarrow X$

$$\text{inf Aut} = \text{Ext}^0(\mathcal{F}, \mathcal{F})$$

$$\text{Def} = \text{Ext}^1(\mathcal{F}, \mathcal{F})$$

$$\text{Obs} = \text{Ext}^2(\mathcal{F}, \mathcal{F})$$

+ higher obstructions $\text{Ext}^k(\mathcal{F}, \mathcal{F})$

Mostly killed by sheaf stability

dim constraints on X are needed to kill \uparrow

Dimension 1

Let \mathcal{X} be a nonsingular projective curve of genus g .

- $\mathcal{U}_{\mathcal{X}}(r, d)$ moduli of stable bundles
(r, d) = 1, already nonsingular of the expected dimension since

$$\text{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$$

many variations:
Higgs bundles

- Quot scheme $\text{Quot}_{\mathcal{X}}(\mathcal{F}^n, r, d)$

$$0 \rightarrow G \rightarrow \mathcal{F}^n \otimes \mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{F} \rightarrow 0$$

rank r , degree d

Marian
Oprea

$$\text{Def} = \text{Ext}^0(G, \mathcal{F})$$

$$\text{Obs} = \text{Ext}^1(G, \mathcal{F})$$

$$\text{Ext}^{\geq 2}(G, \mathcal{F}) = 0$$

since $\dim_{\mathbb{C}} \mathcal{X} = 1$

$\text{Quot}_\chi(\mathbb{F}^n, r, d)$ is generally

singular of mixed dimension, but

carries a virtual fundamental class.

Exercise: Compute the virtual dimension,
 $\text{vir dim } \text{Quot}_\chi(\mathbb{F}^n, r, d)$
" "
 $r(n-r)(1-g) + nd$.

On an open set, $\text{Quot}_\chi(\mathbb{F}^n, r, d)$

is a moduli space of bundles with sections.

Marian-Oprea transfer integrals on

$\mathcal{U}_\chi(n-r, d)$ to $\text{Quot}_\chi(\mathbb{F}^n, r, d)$ against

the virtual class

\Rightarrow leads to a proof of Verlinde formulas.

- $\text{Quot}_X(\mathcal{F}^1, \mathcal{O}, d) = \text{Sym}^d X$

$$\text{Quot}_X(\mathcal{F}^n, \mathcal{O}, d) = \text{functional Quot Schemes of the Curve } X$$

Exercise: $\text{Quot}_X(\mathcal{F}^n, \mathcal{O}, d)$ is nonsingular
of dimension nd and
virtual class is the usual
fundamental class.

Tautological bundles on $\text{Quot}_X(\mathcal{F}^n, \mathcal{O}, d)$

can be constructed as follows.

$E \rightarrow X$ vector bundle of rank e



$E^{[d]} \rightarrow \text{Quot}_X(\mathcal{F}^n, \mathcal{O}, d)$ vector bundle of rank de
with fiber $H^0(X, \mathcal{F} \otimes E)$

Interesting property: for $L \rightarrow X$ line bundle,

$$\int \Delta(L^{[d]})^1 = (-1)^{(n-1)d} \int \Delta(L^{[d]})^n$$

$\text{Quot}_X(\Phi^n, 0, d)$
 $\text{Quot}_X(\Phi^1, 0, d)$
← Segre class

$\text{Sym}^d X$
 $\Delta(B) = \frac{1}{c(B)}$

Oprea-P 2019

Challenge: find a conceptual proof.

Dimension 2

Let X be a nonsingular projective surface

- The simplest theory is again for the Quot scheme

$\text{Quot}_\chi(\Phi^n, \beta, d)$ of quotients

$$0 \rightarrow G \rightarrow \Phi^n \otimes \mathcal{O}_X \rightarrow \mathcal{F} \rightarrow 0$$

rank 0 [supported on curves]

$$c_1(\mathcal{F}) = \beta, \quad \chi(\mathcal{F}) = d$$

Marian
Oprea
P 20

$$\text{Def} = \text{Ext}^0(G, \mathcal{F})$$

$$\text{Obs} = \text{Ext}^1(G, \mathcal{F})$$

$$\text{Ext}^2(G, \mathcal{F}) = \text{Ext}^0(\mathcal{F}, G \otimes k_X)^*$$

= 0 Since \mathcal{F}
is torsion

Serre duality

We can remove the \mathcal{F} is torsion assumption
if χ is Fano - a mostly unexplored
direction

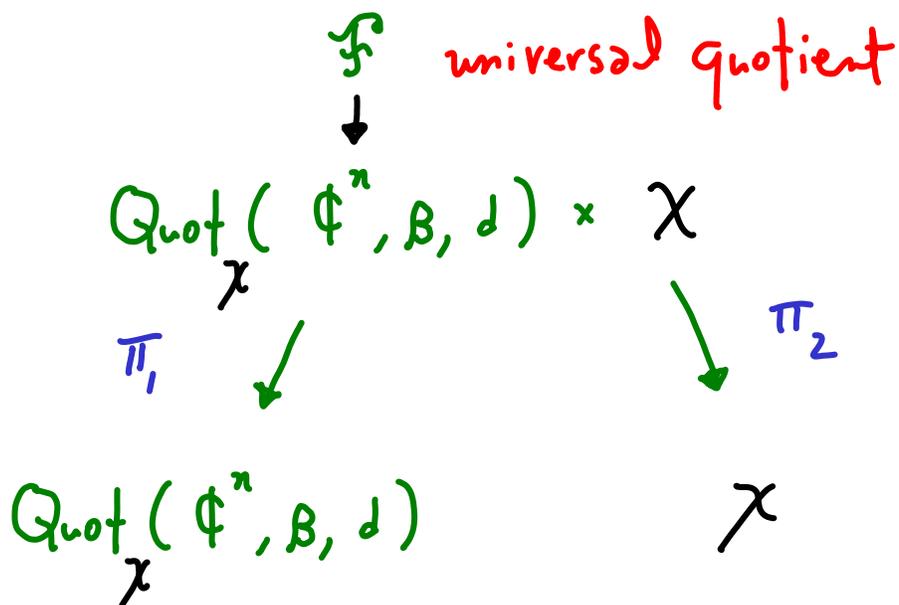
$$\text{vir dim } \text{Quot}_\chi(\Phi^n, \beta, d) = nd + \int_X \beta^2$$

↑
grows with d

What are the integrals?

For $\alpha \in K^0(X)$, define

$$q^{[d]} = R\pi_{1*}(\mathcal{F} \otimes \pi_2^* \alpha) \in K^0(\text{Quot}_X)$$



$$\mathbb{Z}_{n,\beta}^X(\alpha_1, \dots, \alpha_\ell \mid k_1, \dots, k_\ell)$$

=

Chern char
here viewed
as descendent
insertion

$$\sum_{d \in \mathbb{Z}} q^d \int \prod_{i=1}^{\ell} \text{ch}_{k_i}(q_i^{[d]}) c(T^{\text{vir}}(\text{Quot}_X(\Phi^n, \beta, d)))$$

$$d \in \mathbb{Z} \quad [\text{Quot}_X(\Phi^n, \beta, d)]^{\text{vir}}$$

total Chern class

Two basic ideas in the theory

(A) Rationality

Conjecture: $Z_{n,\beta}^X(\alpha_1, \dots, \alpha_\ell \mid k_1, \dots, k_\ell)$

is the Laurent expansion of
a rational function in q .

Oprea-P

Johnson-Oprea-P

W. Lim

Arbesfeld-J-L-O-P

\Rightarrow proof in many cases,
but not yet all

(B) Exact solutions for

χ = simply connected minimal surface
of general type with nonsingular
canonical curve.

Theorem [Oprea-P] :

genus of
canonical
curve

$$Z_{n, l, k_x}^x(q) = (-1)^{l \cdot \chi(\mathcal{O}_x)} q^{l(1-g)} .$$

$$\sum A(r_{i_1}, \dots, r_{i_{n-l}})^{1-g}$$

$$1 \leq i_1 < \dots < i_{n-l} \leq n$$

where the sum is taken over all

$\binom{n}{n-l}$ choices of $n-l$ distinct roots

$$\omega = r_i(q)$$

of the equation $\omega^n - q(\omega-1)^n = 0$,

$$A(x_1, \dots, x_{n-l}) = \frac{(-1)^{\binom{n-l}{2}}}{n^{n-l}} \cdot \prod_{i=1}^{n-l} \frac{(1+x_i)^n (1-x_i)}{x_i^{n-1}} \cdot \prod_{i < j} \frac{(x_i - x_j)^2}{1 - (x_i - x_j)^2}$$

The result suggests a connection
to Gromov-Witten Curve Counting
via the appearance of $(-1)^{\chi(\theta_x)}$ and g

$$\langle 1 \rangle_{g, k_x}^x \quad \text{Gromov} = \text{SW} \quad \text{Torbes}$$

- A more sophisticated sheaf counting theory of surfaces was proposed by Vafa-Witten and defined mathematically by Tanaka-Thomas.

Sheaf counting on \mathcal{X} approached

via counting sheaves on the 3-fold

total space $\rightarrow \mathcal{K}_x \rightarrow x$

Much harder to calculate, rational functions replaced by modular forms, many results/conjectures by Göttsche-Kool

Dimension 3

Three is the most interesting dimension for counting, and there are many directions of study: *Mirror symmetry, DT wallcrossing / Stability conditions, refined invariants, ...*

The simplest place to start is with the Hilbert scheme of Curves.

Let X be a nonsingular projective 3-fold.

$I_n(X, \beta) =$ Hilbert scheme of
curves $C \subset X$

$$0 \rightarrow \mathcal{I}_C \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0$$

$\eta = \chi(\mathcal{O}_C)$
 $\beta = [C] \in H_2(X, \mathbb{Z})$

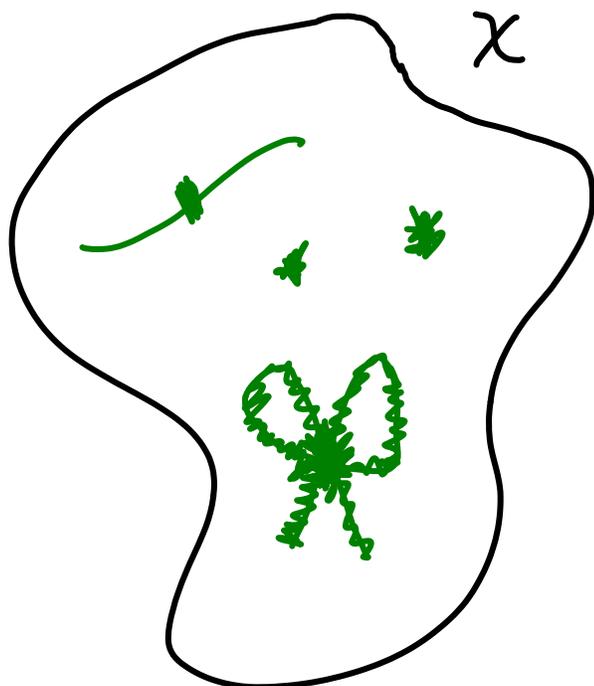
We can consider the Hilbert scheme as a moduli space of ideal sheaves.

$$0 \rightarrow \mathcal{I}_C \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0$$

We view Hilb as a moduli of ideal sheaves (with trace free defs)

usually Hilb is viewed as a moduli of quotients

We really consider the entire Hilbert Scheme



R. Thomas
Phd Thesis

consider the Def-Obs theory

$$\text{Ext}^0(\mathcal{d}, \mathcal{d}) = \mathbb{C} \quad \text{scalars}$$

$$\text{Ext}^1(\mathcal{d}, \mathcal{d}) = \text{Def}$$

$$\text{Ext}^2(\mathcal{d}, \mathcal{d}) = \text{Obs}$$

killed by
traceless
def theory

$$\text{Ext}^3(\mathcal{d}, \mathcal{d}) \cong \text{Ext}^0(\mathcal{d}, \mathcal{d} \otimes k_x)^*$$

Conclusion: $\text{Ext}_0^1(\mathcal{d}, \mathcal{d}) = \text{Def}$

traceless
Ext

$$\text{Ext}_0^2(\mathcal{d}, \mathcal{d}) = \text{Obs}$$

$\mathcal{I}_n(x, \beta)$ has a virtual fundamental class

Exercise: Calculate the virtual
dimension

independent
of n !

$$\text{vir dim } \mathcal{I}_n(x, \beta) = \int_B c_1(x)$$

Integration against $[\mathcal{I}_n(x, \beta)]^{\text{vir}}$

is Donaldson-Thomas theory.

Gromov-Witten theory also has an independence property for vir dim in dimension 3:

$$\text{vir dim } \overline{\mathcal{M}}_g(x, \beta) = \int_{\beta} c_1(x)$$

Moreover the vir dim formula

is the same.

↑
independent
of g !

Calabi-Yau 3-folds

CY3s are the perfect location

for enumerative geometry: all

problems have virtual dimension 0

Let X be a CY3

Let $\beta \in H_2(X, \mathbb{Z})$

$$\text{vir dim } \overline{M}_g(X, \beta) = \text{vir dim } I_n(X, \beta) = 0$$

Question: Is there a relationship

$$N_{g, \beta} = \int [\overline{M}_g(X, \beta)]^{\text{vir}} \quad \overset{?}{\sim} \quad I_{n, \beta} = \int [I_n(X, \beta)]^{\text{vir}}$$

Both sides virtually count curves,
but some differences.

- Simplest hope:

Assume β is an indecomposable class

Can hope $N_{g, \beta} \stackrel{?}{=} I_{1-g, \beta}$



$\chi(C_g) = 1-g$

Let analyse the simplest
case of such a geometry

$$C \subset X, C \cong \mathbb{P}^1 \text{ with normal bundle } N_{X/C} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$$

GW calculation Faber-P 2000

$$\sum_{g \geq 0} u^{2g-2} N_{g, [C]} = \left(\frac{u/2}{\sin(u/2)} \right)^2 \frac{1}{u^2}$$

How are these integrals computed?

Everything can be moved

to the moduli of maps to $C \subseteq \mathbb{P}^1$

Then the techniques are

- Localization (of the virtual class)
- Hodge integrals $\int_{\overline{\mathcal{M}}_{g,1}} c(\mathbb{E}) \cdot \psi_1^k$
↑ Hodge bundle
- Tricks

DT calculation MNOP I, II

Euler
char

$$\sum_{n \in \mathbb{Z}} q^n I_{n, [C]} = \frac{q}{(1+q)^2} \cdot M(-q)$$

$e(X)$

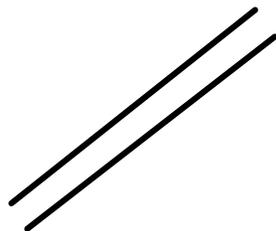
$$M(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-n}$$

- Localization (of the virtual class)
- Box counting in 3-dimensions
- Tricks

Conclusion: simple hope taken
literally fails.

- More sophisticated hope

$$M(-q)^{-e(x)} \sum_{n \in \mathbb{Z}} q^n I_{n, [c]} = \frac{q}{(1+q)^2}$$



Substitute

$$q = -e^{iu}$$

$$\frac{-e^{iu}}{(1 - e^{iu})^2} = -\frac{1}{(e^{iu/2} - e^{-iu/2})^2}$$

$$= \left(\frac{2i}{e^{iu/2} - e^{-iu/2}} \right)^2 \frac{1}{2^2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$= \left(\frac{u/2}{\sin(u/2)} \right)^2 \frac{1}{u^2}$$

$$= \sum_{g \geq 0} u^{2g-2} N_{g, [C]}$$

- GW/DT Correspondence of MNOP

Maulik
Nekrasov
Okounkov
P

Conjecture : The relationship found

for $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$



\mathbb{P}^1

holds in general

Topological Vertex
and Box Counting
allow for further
examples

Aganagic Klemm Mariño Vafa

Let us write the conjecture precisely.

X is a CY3 fold

$$\mathcal{F}'_{\text{GW}} = \sum_{g \geq 0} \sum_{\beta \neq 0} N_{g, \beta} u^{2g-2} v^\beta$$

GW theory of connected, non constant maps

$$\mathcal{Z}'_{\text{GW}} = \exp(\mathcal{F}')$$

disconnected theory, but nonconstant on every component

$$\mathcal{Z}'_{\text{GW}} = 1 + \sum_{\beta \neq 0} \mathcal{Z}'_{\text{GW}}(x, u)_\beta v^\beta$$

$$\mathbb{Z}_{DT} = \sum_{\text{all } \beta} \sum_{n \in \mathbb{Z}} I_{n, \beta} q^n v^\beta$$

$$\mathbb{Z}_{DT} = \sum_{\text{all } \beta} \mathbb{Z}_{DT}(\mathcal{X}, q)_\beta v^\beta$$

MNOP Conjecture 1:

$$\mathbb{Z}_{DT}(\mathcal{X}, q)_0 = \mathcal{M}(-q)^{e(\mathcal{X})}$$



$\beta=0$, hence about
Hilbert schemes of
points on \mathcal{X}

STATUS:

Proven

Jun Li

Behrend-Fantechi

Levine-P

$$Z'_{DT} = Z_{DT} / Z_{DT}(x, q)_0$$

idea: remove the constant contributions

$$Z'_{DT} = 1 + \sum_{\beta \neq 0} Z'_{DT}(x, q)_\beta v^\beta$$

MNOP Conjecture 2:

$Z'_{DT}(x, q)_\beta$ is the Laurent expansion of a rational function in q .

Also: $Z'_{DT}(x, q) = Z'_{DT}(x, \frac{1}{q})$

Status: Proven

Bridgeland, Toda
wallcrossing

MNOP Conjecture 3 :

$$Z'_{\text{GW}}(\chi, u) = Z'_{\text{DT}}(\chi, q)$$

after $-e^{iu} = q$

Status: Open, but proven

in many cases

CY3 toric geometries

MNOP
MOOP

Complete intersection CY3s

Pixton-P

Enumerative Geometry of
Curves, Maps, and Sheaves

Part IV : Stable Pairs Descendants

Rahul Pandharipande

ETH ZÜRICH

15 July 2021

I. Descendants for curves and sheaves

We have discussed descendants for moduli spaces of stable maps to X .

Let us revisit the construction to define

$$\tau_k(\gamma) \in H^{2k+\delta-2}(\bar{\mathcal{M}}_g(x, \beta))$$

power of ψ $\gamma \in H^\delta(x)$

Idea is to use the correspondence

$$\begin{array}{ccc} \bar{\mathcal{M}}_{g,1}(x, \beta) & & \\ \pi \swarrow & & \searrow \text{ev} \\ \bar{\mathcal{M}}_g(x, \beta) & & X \end{array} \quad \tau_k(\gamma) = \pi_* (\psi_1^k \cdot \text{ev}^*(\gamma))$$

For moduli spaces of sheaves on X , there is a parallel construction

\uparrow $\dim_{\mathbb{C}} = r$

universal sheaf

$$\rightarrow \mathcal{O}$$

$\gamma \in H^{\delta}(X)$

$$\downarrow$$

$$I \times X$$

π_1

π_2

$$I$$

$$X$$

moduli space of sheaves

$$T_k(\gamma) = \pi_{1*} \left(c_{h_{k+r-1}}(\mathcal{O}) \cdot \pi_2^*(\gamma) \right)$$

\cap

$$H^{2k+\delta-2}(I)$$

descendent in sheaf theory

Examples of descendents in sheaf theory

- X is a nonsingular projective curve
 $\mathcal{L} \rightarrow X$ is a line bundle

$\mathcal{U}_{X, 2, \mathcal{L}}$ = moduli space of rank 2
Stable bundles on X
with fixed $\det = \mathcal{L}$

Descendents defined

deg $\mathcal{L} = 1$
[no semistables]

via the universal bundle

$$\mathcal{E} \rightarrow \mathcal{U}_{X, 2, \mathcal{L}}$$

Theorem: $H^*(\mathcal{U}_{X, 2, \mathcal{L}})$ generated by

descendents. [Mumford, Kirwan, Zagier
also found relations]

- X is a surface

Exactly parallel construction for
moduli of sheaves on a surface

\Rightarrow used in the theory
of Donaldson invariants

We already saw related descendents
in our discussion of

$$\int \prod \text{ch}_{k_i}(\alpha_i^{[d]})$$

$$[\text{Quot}_X(\mathcal{F}^n, \beta, d)]^{\text{vir}}$$


Chern classes after $R\pi_*$,
so need GRR to relate to
the descendents defined here

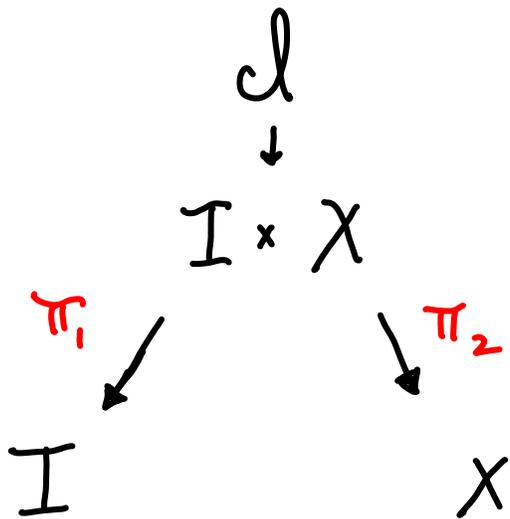
• \mathcal{X} is a 3-fold

differs slightly
from the
previous

GW theory : $\int \prod \tau_{k_i}(\gamma_i)$
 $[\bar{M}_g(x, \beta)]^{\text{vir}}$

$\langle \tau_{k_1}(\gamma_1) \dots \tau_{k_r}(\gamma_r) \rangle$
 defined
 using $\bar{M}_{g,r}(x, \beta)$

DT theory : $\int \prod \tau_{k_i}(\gamma_i)$
 (ideal sheaves)
 $[\mathcal{I}_n(x, \beta)]^{\text{vir}}$



$$\tau_k(\gamma) = \pi_{1*} \left(c_{h_{k+2}}(d) \cdot \pi_2^*(\gamma) \right)$$

n

$$H^{2k+\delta-2}(\mathcal{I}_n(x, \beta))$$

Question: Can we extend the GW/DT

Correspondence of MNOP to descendants?

II. Stable pairs

The Hilbert scheme $I_n(\chi, \beta)$ has

Some shortcomings for the study of descendents. The moduli of stable pairs is better behaved.

Let χ be a nonsingular projective 3 fold,

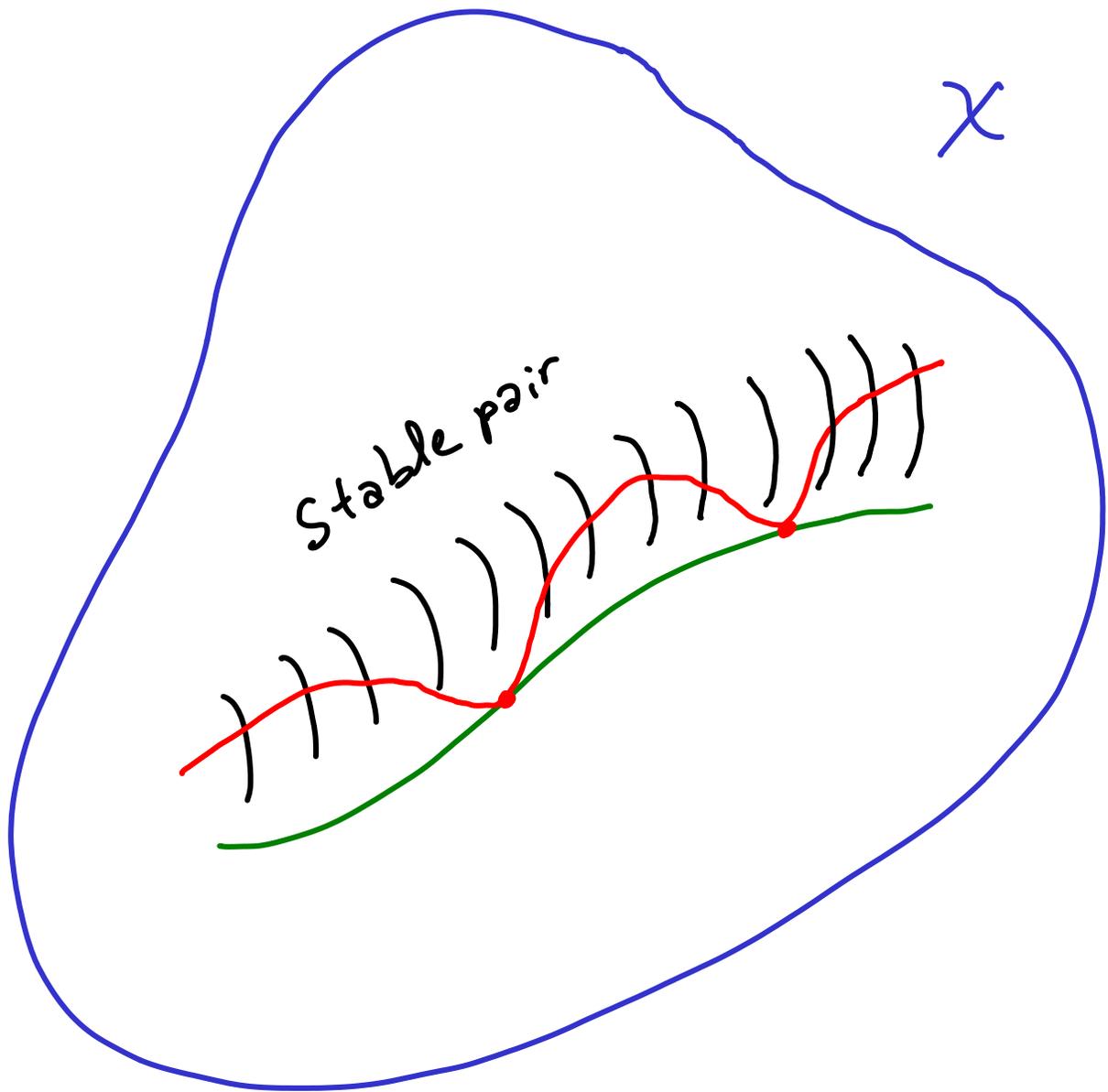
$$\beta \in H_2(\chi, \mathbb{Z}),$$

$$n \in \mathbb{Z}.$$

$\mathcal{P}_n(\chi, \beta)$ is the moduli of stable pairs:

$$[\mathcal{F}, s] \in \mathcal{P}_n(\chi, \beta)$$

- \mathcal{F} is pure sheaf of dimension 1
- $\mathcal{O}_\chi \xrightarrow{s} \mathcal{F}$ is a section with coker of dimension 0



\mathcal{F} sheaf $n = \chi(\mathcal{F})$
 \downarrow \downarrow
 $\text{Supp}(\mathcal{F})$ $B = [\text{Supp}(\mathcal{F})]$

Construction of $P_n(\chi, \beta)$: use Le Potier,

See Papers by P-R. Thomas

Example: $\mathcal{X} = \mathbb{P}^3$

Then $P_n(x, d) \supset$ classical locus

which parameterizes ideal objects

$C \subset \mathbb{P}^3$ nonsingular
irreducible curve of
degree d

$\mathcal{F} \rightarrow C$ line bundle of
degree l

$s \in H^0(C, \mathcal{F})$ a nonzero section

$$n = l - \text{genus}(C) + 1$$

Of course $P_n(x, d)$ also parameterizes more
degenerate objects

We view $\mathbb{I} = [\mathcal{O}_X \xrightarrow{\Delta} \mathbb{F}]$ as

an object in $D_{\text{Coh}}^b(X)$. Then

$$\text{Def} = \text{Ext}_0^1(\mathbb{I}, \mathbb{I})$$

$$\text{Obs} = \text{Ext}_0^2(\mathbb{I}, \mathbb{I})$$

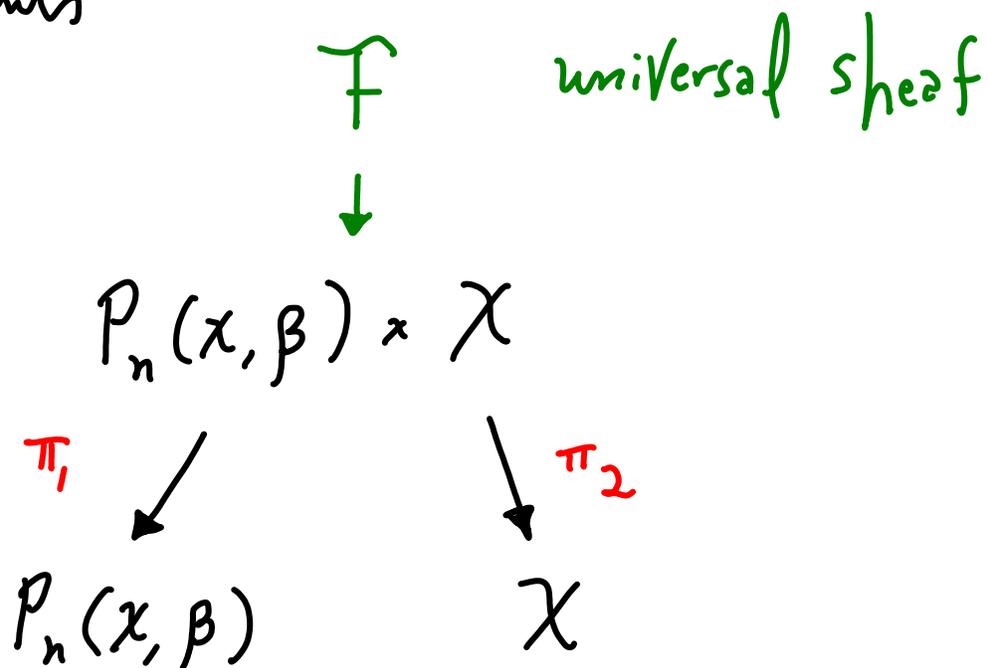
higher Ext_0^i 's vanish

We have a virtual fundamental class

$$[P_n(X, \beta)]^{\text{vir}} \text{ of dimension } \int_{\beta} c_1(X)$$

See "Counting curves via stable pairs"
with R. Thomas

Descendents



$$T_k(\gamma) = \pi_{1*} \left(ch_{k+2}(\mathcal{F}) \cdot \pi_2^*(\gamma) \right)$$

\uparrow
 $\gamma \in H^*(\mathcal{X})$

We will use a better convention

$$ch_k(\gamma) = \pi_{1*} \left(ch_k(\mathcal{F} - \mathcal{O}) \cdot \pi_2^*(\gamma) \right)$$

\uparrow
no shift now

Conjectures for the descendent theory of stable pairs

MNOP 2005

Pixton-P 2012

OO P 2019

GW/Pairs

Correspondence

MNOP 2005

Pixton-P 2012

Rationality

Virasoro

OO P 2019

Moreira OOP 2020

Moreira 2020

M = Maulik

N = Nekrasov

O = Okounkov

OO = Oblomkov, Okounkov

III. Rationality

Let X be a nonsingular projective 3fold

Define descendent generating series:

$$\left\langle \text{ch}_{k_1}(\gamma_1) \cdots \text{ch}_{k_r}(\gamma_r) \right\rangle_B^X \quad \gamma_i \in H^*(X)$$

\equiv

$$\sum_{n \in \mathbb{Z}} q^n \int \text{ch}_{k_1}(\gamma_1) \cdots \text{ch}_{k_r}(\gamma_r) [P_n(X, \beta)]^{\text{vir}}$$

moduli space are empty for $n < 0$

Rationality Conjecture I (P-R. Thomas):

$$\left\langle \text{ch}_{k_1}(\gamma_1) \cdots \text{ch}_{k_r}(\gamma_r) \right\rangle_B^X \in \mathbb{Q}((q))$$

is the Laurent expansion of

a rational function in q .

Example (from my paper

"Descendants for stable pairs
on 3 folds")

with help
from Oblomkov

$$Z_P(\mathbb{P}^3; q | \tau_9(1))_{2L} =$$

$$\frac{(73q^{12} - 825q^{11} - 124q^{10} + 5945q^9 + 779q^8 - 36020q^7 + 60224q^6 - 36020q^5 + 779q^4 + 5945q^3 - 124q^2 - 825q + 73)q}{60480(1+q)^3(-1+q)^3}$$

Rationality Conjecture II (formulated with Pixton)

$$\left\langle \text{ch}_{k_1}(\gamma_1) \cdots \text{ch}_{k_r}(\gamma_r) \right\rangle_{\mathcal{B}}^x = Z(q) \in \mathbb{Q}(q)$$

has poles only at roots of unity and 0 and satisfies a

functional equation

$$Z\left(\frac{1}{q}\right) = (-1)^{\sum_{i=1}^r k_i} q^{-d_{\mathcal{B}}} Z(q)$$

where $d_{\mathcal{B}} = \int_{\mathcal{B}} c_1(x)$.

Failure of Rationality for the Hilbert Scheme:

$$\begin{array}{ccc}
 & \mathcal{I} & \text{universal sheaf} \\
 & \downarrow & \\
 & \mathbb{I}_n(x, \beta) \times X & \\
 \pi_1 \swarrow & & \searrow \pi_2 & \gamma \in H^*(X) \\
 \mathbb{I}_n(x, \beta) & & X &
 \end{array}$$

$$\bullet \text{ch}_k(\gamma) = \pi_{1*}(\text{ch}_k(\mathcal{I}) \cdot \pi_2^*(\gamma))$$

$$\bullet \left\langle \text{ch}_{k_1}(\gamma_1) \cdots \text{ch}_{k_r}(\gamma_r) \right\rangle_{\mathbb{B}}^{X, \mathbb{I}}$$

$=$

$$\sum_{n \in \mathbb{Z}} q^n \int \text{ch}_{k_1}(\gamma_1) \cdots \text{ch}_{k_r}(\gamma_r) [\mathbb{I}_n(x, \beta)]^{\text{vir}}$$

Since $\langle 1 \rangle_0^{\chi, \mathbb{I}} = \sum_x c_3 - c_1 c_2 M(-q),$

$\langle 1 \rangle_0^{\chi, \mathbb{I}}$ not rational in q

- We are interested in (see MNOP)

$$\frac{\langle ch_{k_1}(\gamma_1) \cdots ch_{k_r}(\gamma_r) \rangle_0^{\chi, \mathbb{I}}}{\langle 1 \rangle_0^{\chi, \mathbb{I}}}$$

But still not rational in q

Conjecture (Oblomkov - Okounkov - P):

Normalized series is a polynomial in

$\left\{ \left(q \frac{d}{dq} \right)^i F_3(-q) \right\}_i$ with coefficients in $\mathbb{Q}(q)$.

$$F_3(q) = \sum_{n=1}^{\infty} n^2 \frac{q^n}{1-q^n}$$

Enumerative Geometry of Curves, Maps, and Sheaves

Part V : Virasoro Constraints for Stable Pairs

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16 July 2021

Let \mathcal{X} be a nonsingular
projective 3 fold with only
(p,p) cohomology

Main Example: \mathcal{X} is a toric 3 fold

Virasoro Constraints will take
the form of universal relations
among descendent series

$$\left\langle ch_{k_1}(\gamma_1) \cdots ch_{k_r}(\gamma_r) \right\rangle_{\beta}^{\mathcal{X}}$$

Algebraic form is simpler than for GW

- Constraints are conjectural in almost all cases

Theorem: Stationary Constraints
 Moreira OOP 2020 hold for X toric.

- The formulas here assume only (p, p) cohomology for X .

Moreira 2020 \Rightarrow Proposes parallel Virasoro Constraints for all simply connected 3 folds X

Theorem: Virasoro Constraints hold for descendent integrals on $\text{Hilb}^n(S)$ for simply connected surfaces S

Moreira 2020

Hilbert scheme of points \rightarrow

Algebraic constructions

Let \mathbb{D}^x be the commutative \mathbb{Q} -algebra with generators

$$\left\{ ch_i(\gamma) \mid i \geq 0, \gamma \in H^*(x) \right\}$$

subject to the basic relations

$$ch_i(\lambda \cdot \gamma) = \lambda \cdot ch_i(\gamma), \quad \lambda \in \mathbb{Q}$$

$$ch_i(\gamma + \hat{\gamma}) = ch_i(\gamma) + ch_i(\hat{\gamma}), \quad \gamma, \hat{\gamma} \in H^*(x)$$

In order to define the Virasoro constraints, we require three constructions in \mathbb{D}^x :

(i) Define \mathbb{Q} -derivations for $k \geq -1$

$$R_k : \mathbb{D}^x \rightarrow \mathbb{D}^x$$

by action on the generators

$$R_k (ch_i(\gamma)) = \prod_{n=0}^k (i + d(\gamma) - 3 + n) ch_{i+k}(\gamma)$$

is complex degree
 $\gamma \in H^{2d(\gamma)}(X)$

$$R_{-1} (ch_i(\gamma)) = ch_{i-1}(\gamma)$$

Convention $ch_{j < 0}(\gamma) = 0$

(ii) Define $ch_a ch_b(\gamma) \in \mathbb{D}^X$

by the following formula

$$ch_a ch_b(\gamma) = \sum_i ch_a(\gamma_i^L) ch_b(\gamma_i^R)$$

where $\sum_i \gamma_i^L \otimes \gamma_i^R$

is the Künneth decomposition of

$$\gamma \cdot \Delta \in \mathcal{H}^*(X \times X)$$

↑ diagonal

The notation

$$(-1)^{d^L d^R} (a + d^L - 3)! (b + d^R - 3)! \text{ch}_a \text{ch}_b(\sigma)$$

will be used for

$$\sum_i (-1)^{d(\gamma_i^L) d(\gamma_i^R)} \cdot (a + d(\gamma_i^L) - 3)! (b + d(\gamma_i^R) - 3)! \cdot \text{ch}_a(\gamma_i^L) \text{ch}_b(\gamma_i^R)$$

factorials with negative arguments are defined to vanish.

d is always the complex degree

(iii) Define the operator

$$T_k : \mathbb{D}^x \rightarrow \mathbb{D}^x$$

by multiplication by the element

$$T_k = -\frac{1}{2} \sum_{a+b=k+2} (-1)^{d^L d^R} (a+d^L-3)! (b+d^R-3)! ch_a ch_b (c_1)$$

$$+ \frac{1}{24} \sum_{a+b=k} a! b! ch_a ch_b (c_1, c_2)$$

- in sums, we require $a, b \geq 0$
- $c_1, c_2 \in H^*(X)$ are the Chern classes of T_X

Virasoro Constraints

Define the constraint operator

$$L_k = T_k + R_k + (k+1)! R_{-1} ch_{k+1}(p)$$

for $k \geq -1$

Virasoro Conjecture [Moreira 00P]

X has only (p, p) cohomology

$\beta \in H_2(X, \mathbb{Z})$ curve class

$D \in \mathbb{D}^X$ is any element

Then, $\left\langle L_k(D) \right\rangle_{\beta}^X = 0$ for $k \geq -1$.

Example : $X = \mathbb{P}^3$

$$L_1(D) = (-4 \text{ch}_3(H) + R_1 + 2 \text{ch}_2(p) R_{-1}) D$$

Try $D = \text{ch}_3(p)$ and $\beta = \text{Line class } L$

Then, we obtain

$$-4 \left\langle \text{ch}_3(H) \text{ch}_3(p) \right\rangle_L^{\mathbb{P}^3}$$

$$+ 12 \left\langle \text{ch}_4(p) \right\rangle_L^{\mathbb{P}^3}$$

$$+ 2 \left\langle \text{ch}_2(p) \text{ch}_2(p) \right\rangle_L^{\mathbb{P}^3}$$

\parallel

0

Check

$$-3q + 6q^2 - 3q^3$$

$$+ 9 - 10q^2 + q^3$$

$$+ 2q + 4q^2 + 2q^3$$

\parallel

0

Theorem (Morcira OOP 2020)

Nonsingular
Projective

Let X be a toric 3fold.

For all $D \in \mathbb{D}_+^X$, ← stationary case

the Virasoro Constraints hold

$$\left\langle L_k(D) \right\rangle_B^X = 0 \text{ for } k \geq -1.$$

Define $\mathbb{D}_+^X \subset \mathbb{D}^X$ subalgebra

generated by

Stationary
descendants



$$\left\{ ch_i(\gamma) \mid i \geq 0, \gamma \in H^{>0}(X) \right\}$$

Path of proof: \mathcal{X} is a toric 3 fold

GW Virasoro constraints hold
Semisimple / Givental-Teleman 2010



Lose
Control of
descendants
of 1
here

GW/Pairs descendent
Correspondence Pixton-P 2012
formula in the OOP 2019
Stationary case



Transfer Virasoro constraints
from GW theory to stable pairs
Moreira OOP 2020

Actually, we would like to run the whole argument in the other direction.

Main Challenge: Prove the Virasoro constraints for stable pairs directly using the geometry of $P_n(x, \beta)$.

Sub challenge: Control the descendants of $1 \in H^*(X)$.

 $ch_k(1)$ insertions

for the GW/descendent Correspondence:

subject to the natural relations

$$\begin{aligned}\tau_i(\lambda \cdot \gamma) &= \lambda \tau_i(\gamma), \\ \tau_i(\gamma + \hat{\gamma}) &= \tau_i(\gamma) + \tau_i(\hat{\gamma})\end{aligned}$$

for $\lambda \in \mathbb{Q}$ and $\gamma, \hat{\gamma} \in H^*(X)$. The subalgebra $\mathbb{D}_{\text{GW}}^{X+} \subset \mathbb{D}_{\text{GW}}^X$ of stationary descendents is generated by

$$\{\tau_i(\gamma) \mid i \geq 0, \gamma \in H^{>0}(X, \mathbb{Q})\}.$$

We will use Getzler's renormalization \mathbf{a}_k of the Gromov-Witten descendents⁷:

$$(9) \quad \sum_{n=-\infty}^{\infty} z^n \tau_n = Z^0 + \sum_{n>0} \frac{(uz)^{n-1}}{(1+zc_1)_n} \mathbf{a}_n + \frac{1}{c_1} \sum_{n<0} \frac{(uz)^{n-1}}{(1+zc_1)_n} \mathbf{a}_n,$$

$$Z^0 = \frac{z^{-2}u^{-2}}{\mathcal{S}\left(\frac{zu}{\theta}\right)} - z^{-2}u^{-2},$$

On the
GW side

where we use standard notation for the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

$$\{\tau_k(\gamma)\} \leftrightarrow \{\mathbf{a}_{k+1}(\gamma)\}$$

For example⁸,

$$(10) \quad \tau_0(\gamma) = \mathbf{a}_1(\gamma) + \frac{1}{24} \int_X \gamma c_2,$$

$$(11) \quad \tau_1(\gamma) = \frac{zu}{2} \mathbf{a}_2(\gamma) - \mathbf{a}_1(\gamma \cdot c_1).$$

For $k \geq 2$ and $\gamma \in H^{>0}(X)$, we have the general formula

$$(12) \quad \tau_k(\gamma) = \frac{(zu)^k}{(k+1)!} \mathbf{a}_{k+1}(\gamma) - \frac{(zu)^{k-1}}{k!} \left(\sum_{i=1}^k \frac{1}{i} \right) \mathbf{a}_k(\gamma \cdot c_1) \\ + \frac{(zu)^{k-2}}{(k-1)!} \left(\sum_{i=1}^{k-1} \frac{1}{i^2} + \sum_{1 \leq i < j \leq k-1} \frac{1}{ij} \right) \mathbf{a}_{k-1}(\gamma \cdot c_1^2).$$

0.6. The GW/PT correspondence for essential descendents. The subalgebra

$$\mathbb{D}_{\text{PT}}^{X\star} \subset \mathbb{D}_{\text{PT}}^{X+}$$

of essential descendents is generated by

$$\{\tilde{\text{ch}}_i(\gamma) \mid (i \geq 3, \gamma \in H^{>0}(X, \mathbb{Q})) \text{ or } (i = 2, \gamma \in H^{>2}(X, \mathbb{Q}))\}.$$

While closed formulas for the full GW/PT descendent transformation of [25] are not known in full generality, the stationary theory is much better understood [17].⁹ The transformation takes the simplest form when restricted to essential descendents.

⁷We use ι for the square root of -1 . The genus variable u will usually occur together with ι .

⁸The constant term $\frac{1}{24} \int_X \gamma c_2$ in the formula does not contribute unless $\gamma \in H^2(X)$.

⁹See [13, 14] for an earlier view of descendents and descendent transformations.

Stationary GW/Pairs descendent Correspondence

The GW/PT transformation restricted to the essential descendents is a linear map

$$\mathfrak{e}^\bullet : \mathbb{D}_{\text{PT}}^{X^\star} \rightarrow \mathbb{D}_{\text{GW}}^X$$

satisfying

$$\mathfrak{e}^\bullet(1) = 1$$

and is defined on monomials by

$$\mathfrak{e}^\bullet(\tilde{\text{ch}}_{k_1}(\gamma_1) \dots \tilde{\text{ch}}_{k_m}(\gamma_m)) = \sum_{P \text{ set partition of } \{1, \dots, m\}} \prod_{S \in P} \mathfrak{e}^\circ\left(\prod_{i \in S} \tilde{\text{ch}}_{k_i}(\gamma_i)\right).$$

The operations \mathfrak{e}° on $\mathbb{D}_{\text{PT}}^{X^\star}$ are

$$(13) \quad \mathfrak{e}^\circ(\tilde{\text{ch}}_{k_1+2}(\gamma)) = \frac{1}{(k_1+1)!} \mathbf{a}_{k_1+1}(\gamma) + \frac{(vu)^{-1}}{k_1!} \sum_{|\mu|=k_1-1} \frac{\mathbf{a}_{\mu_1} \mathbf{a}_{\mu_2}(\gamma \cdot c_1)}{\text{Aut}(\mu)} \\ + \frac{(vu)^{-2}}{k_1!} \sum_{|\mu|=k_1-2} \frac{\mathbf{a}_{\mu_1} \mathbf{a}_{\mu_2}(\gamma \cdot c_1^2)}{\text{Aut}(\mu)} + \frac{(vu)^{-2}}{(k_1-1)!} \sum_{|\mu|=k_1-3} \frac{\mathbf{a}_{\mu_1} \mathbf{a}_{\mu_2} \mathbf{a}_{\mu_3}(\gamma \cdot c_1^2)}{\text{Aut}(\mu)},$$

$$(14) \quad \mathfrak{e}^\circ(\tilde{\text{ch}}_{k_1+2}(\gamma) \tilde{\text{ch}}_{k_2+2}(\gamma')) = -\frac{(vu)^{-1}}{k_1!k_2!} \mathbf{a}_{k_1+k_2}(\gamma\gamma') - \frac{(vu)^{-2}}{k_1!k_2!} \mathbf{a}_{k_1+k_2-1}(\gamma\gamma' \cdot c_1) \\ - \frac{(vu)^{-2}}{k_1!k_2!} \sum_{|\mu|=k_1+k_2-2} \max(\max(k_1, k_2), \max(\mu_1+1, \mu_2+1)) \frac{\mathbf{a}_{\mu_1} \mathbf{a}_{\mu_2}}{\text{Aut}(\mu)}(\gamma\gamma' \cdot c_1),$$

$$(15) \quad \mathfrak{e}^\circ(\tilde{\text{ch}}_{k_1+2}(\gamma) \tilde{\text{ch}}_{k_2+2}(\gamma') \tilde{\text{ch}}_{k_3+2}(\gamma'')) = \frac{(vu)^{-2}|k|}{k_1!k_2!k_3!} \mathbf{a}_{|k|-1}(\gamma\gamma'\gamma''), \quad |k| = k_1 + k_2 + k_3.$$

The above sums are over *partitions* of μ of length 2 or 3. The parts of μ are *positive* integers, and we always write

$$\mu = (\mu_1, \mu_2) \quad \text{and} \quad \mu = (\mu_1, \mu_2, \mu_3)$$

with weakly decreasing parts. In equations (13)-(15), we have $k_i \geq 0$, and all occurrences of \mathbf{a}_0 and \mathbf{a}_{-1} are set to 0.

The above formulas for the GW/PT descendent correspondence are proven here from the vertex operator formulas of [17] by a direct evaluation of the leading terms. In the toric case, we have the following explicit correspondence statement^[10]

Theorem 6. *Let X be a nonsingular projective toric 3-fold. Let*

$$\prod_{i=1}^m \tilde{\text{ch}}_{k_i}(\gamma_i) \in \mathbb{D}_{\text{PT}}^{X^\star}.$$

¹⁰A straightforward exercise using our new conventions is to show the abstract correspondence of Theorem 6 is a consequence of [25, Theorem 4]. The novelty of Theorem 6 is the closed formula for the transformation.

Let $\beta \in H_2(X, \mathbb{Z})$ with $d_\beta = \int_\beta c_1(X)$. Then, the GW/PT correspondence defined by formulas (13)-(15) holds:

$$(-q)^{-d_\beta/2} \left\langle \prod_{i=1}^m \tilde{c}h_{k_i}(\gamma_i) \right\rangle_{\beta}^{X, PT} = (-uu)^{d_\beta} \left\langle e^\bullet \left(\prod_{i=1}^m \tilde{c}h_{k_i}(\gamma_i) \right) \right\rangle_{\beta}^{X, GW},$$

after the change of variables $-q = e^{uu}$.

What is $\tilde{c}h_k(\gamma)$?

Definition: $\tilde{c}h_k(\gamma) = ch_k(\gamma) + \frac{1}{24} ch_{k-2}(\gamma \cdot c_2)$

\uparrow
 2nd Chern class
 of T_X

These formulas (and their proof in the toric case) use a lot of previous work over the past 15 years.

Okounkov-P	GW/Hurwitz
Moop	GW/DT toric
Pixton-P	Toric descendant GW/Pairs
OOP / Mor OOP	Final formulas

Hilbⁿ(S) of a surface S

If the 3fold \mathcal{X} is of the form

$$\mathcal{X} = S \times \mathbb{P}^1$$



Simply connected

nonsingular projective surface

and the curve class is $\beta = n[\mathbb{P}^1]$

then $\mathcal{P}_n(S \times \mathbb{P}^1, n[\mathbb{P}^1]) = \text{Hilb}^n(S)$.

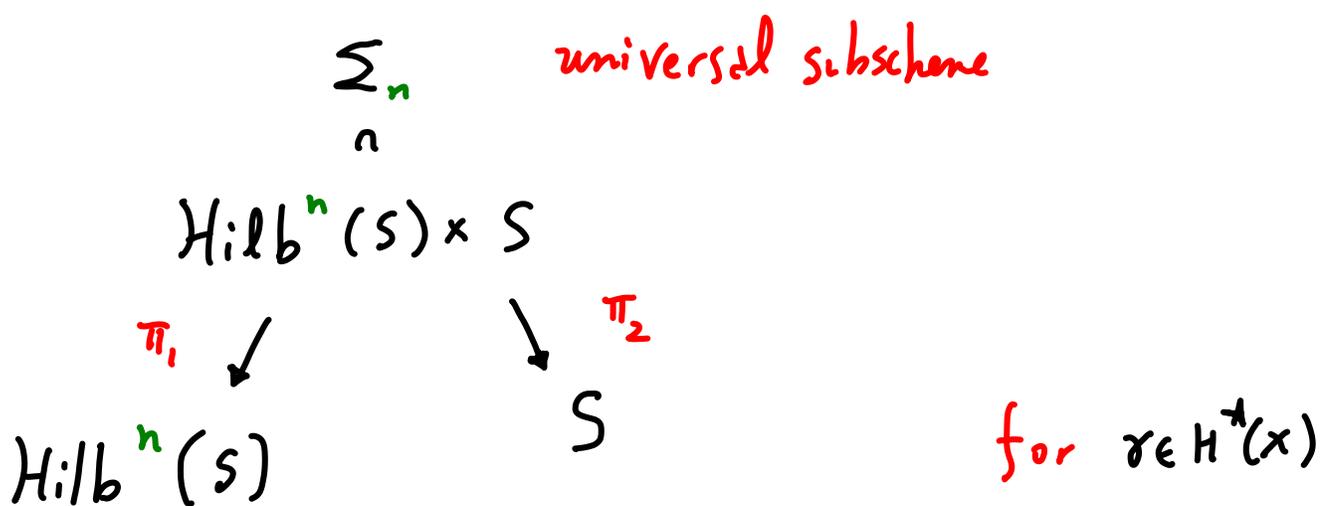
Moreover $[\mathcal{P}_n(S \times \mathbb{P}^1, n[\mathbb{P}^1])]^{\text{vir}}$ is

the usual fundamental class of $\text{Hilb}^n(S)$.

The Virasoro constraints for stable pairs on $S \times \mathbb{P}^1$ specialize to Virasoro constraints for certain descendent integrals on $\text{Hilb}^n(S)$.

Morcira's paper "Virasoro conjecture for stable pairs descendent theory of simply connected 3 folds"

What is a descendent for $\text{Hilb}^n(S)$?



$$\text{Ch}_k(\sigma) = \pi_{1*} \left(\text{Ch}_k(\Theta_{\Sigma_n} - \Theta_{H \times S}) \cdot \pi_2^*(\sigma) \right)$$

Theorem [Moreira 2020]

S is a simply connected surface

$$\int \mathcal{L}_k \left(\text{ch}_{k_1}(\gamma_1) \cdots \text{ch}_{k_r}(\gamma_r) \right) = 0$$

$\text{Hilb}^n(S)$

where

$$\mathcal{L}_k = T_k + R_k + S_k$$

very similar

to T_k, R_k

for 3-folds,

but now involve

the Hodge grading

slightly different

To define S_k :

$$R_{-1}[\alpha](ch_i(\sigma)) = ch_{i-1}(\alpha \cdot \sigma)$$

derivation on algebra \mathbb{D}^S with generators $\{ch_i(\sigma)\}$

$$S_k = (k+1)! \sum_{P_i^L=0} R_{-1}[\gamma_i^L] ch_{k+1}(\gamma_i^R)$$

where the sum runs over the terms

$\gamma_i^L \otimes \gamma_i^R$ of the Künneth decomposition

of the diagonal $\Delta \subset S \times S$ where

$$\gamma_i^L \in H^{0, 2i}(\mathbb{S}).$$



The End

Moduli Space of

Stable maps

(1) Some references (see webpage)

- Fulton - P Notes on
stable maps

- Koch - Vainsencher

Invitation to
Quantum Cohomology

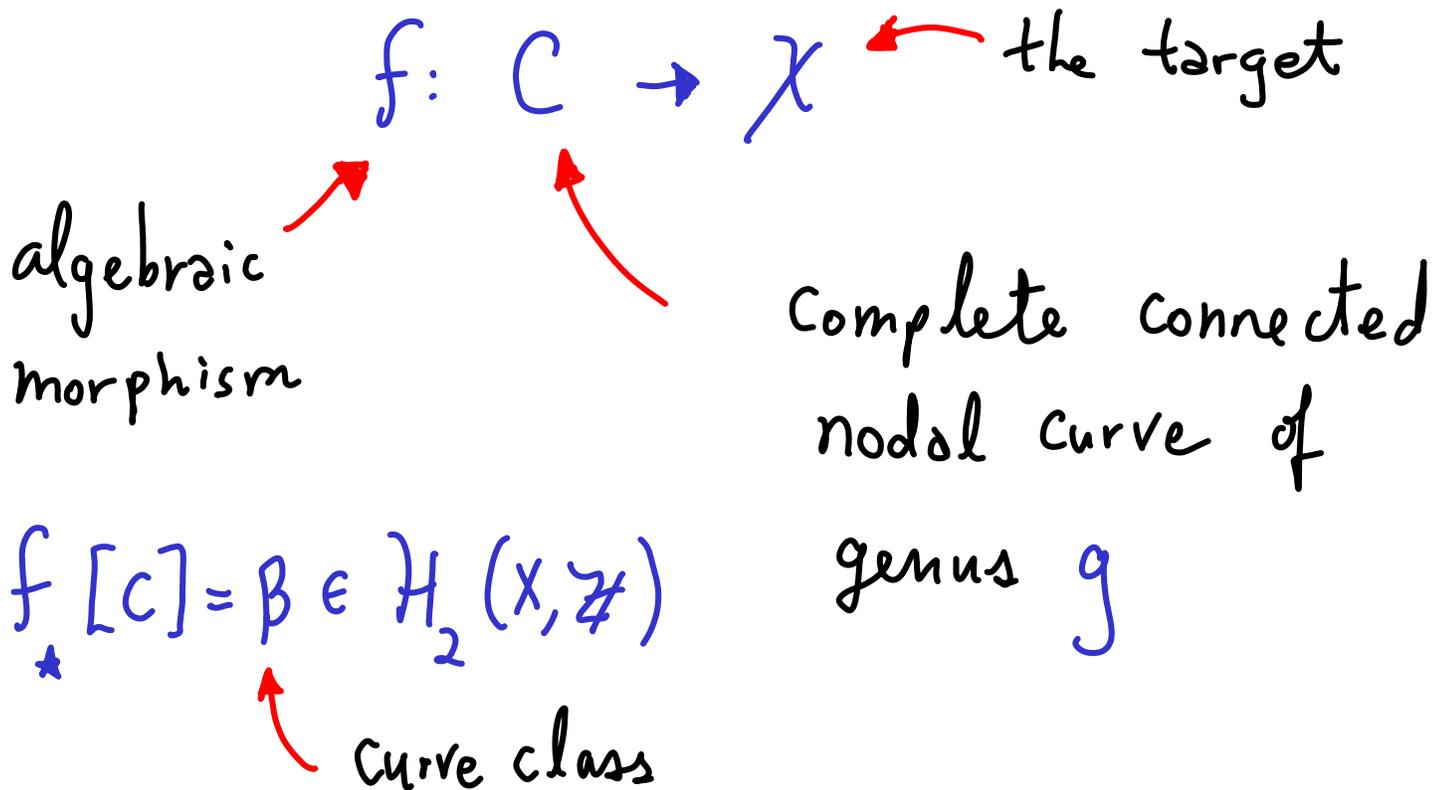
- P - Thomas $1\frac{3}{2}$ ways of
Counting Curves

(2) Definition

Let X be a Variety

nonsingular projective
is the best case

We will consider maps



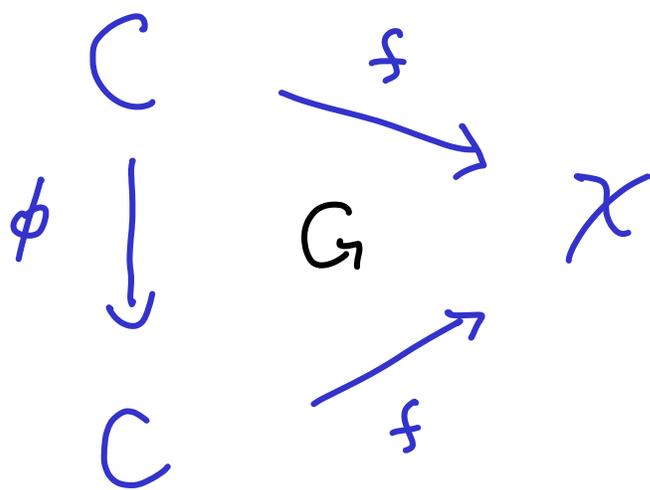
- $\overline{\mathcal{M}}_g(\mathcal{X}, \beta)$ is the moduli space of stable maps of genus g curves to \mathcal{X} representing the class $\beta \in H_2(\mathcal{X}, \mathbb{Z})$

- $[f: C \rightarrow \mathcal{X}] \in \overline{\mathcal{M}}_g(\mathcal{X}, \beta)$

is stable if and only if

$$|\text{Aut}(f)| < \infty .$$

- An automorphism of f is an automorphism of C which commutes with f :



$$\text{Aut}(f) \subset \text{Aut}(C)$$

C stable curve \Rightarrow

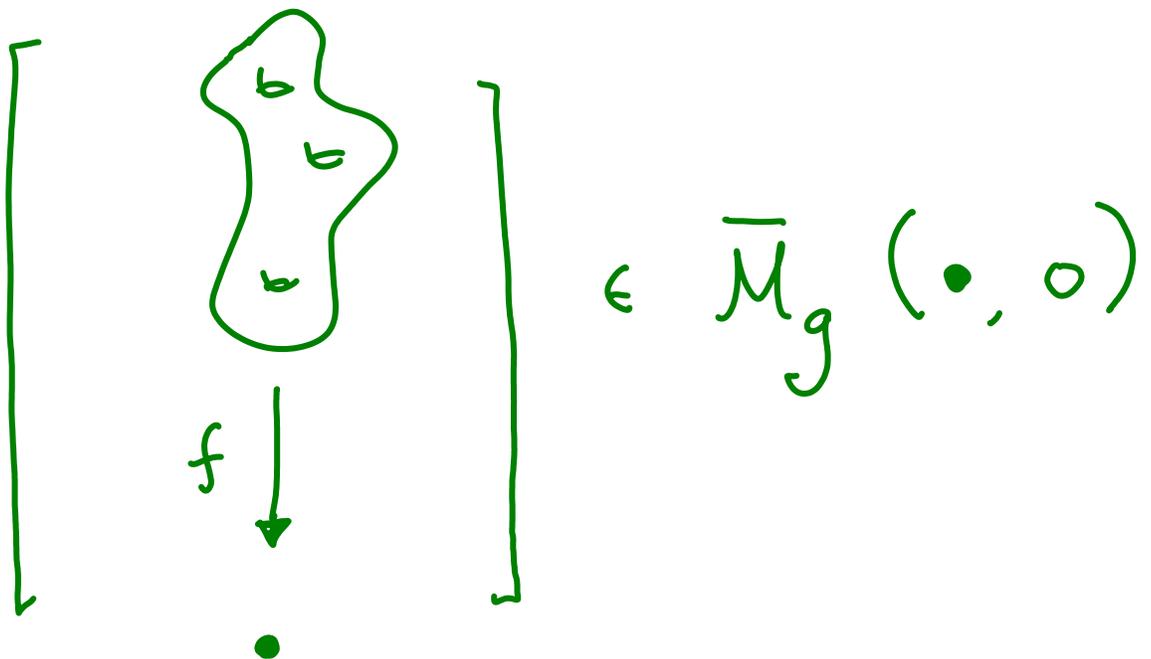
f is a stable map

Example: Maps to a point

Consider $\overline{\mathcal{M}}_g(\bullet, 0)$

Target X is
the point

$$H_2(\bullet, \mathbb{Z}) = 0$$



The morphism f carries no data
for point targets.

We have an isomorphism ($g \geq 2$)

$$\overline{\mathcal{M}}_g(\bullet, 0) \cong \overline{\mathcal{M}}_g$$

For $g = 0, 1$, the space is empty

$$\overline{\mathcal{M}}_0(\bullet, 0) \cong \emptyset$$

$$\overline{\mathcal{M}}_1(\bullet, 0) \cong \emptyset$$

because of the stability
condition.

We have discussed Automorphisms,
 but what is an isomorphism of
 stable maps?

The maps

$$f : C \rightarrow X$$

$$\hat{f} : \hat{C} \rightarrow X$$

are isomorphic if and only if

$$\exists \phi : C \xrightarrow[\text{Isom}]{} \hat{C} \quad \text{which commutes}$$

with f, \hat{f} :

$$\begin{array}{ccc}
 C & \xrightarrow{f} & X \\
 \phi \downarrow & \curvearrowright G & \\
 \hat{C} & \xrightarrow{\hat{f}} & X
 \end{array}$$

Example: genus 0 with target \mathbb{P}^1

Since $H_2(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z} \cdot [\mathbb{P}^1]$

We can represent β by an integer.

What is $\overline{\mathcal{M}}_0(\mathbb{P}^1, 1)$?

$$f: \mathbb{C} \cong \mathbb{P}^1 \xrightarrow{\text{deg } 1} \mathbb{P}^1$$

All differ by reparameterization of the domain \mathbb{P}^1 .

[Check that nothing else is stable

We see

$$\overline{\mathcal{M}}_0(\mathbb{P}^1, 1) \cong \text{point}$$

What about $\overline{\mathcal{M}}_0(\mathbb{P}^1, 2)$?

A much more interesting case.

First approach as coarse moduli

(forgetting the stack structure)

Suppose we have a map

$$f: C \cong \mathbb{P}^1 \xrightarrow{\text{deg } 2} \mathbb{P}^1,$$

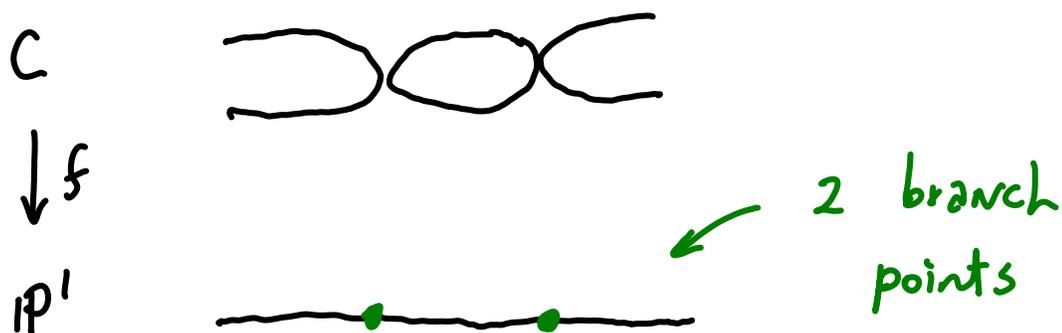
how can we classify f up
to isomorphism of stable maps?

Riemann-Hurwitz:

$$\text{br}(f) + 2(-2) = -2$$

Number of branch points of f degree of f degree of $K_{\mathbb{P}^1}$ degree of K_C

We see $\text{br}(f) = 2$.



$$\text{If } f: C \cong \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

$$\text{and } \hat{f}: \hat{C} \cong \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

are isomorphic then the two

branch points in the target \mathbb{P}^1

must be the same.

Exercise: f and \hat{f} are isomorphic

if and only if the two branch points

in the target \mathbb{P}^1 are the same.

Hint: if the two branch points are

$0, \infty \in \mathbb{P}^1$, then the map $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$

must be of the form

$$[x, y] \xrightarrow{f} [x^2, \lambda y^2] \quad \lambda \in \mathbb{C}^*$$

The locus $\mathcal{M}_0(\mathbb{P}^1, 2) \subset \overline{\mathcal{M}}_0(\mathbb{P}^1, 2)$

is therefore (coarsely) isomorphic to

$$\left(\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta \right) / \mathbb{Z}_2 \cong \mathbb{P}^2 \setminus \text{Conic}$$

↗ diagonal
 ↗ discriminant

We must then examine the broken domains

$$f: \mathbb{P}^1 \cup \mathbb{P}^1 \rightarrow \mathbb{P}^1$$



only invariant
is image of
the node

These yield a \mathbb{P}^1 parameterizing
the image of the node.

All together

$$\bar{\mathcal{M}}_0(\mathbb{P}^1, 2) \underset{\text{Coarse}}{\cong} \mathbb{P}^2$$

(3) Marked points

• $\bar{\mathcal{M}}_{g,n}(\chi, \beta)$ is the

moduli space of stable maps

to χ of genus g curves

with n markings (distinct, nonsingular)

representing the class $\beta \in H_2(\chi, \mathbb{Z})$

- $[f: (C, p_1, \dots, p_n) \rightarrow \chi] \in \overline{\mathcal{M}}_{g,n}(X, \beta)$

is stable if and only if

$$|\text{Aut}(f)| < \infty .$$

- An automorphism of f is an automorphism of C which commutes with f :

$$\begin{array}{ccc}
 C & \xrightarrow{f} & \chi \\
 \phi \downarrow & \cong & \\
 C & \xrightarrow{f} & \chi
 \end{array}$$

and fixes the markings

$$f(p_i) = p_i$$

We have an isomorphism ($2g-2+n > 0$)

$$\overline{\mathcal{M}}_{g,n}(\cdot, 0) \cong \overline{\mathcal{M}}_{g,n}$$

Stable maps are a significant generalization of the theory of stable curves.

Theorem: If \mathcal{X} is a projective variety,

$$\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$$

is a proper Deligne-Mumford Stack.

(4) Genus 0 maps to \mathbb{P}^m

The moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$

is a nonsingular Deligne-Mumford

stack (we will discuss the

deformation theory later).

What is the dimension?

$$f: (C, p_1, \dots, p_n) \rightarrow \mathbb{P}^m$$

generically $C \cong \mathbb{P}^1$

and the map f is determined

by $m+1$ polynomials

$$[x, y] \xrightarrow{f} [P_0(x, y), \dots, P_m(x, y)]$$

homogeneous of degree d

Dimension count:

- Vary $P_0, \dots, P_m \Rightarrow (m+1)(d+1)$
- projective equivalence $\Rightarrow -1$
- Reparametrization of domain $\Rightarrow -3$
- markings $\Rightarrow n$

$$\text{Total: } (m+1)(d+1) - 4 + n$$

Exercise: Use the ideas
of expressing maps

$$f: (C, p_1, \dots, p_n) \rightarrow \mathbb{P}^r$$

explicitly in terms of polynomials

to prove $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$

is a nonsingular irreducible

Deligne Mumford stack of dim

$$(n+1)(d+1) - 4 + n$$

[Fulton-Pr]

We can write the dimension formula as.

$$\dim \bar{M}_{0,n}(\mathbb{P}^m, d) = \chi(f^* T_{\mathbb{P}^m}) + 3 \cdot 0 - 3 + n$$

genus 0
↓

↓ Riemann-Roch

$$\begin{aligned} \text{degree}(f^* T_{\mathbb{P}^m}) + \text{rank}(f^* T_{\mathbb{P}^m}) \cdot (1 - 0) \\ \parallel \qquad \qquad \qquad \parallel \\ m(d+1) \qquad \qquad \qquad m \end{aligned}$$

genus 0
↓

use the Euler sequence

$$0 \rightarrow \mathcal{O} \rightarrow \bigoplus_0^m \mathcal{O}(1) \rightarrow T_{\mathbb{P}^m} \rightarrow 0$$

Definition: The virtual dimension of

$\bar{M}_{g,n}(\alpha, \beta)$ is

$$\text{vir dim} = \int_{\beta} c_1(T_x) + \dim_{\mathbb{C}}(x)(1-g) + 3g - 3 + n$$

Theorem: Every irreducible

Component of $\bar{M}_{g,n}(\alpha, \beta)$

has dimension \geq vir dim.

(5) Moduli of genus 0 maps to \mathbb{P}^2

$\bar{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$

nonsingular DM stack
of dimension $3d-1+n$

\uparrow
n markings yield
n evaluation maps

$$ev_i: \bar{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) \rightarrow \mathbb{P}^2$$

$$ev_i \left[f: (C, p_1, \dots, p_i, \dots, p_n) \rightarrow \mathbb{P}^2 \right]$$

$\equiv \text{def}$

$$f(p_i)$$

Gromov-Witten invariants :

$$\langle \gamma_1, \dots, \gamma_n \rangle_{\mathbb{P}^2, 0, d} = \int \prod_{i=1}^n ev_i^*(\gamma_i)$$

$\gamma_i \in H^*(\mathbb{P}^2)$ (green arrows)
 $0, d$ (red arrows)
 genus degree $\bar{M}_{0,n}(\mathbb{P}^2, d)$ (red arrows)
 Cohomology class (red arrow)

Meaning of the integral is :

S_n symmetric construction

$$\prod_{i=1}^n ev_i^*(\gamma_i) \cap [\bar{M}_{0,n}(\mathbb{P}^2, d)] \in H_* (\bar{M}_{0,n}(\mathbb{P}^2, d))$$

Cohomology (red arrow)
 fundamental class in Homology (red arrow)
 Project (red arrow)
 $H_0 (\bar{M}_{0,n}(\mathbb{P}^2, d))$
 \cong
 \mathbb{Q}
 value of the integral \int (red arrow)

- Dimension constraint:

$$\langle \gamma_1, \dots, \gamma_n \rangle_{0,d}^{\mathbb{P}^2} = 0$$

Unless $3d-1+n = \sum_{i=1}^n \text{degree}_d(\gamma_i)$.

- Symmetry:

$$\langle \gamma_1, \dots, \gamma_n \rangle_{0,d}^{\mathbb{P}^2} = \langle \gamma_{\sigma(1)}, \dots, \gamma_{\sigma(n)} \rangle_{0,d}^{\mathbb{P}^2}$$

$\sigma \in S_n$
symmetric group

- Fundamental class:

If $\gamma_i = 1 \in H^*(\mathbb{P}^2)$

Then $\langle \gamma_1, \dots, \gamma_n \rangle_{0,d}^{\mathbb{P}^2} = 0$

Unless $n=3$ and $d=0$,

$$\langle \gamma_1, \gamma_2, 1 \rangle = \int_{\mathbb{P}^2} \gamma_1 \cup \gamma_2$$

- Divisor equation

If $\delta_1 = H$ then

Cohomology of \mathbb{P}^2

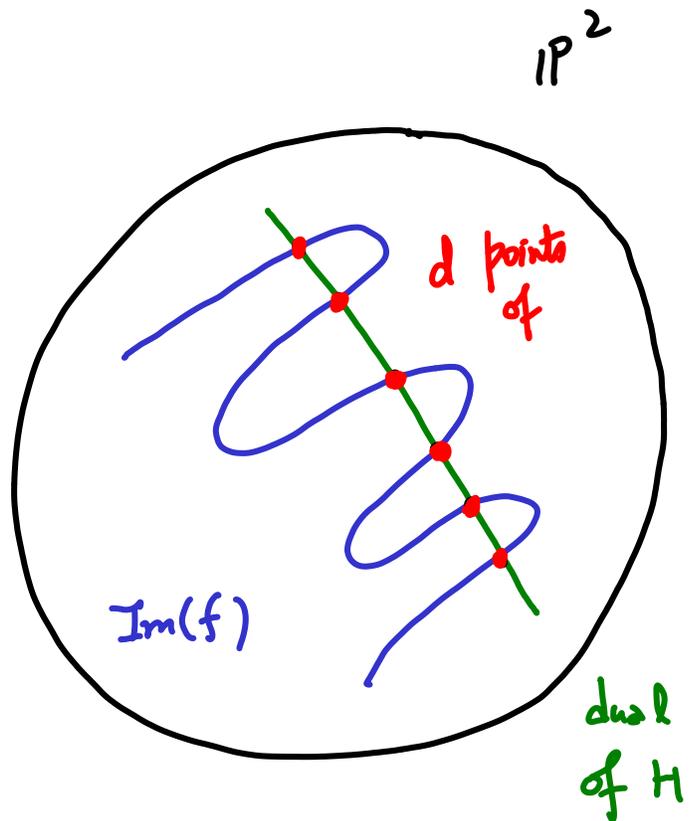
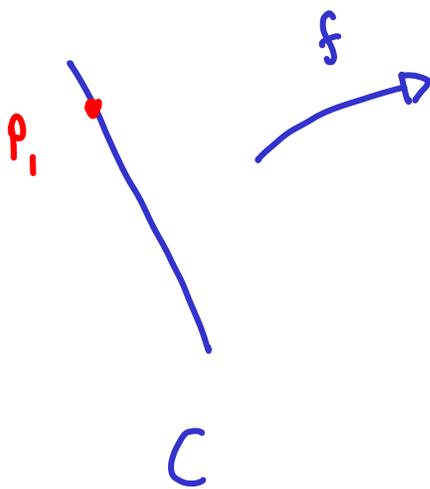
$$H^0 \cong \mathbb{Q} \cdot 1$$

$$H^2 \cong \mathbb{Q} \cdot H$$

$$H^4 \cong \mathbb{Q} \cdot P$$

$$\langle H, \gamma_2, \dots, \gamma_n \rangle_{0,d}^{\mathbb{P}^2} = d \langle \gamma_2, \dots, \gamma_n \rangle_{0,d}^{\mathbb{P}^2}$$

Why?



$$\left[\text{exceptional case } \langle H, \gamma_2, \gamma_3 \rangle_{0,3}^{\mathbb{P}^2} = \int_{\mathbb{P}^2} H \cup \gamma_2 \cup \gamma_3 \right]$$

In fact, since $\overline{M}_{0,3}(\mathbb{P}^2, 0) \cong \overline{M}_{0,3} \times \mathbb{P}^2$
 $\cong \mathbb{P}^2$

We easily see:

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{0,3}^{\mathbb{P}^2} = \int_{\mathbb{P}^2} \gamma_1 \cup \gamma_2 \cup \gamma_3$$

What is left?

$$\langle \underbrace{p, \dots, p}_{3d-1} \rangle_{0,d}^{\mathbb{P}^2} \stackrel{\text{def}}{=} \mathcal{N}_d$$

Can we compute \mathcal{N}_d ?

Yes! But we require an additional idea.

Consider the forgetful morphism

$$\varepsilon: \bar{M}_{0,n}(\mathbb{P}^2, d) \rightarrow \bar{M}_{0,4} \quad \left[\begin{array}{l} \text{Require} \\ n \geq 4 \end{array} \right]$$

which forgets everything except the first 4 markings.

We have the cross ratio relation

$$\left[\begin{array}{c} 1 \\ \circlearrowleft \\ 2 \end{array} \circlearrowright \begin{array}{c} 3 \\ \circlearrowleft \\ 4 \end{array} \right] = \left[\begin{array}{c} 1 \\ \circlearrowleft \\ 3 \end{array} \circlearrowright \begin{array}{c} 2 \\ \circlearrowleft \\ 4 \end{array} \right] \in H^2(\bar{M}_{0,4})$$

We simply pull back the relation by ε .

We obtain

$$\sum_{\substack{A \cup B = [5, 6, \dots, n] \\ d_1 + d_2 = d}} \left[\begin{array}{c} \text{A} \quad \text{B} \\ \begin{array}{ccc} 1 & & 3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 2 & & 4 \end{array} \\ d_1 \quad d_2 \end{array} \right] \quad \text{WDVV relation}$$

=

$$\sum_{\substack{A \cup B = [5, 6, \dots, n] \\ d_1 + d_2 = d}} \left[\begin{array}{c} \text{A} \quad \text{B} \\ \begin{array}{ccc} 1 & & 2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 3 & & 4 \end{array} \\ d_1 \quad d_2 \end{array} \right]$$

in $\mathcal{H}^2(\bar{\mathcal{M}}_{0,n}(\mathbb{P}^2, d))$

After applying $\prod_{i=1}^n \text{ev}_i^*(\gamma_i)$ and

integrating \int we obtain

$$\bar{M}_{0,n}(\mathbb{P}^2, d)$$

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_2, \gamma_{i \in A}, \cdot \rangle_{d_1} \xrightarrow{\text{diagonal}} \langle \cdot, \gamma_{i \in B}, \gamma_3, \gamma_4 \rangle_{d_2}$$

||

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_3, \gamma_{i \in A}, \cdot \rangle_{d_1} \xrightarrow{\text{diagonal}} \langle \cdot, \gamma_{i \in B}, \gamma_2, \gamma_4 \rangle_{d_2}$$

The diagonal has kineth components

$$1 \otimes P + H \otimes H + P \otimes 1$$

So the left side is

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_2, \gamma_{i \in A}, 1 \rangle_{d_1} \langle P, \gamma_{i \in B}, \gamma_3, \gamma_4 \rangle_{d_2} +$$

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_2, \gamma_{i \in A}, H \rangle_{d_1} \langle H, \gamma_{i \in B}, \gamma_3, \gamma_4 \rangle_{d_2} +$$

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_2, \gamma_{i \in A}, P \rangle_{d_1} \langle 1, \gamma_{i \in B}, \gamma_3, \gamma_4 \rangle_{d_2}$$

and the right side is

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_3, \gamma_{i \in A}, 1 \rangle_{d_1} \langle P, \gamma_{i \in B}, \gamma_2, \gamma_4 \rangle_{d_2} +$$

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_3, \gamma_{i \in A}, H \rangle_{d_1} \langle H, \gamma_{i \in B}, \gamma_2, \gamma_4 \rangle_{d_2} +$$

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_3, \gamma_{i \in A}, P \rangle_{d_1} \langle 1, \gamma_{i \in B}, \gamma_2, \gamma_4 \rangle_{d_2}$$

The equality is the WDVV equation in

Gromov-Witten theory.

We apply the above WDVV equation with

$$d \geq 2$$

$$\gamma_1 = H, \gamma_2 = H, \gamma_3 = p, \gamma_4 = p$$

$$\gamma_5 = \dots = \gamma_{3d} = p$$

Total of
 $3d-2$ point
Conditions

Left side

$$N_d + \sum_{\substack{d_1+d_2=d \\ d_i > 0}} d_1^3 d_2 \binom{3d-4}{3d_1-1} N_{d_1} N_{d_2}$$

Right side

$$\sum_{\substack{d_1+d_2=d \\ d_i > 0}} d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N_{d_1} N_{d_2}$$

Together, for $d \geq 2$:

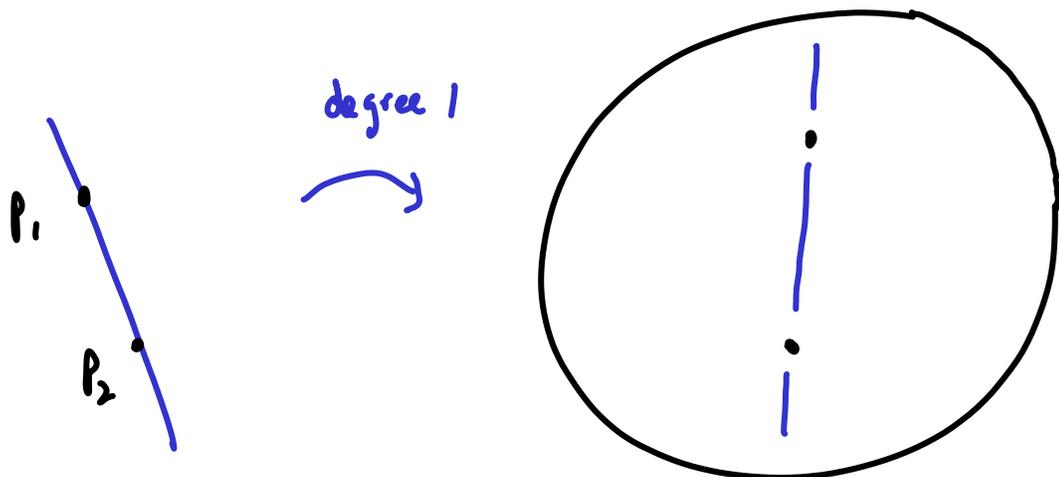
Kontsevich 94

$$N_d + \sum_{\substack{d_1+d_2=d \\ d_i > 0}} d_1^3 d_2^3 \binom{3d-4}{3d_1-1} N_{d_1} N_{d_2}$$

$$= \sum_{\substack{d_1+d_2=d \\ d_i > 0}} d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N_{d_1} N_{d_2}$$

Exercise: $N_1 = 1$

Unique line through
2 points [Euclid]



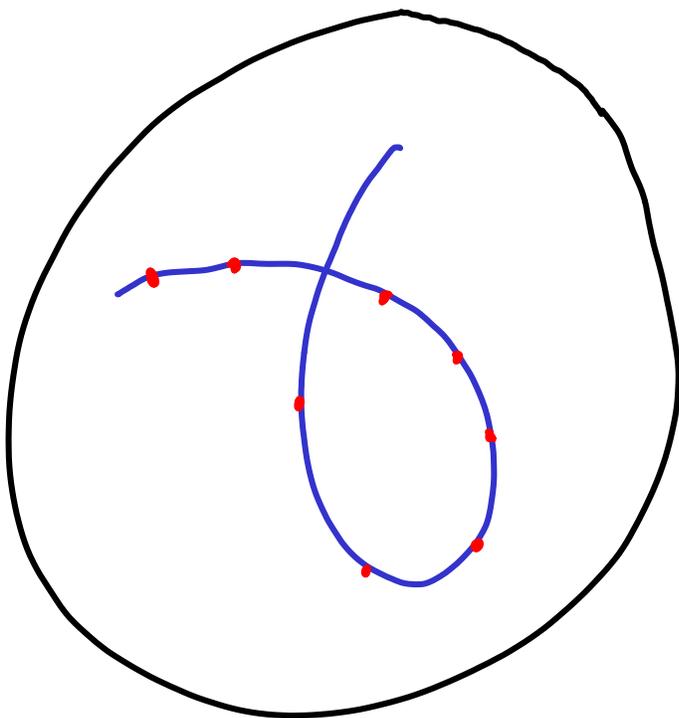
Yes: N_d is the number of

genus 0 curves in \mathbb{P}^2 of degree d

passing through $3d-1$ generic points.

Proof uses
Bertini's Theorem

Example: $N_3 = 12$



12 nodal
cubic passing
through 8
general points

(6) Quantum Cohomology for \mathbb{P}^2

Ordinary Cohomology : $H^*(\mathbb{P}^2)$ is \mathbb{Q} -algebra

Let t_0, t_1, t_2 be coordinates

$$H^*(\mathbb{P}^2) = \{ t_0 \cdot 1 + t_1 \cdot H + t_2 \cdot P \mid t_1, t_2, t_3 \in \mathbb{Q} \}$$

Quantum Cohomology : $QH^*(\mathbb{P}^2)$ is a

$\mathbb{Q}[[t_0, t_1, t_2]]$ -algebra

$QH^*(\mathbb{P}^2)$ is free $\mathbb{Q}[[t_0, t_1, t_2]]$ -module

with basis $1, H, P$

and a $\mathbb{Q}[[t_0, t_1, t_2]]$ -linear

quantum product \star

Quantum product is determined by

$$\langle \gamma_1 * \gamma_2, \gamma_3 \rangle$$

$\gamma_i \in H^*(\mathbb{P}^2)$ quantum product Poincaré pairing

$$\langle \mu, \nu \rangle = \int_{\mathbb{P}^2} \mu \cup \nu$$

Formula for the

Quantum product:

$$\langle \gamma_1 * \gamma_2, \gamma_3 \rangle = \sum_{d=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \langle \gamma_1, \gamma_2, \gamma_3, T^n \rangle_{0,d}^{\mathbb{P}^2}$$

$$T = t_0 \cdot 1 + t_1 \cdot H + t_2 \cdot P$$

Exercise: $\mathbb{Q}H^*(\mathbb{P}^2)$ with \star

is a commutative, associative
ring with unit $1 \in H^*(\mathbb{P}^2)$.

Requires
WDVV equations

Let X be a nonsingular

projective variety. Then the quantum cohomology

$$\left(\mathbb{Q}H^*(X), \star, 1 \right)$$

is defined in the same way

- \mathbb{Z}_2 -grading, supercommutative

- $\overline{M}_{0,n}(X, \beta)$ may not be of

the expected dimension.

Need virtual
fundamental class

(7)

QH^* , QK^*

less well known

Cohomology

All computable

Poincaré poly

Connection to enumerative geometry

All fantological

[YPLee Phd]

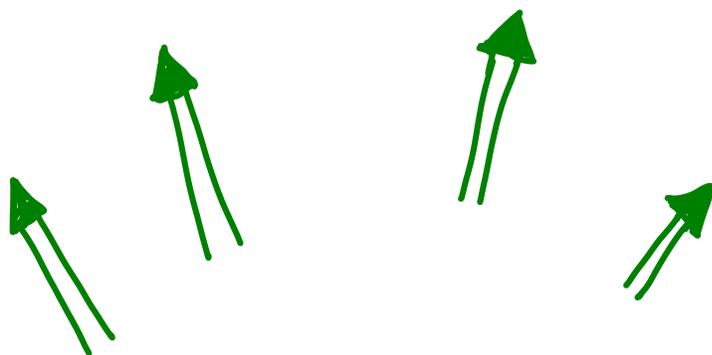
Birational geometry:

[Getzler-P, Oprea]

Rational

$\bar{M}_{0,n}(G/\rho, \beta)$

[B. Kim-P]



A lot known

(ask Andrew Kresch)

$\bar{M}_{0,n}(\mathbb{P}^m, d)$

Proper

Nonsingular

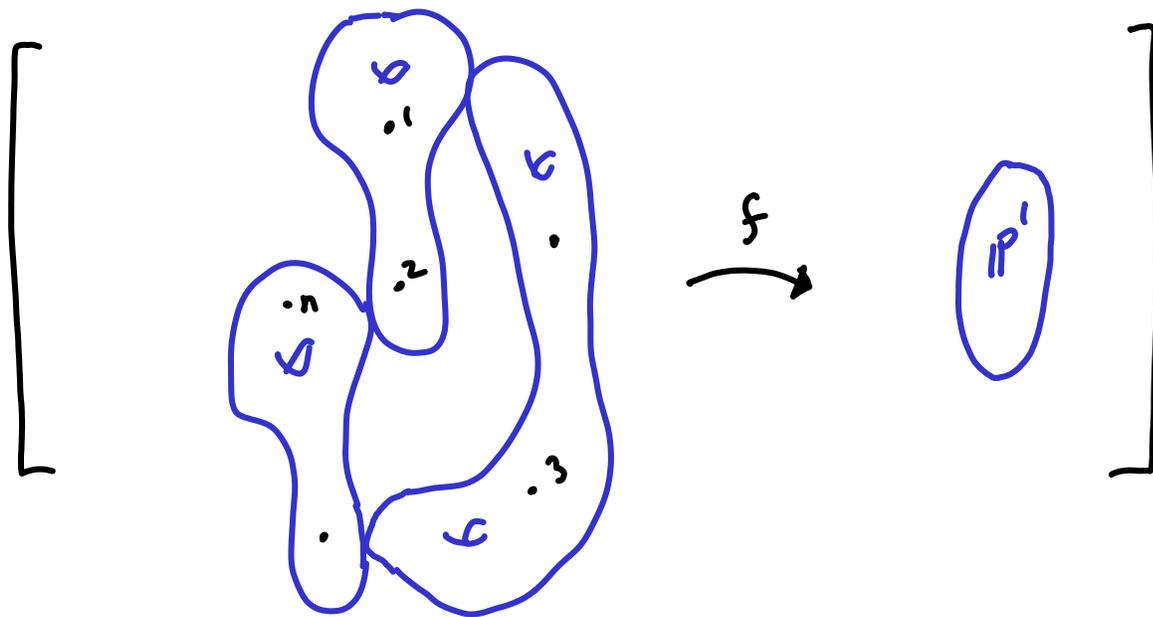
Deligne-Mumford Stack

(8) Maps to \mathbb{P}^1 and the Abel-Jacobi problem

We have already defined and discussed

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$$

\Downarrow



But to address the questions of universal Abel-Jacobi theory, we need a new construction. Why?

Let $A = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$

satisfy $\sum_{i=1}^n a_i = 0$

Assume for
simplicity that
 $\forall i, a_i \neq 0$

Let $(C, p_1, \dots, p_n) \in \mathcal{M}_{g,n}$

satisfy $\Theta_C(\sum a_i p_i) \cong \Theta_C$

Abel-Jacobi
Condition

Then $\exists f$ meromorphic function

$$f: C \rightarrow \mathbb{P}^1.$$

with zero and poles determined

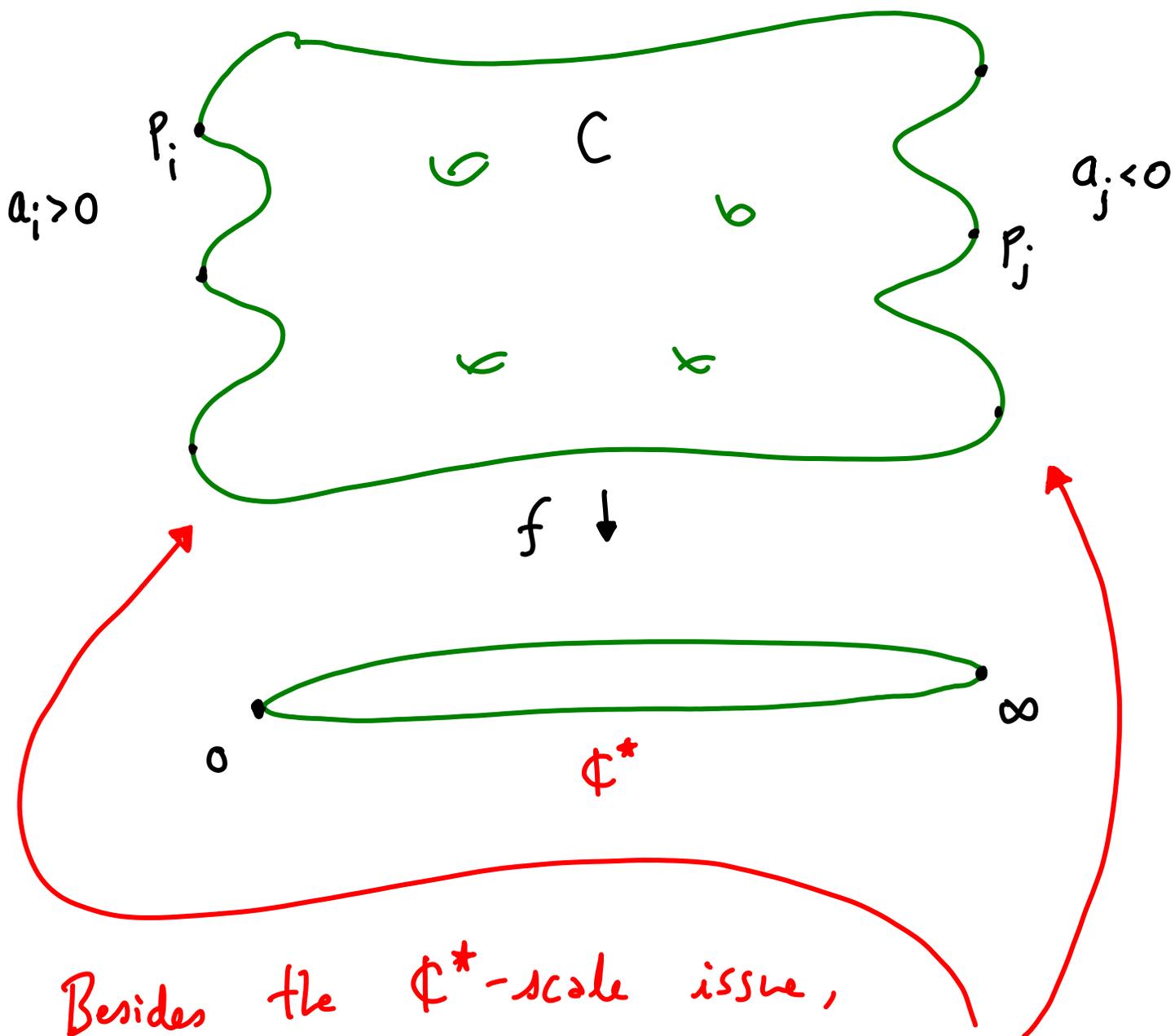
by A . So what is the problem?

f is only determined up to \mathbb{C}^* scale.

Geometrically: $Z \subset M_{g,n}$


locus of solutions
to the Abel-Jacobi
problem for A .

Then, Z is almost
isomorphic to a moduli space of
maps to \mathbb{P}^1 (up to the \mathbb{C}^* -scale).



Besides the ϕ^* -scale issue,
 we also have fixed profiles
 over $0, \infty \in \mathbb{R}^1$

Moduli space of stable relative maps
 solves both problems!

(9) Stable relative maps

Let $g \geq 0$ be the genus

Let μ, ν be two ordered partitions of $d \geq 0$

$$\mu = (\mu_1, \dots, \mu_{l(\mu)})$$

$$\nu = (\nu_1, \dots, \nu_{l(\nu)})$$

Think of these as
coming from the
positive and negative
parts of A

We will define a moduli space

$$\bar{M}_g(\mathbb{P}^1 / 0, \infty)_{\mu, \nu}$$

Relative to the points
 $0, \infty \in \mathbb{P}^1$

determine profiles
over 0 and ∞

ϕ^* equivalence

No bar

Abel-Jacobi solutions

$$M_g(\mathbb{P}^1/0, \infty)_{\mu, \nu}^{\sim} \cong Z \subset M_{g, n}$$

$$A = (a_1, \dots, a_n) \quad a_i \neq 0$$

$$\mu = (\mu_1, \dots) \quad \nu = (\nu_1, \dots)$$

↑
positive parts of A

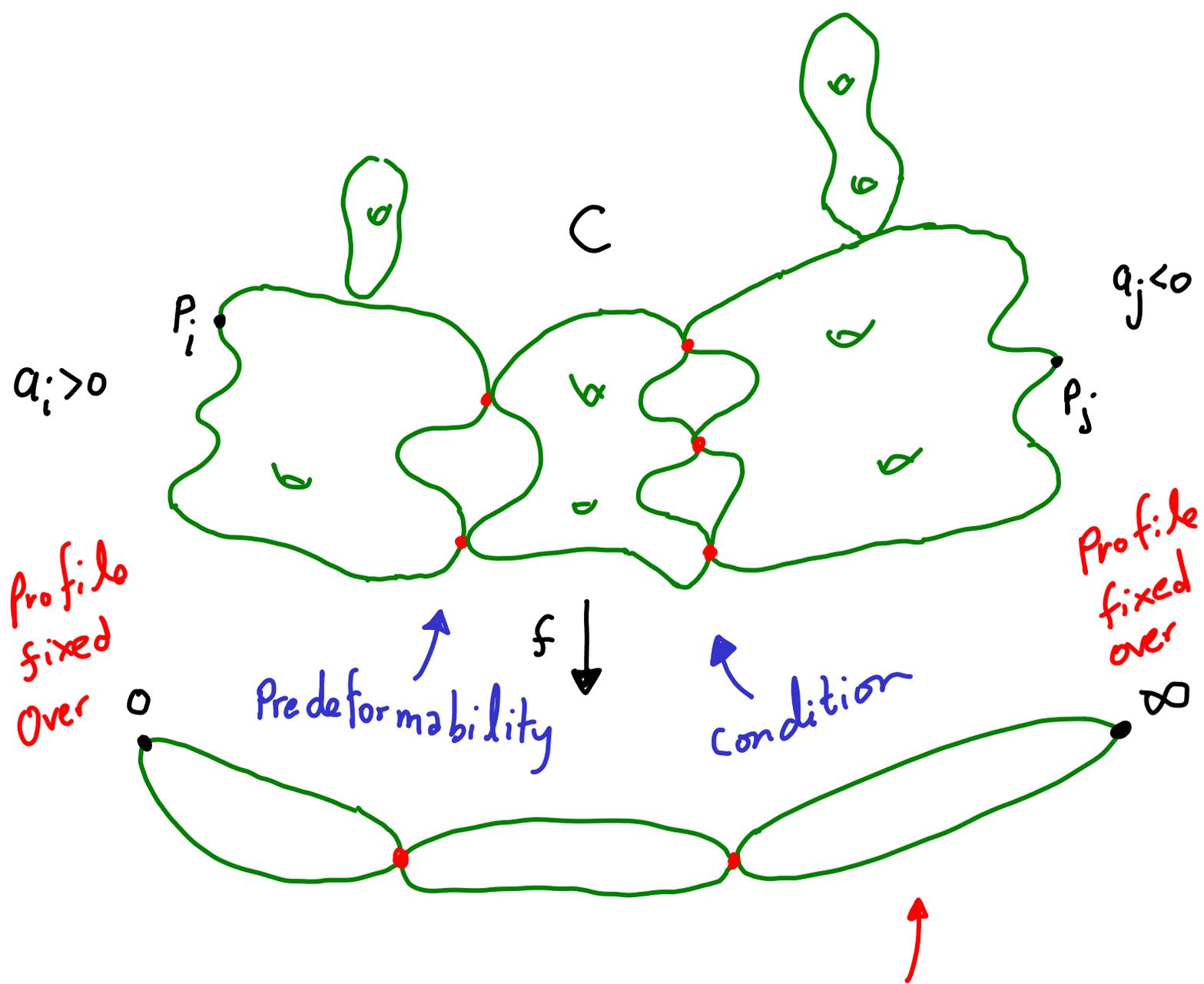
↑
absolute value of the negative parts of A

The geometric matching is perfect on the open set ↷

$$M_g(\mathbb{P}^1/0, \infty)_{\mu, \nu}^{\sim} \subset \bar{M}_g(\mathbb{P}^1/0, \infty)_{\mu, \nu}^{\sim}$$

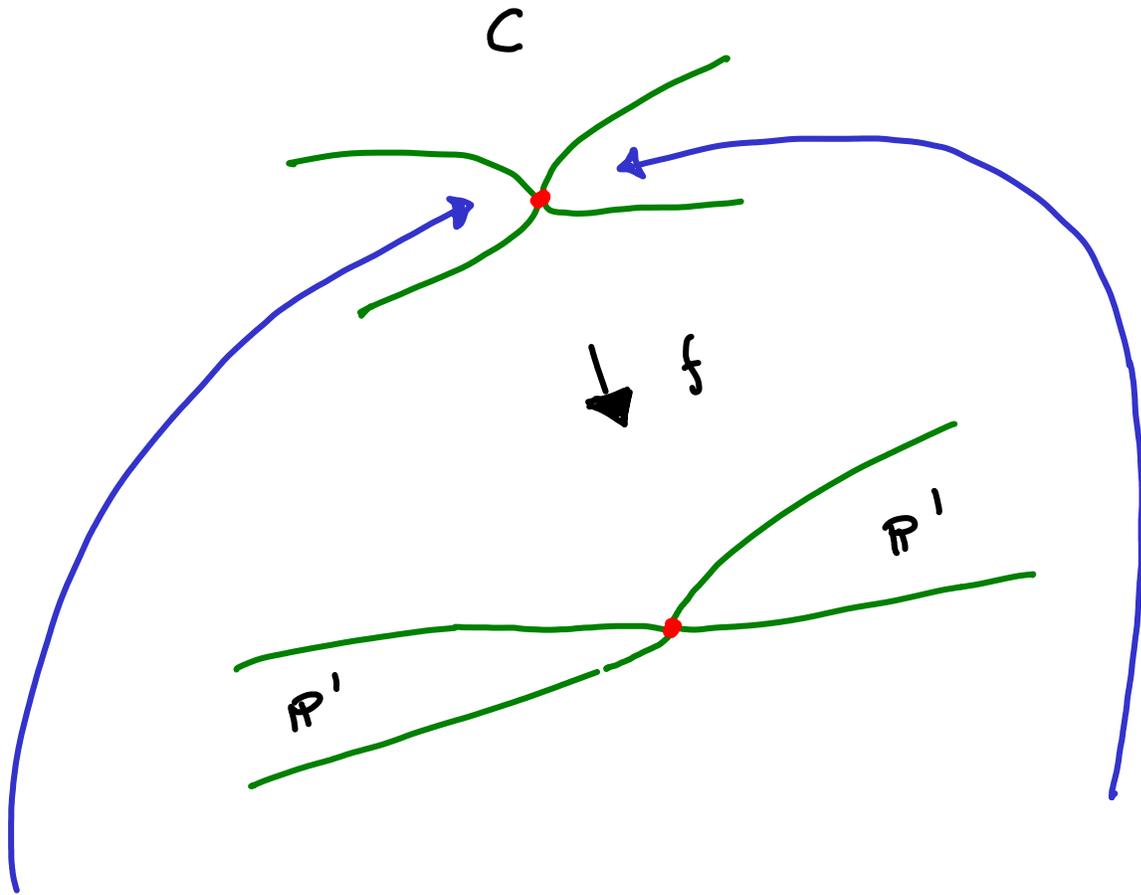
↑
What happens here?

A Stable relative map is of the following form:



Target is a Chain of \mathbb{P}^1_Δ

- Predeformability



Each side has a ramification number.

Predeformability: The two numbers are equal.

Why is this related to
deformability?

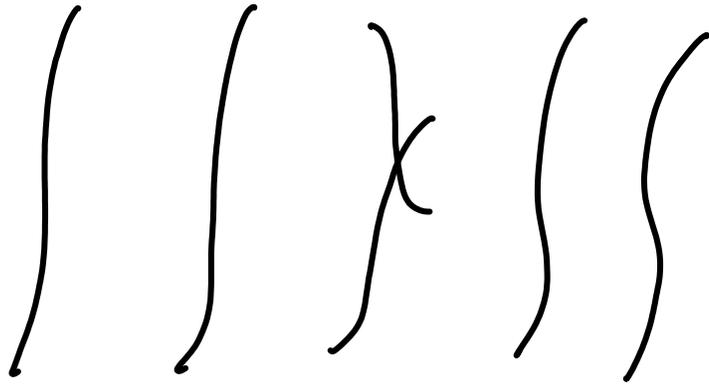
Exercise: Let

\mathcal{C}
 $f \downarrow$
 \mathcal{T}

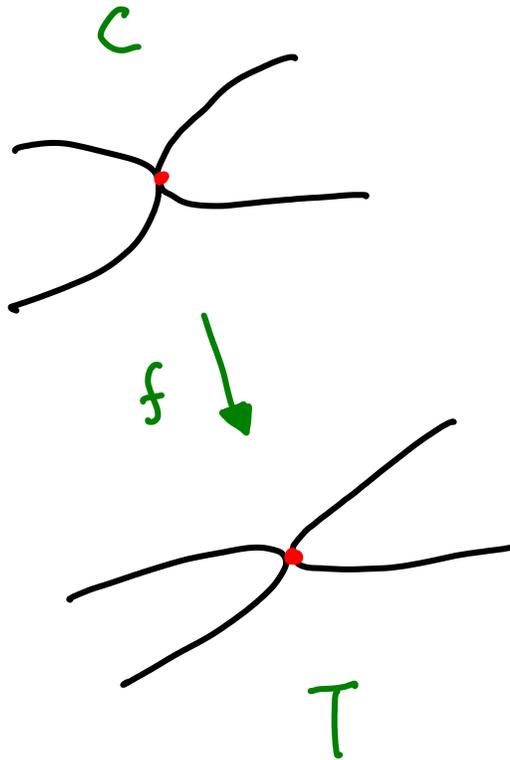
be a flat
family of maps

where \mathcal{T} is a 1-parameter
deformation of a
nodal curve

T



and C looks like

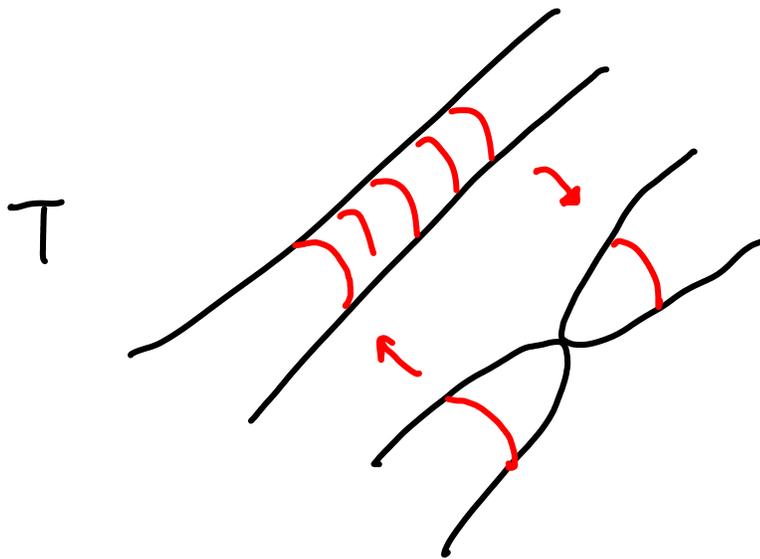


Over the node of T and
étale elsewhere

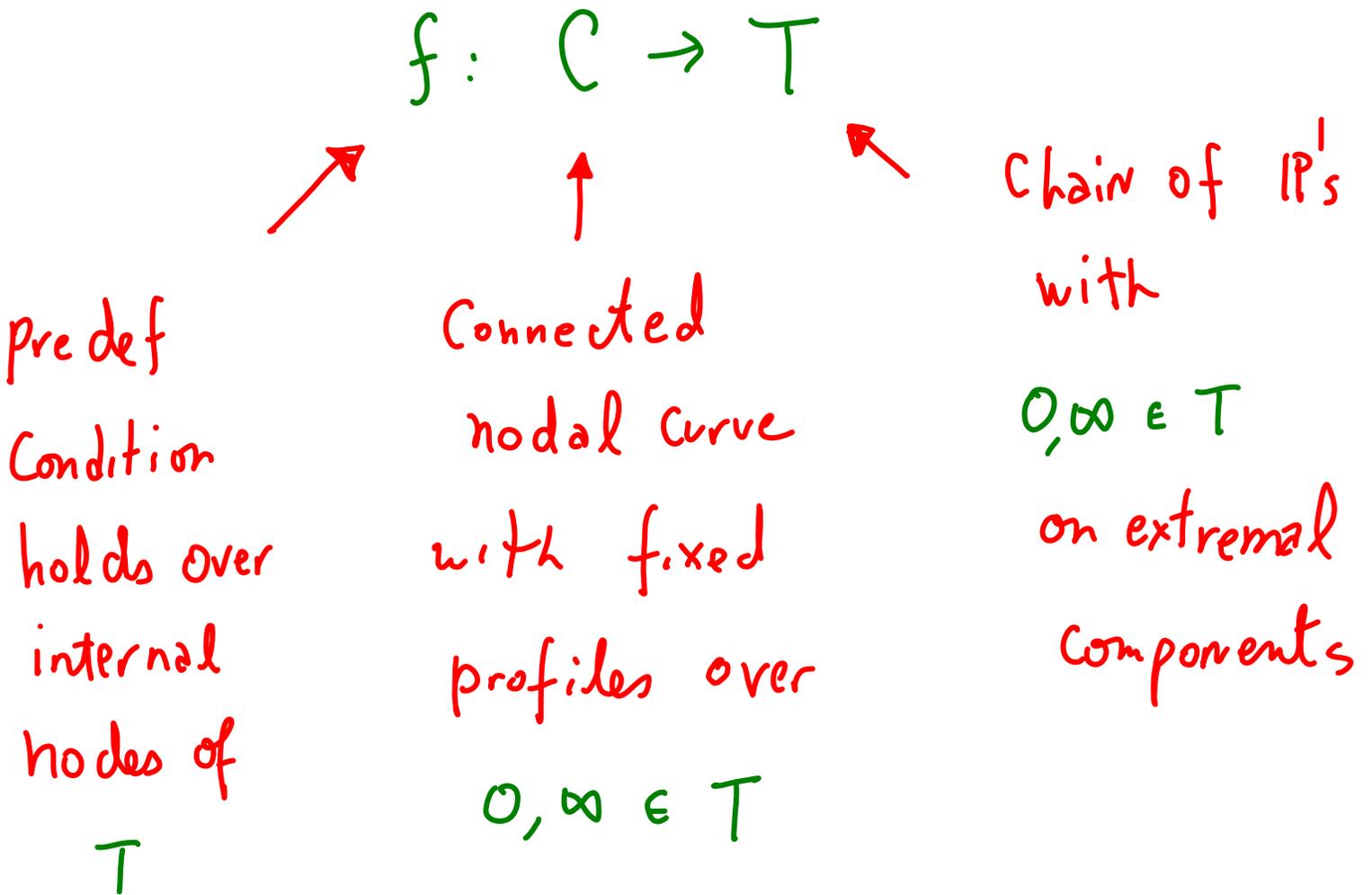
Prove the predeformability

condition holds.

Hint: Look what happens on C
to the preimages of the red loops.



• Stability



Stability Condition:

f has finite Automorphisms.

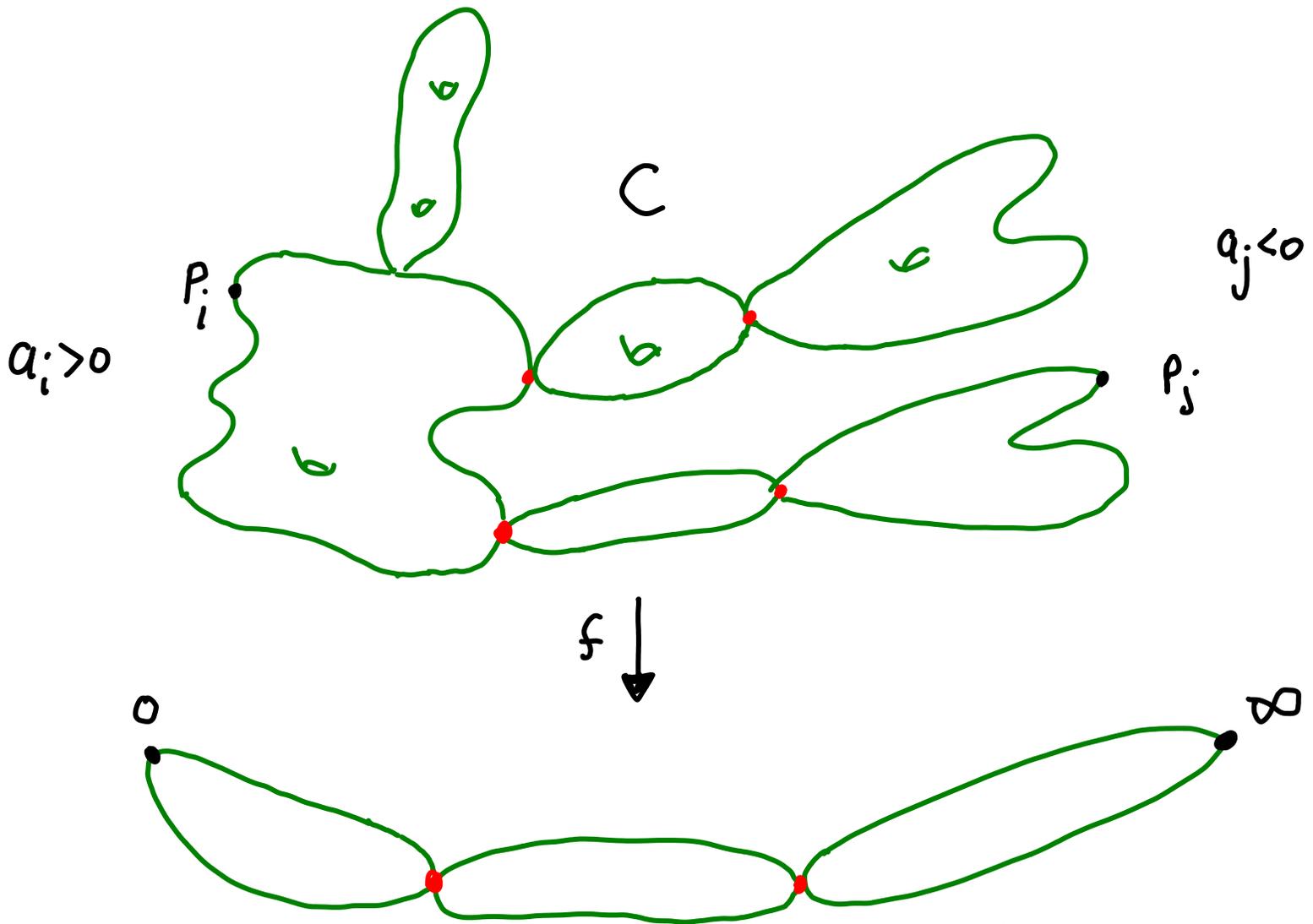
- An automorphism of f is an automorphism of C and a scaling automorphism of T which commutes with f :

$$\begin{array}{ccc}
 C & \xrightarrow{f} & T \\
 \phi \downarrow & G & \downarrow \delta \\
 C & \xrightarrow{f} & T
 \end{array}$$

A scaling Act of T is

ϕ^* scale on each Component.

Another possible picture of a
stable relative map:



Target is a Chain of \mathbb{P}^1_Δ

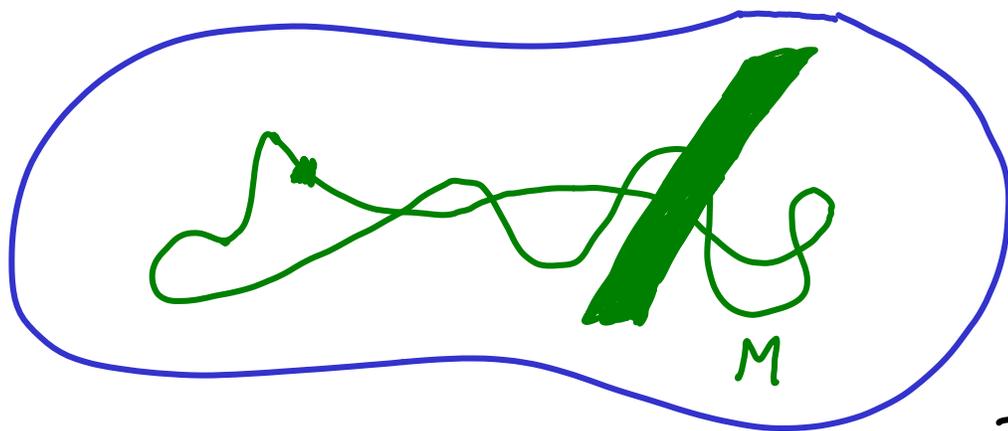
(9) Deformation theory and the virtual fundamental class

General theory (Behrend-Fantechi)

Let M be scheme (or DM stack).

The cotangent complex (2-term cut off):

$$L_M^\bullet = \left[\begin{array}{c} \mathcal{I}_M / \mathcal{I}_M^2 \\ \rightarrow \Omega_Y|_M \end{array} \right] \in D_{\text{Coh}}^b(M)$$



$M \subset Y$ not canonical, but

$L_M^\bullet \in D_{\text{Coh}}^b(M)$ is canonical.

$\mathcal{I}_M \subset \mathcal{O}_Y$ ideal
sheaf defining M

$$h^0(L_M^\bullet) = \text{Coker} \left(\begin{array}{c} I_M \\ \xrightarrow{I_M^2} \end{array} \Omega_Y|_M \right)$$

$$\cong \Omega_M \quad \leftarrow \text{Sheaf of differentials}$$

A 2 term obstruction theory for M is

$$E^\bullet \xrightarrow{\phi} L_M^\bullet$$

where $E^\bullet = [E^{-1} \rightarrow E^0] \in D_{\text{Coh}}^b(M)$

is represented by 2 vector bundles and

ϕ is a morphism in $D_{\text{Coh}}^b(M)$ satisfying:

- ϕ^0 is an isomorphism on h^0
- ϕ^{-1} is surjective on h^1

By the first condition,

$$h^0(E^\bullet) \cong_{\phi^0} \Omega_M$$



Information about the Zariski tangent space of M .

Let $p \in M$, then we have

$$0 \rightarrow \text{Ker} \rightarrow E_p^{-1} \rightarrow E_p^0 \rightarrow \Omega_{M,p} \rightarrow 0$$

We dualize (\mathbb{C} -vector spaces) $E_i = (E^{-i})^\vee$

$$0 \rightarrow T_{M,p} \rightarrow E_{0,p} \rightarrow E_{1,p} \rightarrow \text{Obs} \rightarrow 0$$


Tangent space


obstruction space

We can assume we have maps

$$\phi \quad \begin{array}{ccc} E^{-1} & \rightarrow & E^0 \\ \downarrow & & \downarrow \\ \mathcal{I}_M / \mathcal{I}_M^2 & \rightarrow & \Omega_y|_M \end{array}$$

The condition of an obstruction theory \Rightarrow

$$E^{-1} \rightarrow E^0 \oplus \mathcal{I}_M / \mathcal{I}_M^2 \xrightarrow{\gamma} \Omega_y|_M \rightarrow 0$$

Then we obtain an exact sequence of abelian cones:

$$0 \rightarrow T_y \rightarrow E_0 \times_M (\mathcal{I}_M / \mathcal{I}_M^2) \rightarrow C(Q) \rightarrow 0$$

$C(Q)$ is the cone associated to $\text{Ker}(\gamma)$.

The Normal cone of M in Y is

$$C(M/Y) = \text{Spec} \left(\bigoplus_{k=0}^{\infty} \frac{I_M^k}{I_M^{k+1}} \right)$$

So $C(M/Y) \subset C\left(\frac{I_M}{I_M^2}\right)$

↑ pure dim Y

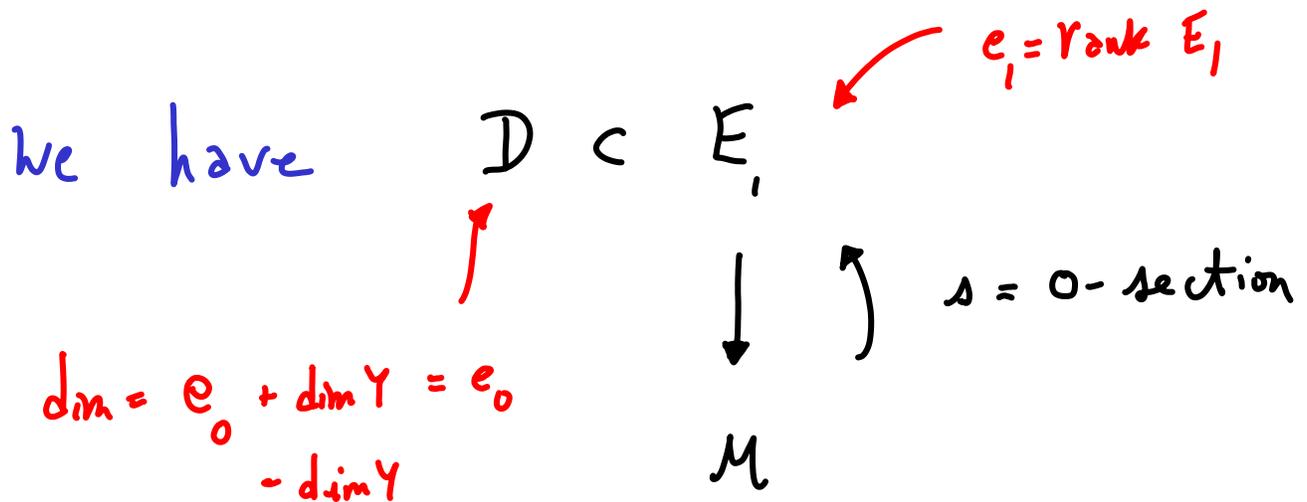
We have

$$E_0 \times_M C(M/Y) \subset E_0 \times_M C\left(\frac{I_M}{I_M^2}\right)$$

Behrend-Fantechi: T_Y -cone

Let $D = \frac{E_0 \times_M C(M/Y)}{T_Y} \subset C(\mathbb{Q})$

Since $C(Q) \subset E$,



Definition: $[M]^{\text{vir}} \in A_{e_0 - e_1}(M)$

virtual dim = $e_0 - e_1$

$$[M]^{\text{vir}} \stackrel{\text{def}}{=} s^*([D])$$

(10) Virtual class of the moduli of maps

There are three stages of the analysis

- Moduli of maps from a fixed curve C to X \leftarrow Nonsingular projective variety

- $\bar{M}_{g,n}(X, \beta)$
 \leftarrow involves varying complex structure on the domain

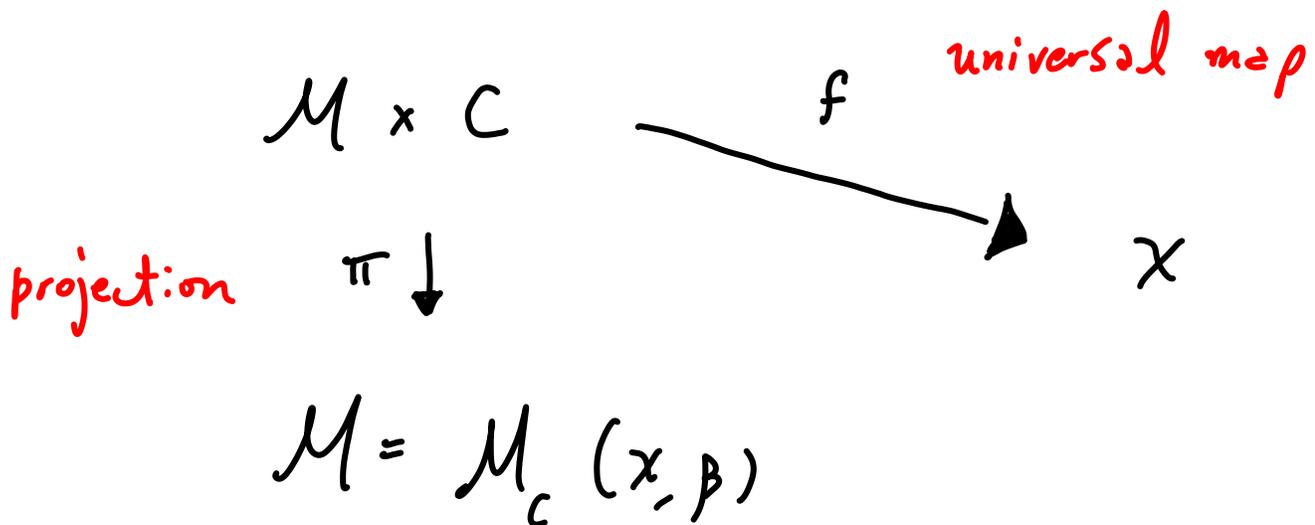
- $\bar{M}_g(\mathbb{P}^1 / 0, \infty)_{\mu, \nu}^{\sim}$
 \leftarrow most relevant for Abel-Jacobi Theory

Start with the simplest:

C \leftarrow fixed nonsingular curve of genus g

X \leftarrow fixed nonsingular projective target variety

$M_C(X, \beta)$ \leftarrow moduli of maps representing
 $\beta \in H_2(X, \mathbb{Z})$



We have

$$f^* \Omega_x \xrightarrow{df} L_{M \times C}^{\bullet} \cong \pi^* L_M^{\bullet} \oplus L_C^{\bullet}$$

Then $f^* \Omega_x \rightarrow \pi^* L_M^{\bullet}$ via projection

so $(\pi^* L_M^{\bullet})^{\vee} \rightarrow f^* T_x$

Then $R\pi_* \left((\pi^* L_M^{\bullet})^{\vee} \right) \rightarrow R\pi_* f^* T_x$

so $(R\pi_* f^* T_x)^{\vee} \rightarrow R\pi_* \left((\pi^* L_M^{\bullet})^{\vee} \right)^{\vee}$

$$R\pi_{\star} \left(\left(\pi^* L_{\mathcal{M}}^{\bullet} \right)^{\vee} \right)^{\vee} =$$

$$\begin{aligned} \text{Then } R\pi_{\star} \left(\pi^* \left(L_{\mathcal{M}}^{\bullet \vee} \right) \right)^{\vee} &= \left(L_{\mathcal{M}}^{\bullet \vee} \otimes R\pi_{\star} \mathcal{O}_{\mathcal{M} \times \mathcal{C}} \right)^{\vee} \\ &= L_{\mathcal{M}}^{\bullet} \otimes \left(R\pi_{\star} \mathcal{O}_{\mathcal{M} \times \mathcal{C}} \right)^{\vee} \end{aligned}$$

$$\text{Then } \mathcal{O}_{\mathcal{M}} \rightarrow R\pi_{\star} \mathcal{O}_{\mathcal{M} \times \mathcal{C}}$$

$$\text{so } \left(R\pi_{\star} \mathcal{O}_{\mathcal{M} \times \mathcal{C}} \right)^{\vee} \rightarrow \mathcal{O}_{\mathcal{M}}$$

$$\text{so } L_{\mathcal{M}}^{\bullet} \otimes \left(R\pi_{\star} \mathcal{O}_{\mathcal{M} \times \mathcal{C}} \right)^{\vee} \rightarrow L_{\mathcal{M}}^{\bullet}$$

$$\text{Finally we have: } \left(R\pi_{\star} f^* T_x \right)^{\vee} \rightarrow L_{\mathcal{M}}^{\bullet}$$

We have a 2 term Obstruction theory
of virtual dimension

Illusie

$$\begin{aligned} \text{vir dim } M_c(x, \beta) &= \chi(C, f^* T_x) \\ &= \int_{\beta} c_1(x) + \dim_x (1-g) \end{aligned}$$

Next step is $\bar{M}_g(x, \beta)$

which is the essentially the same

as for fixed C but studied

relatively over the varying moduli of Curves

Since the moduli of curves is nonsingular,

we obtain

$$[\bar{M}_{g,n}(x,\beta)]^{\text{vir}} \in A_{\text{exp}}(\bar{M}_{g,n}(x,\beta))$$

$$\text{vir dim} = \text{vir dim } M_C(x,\beta) + 3g - 3 + n$$

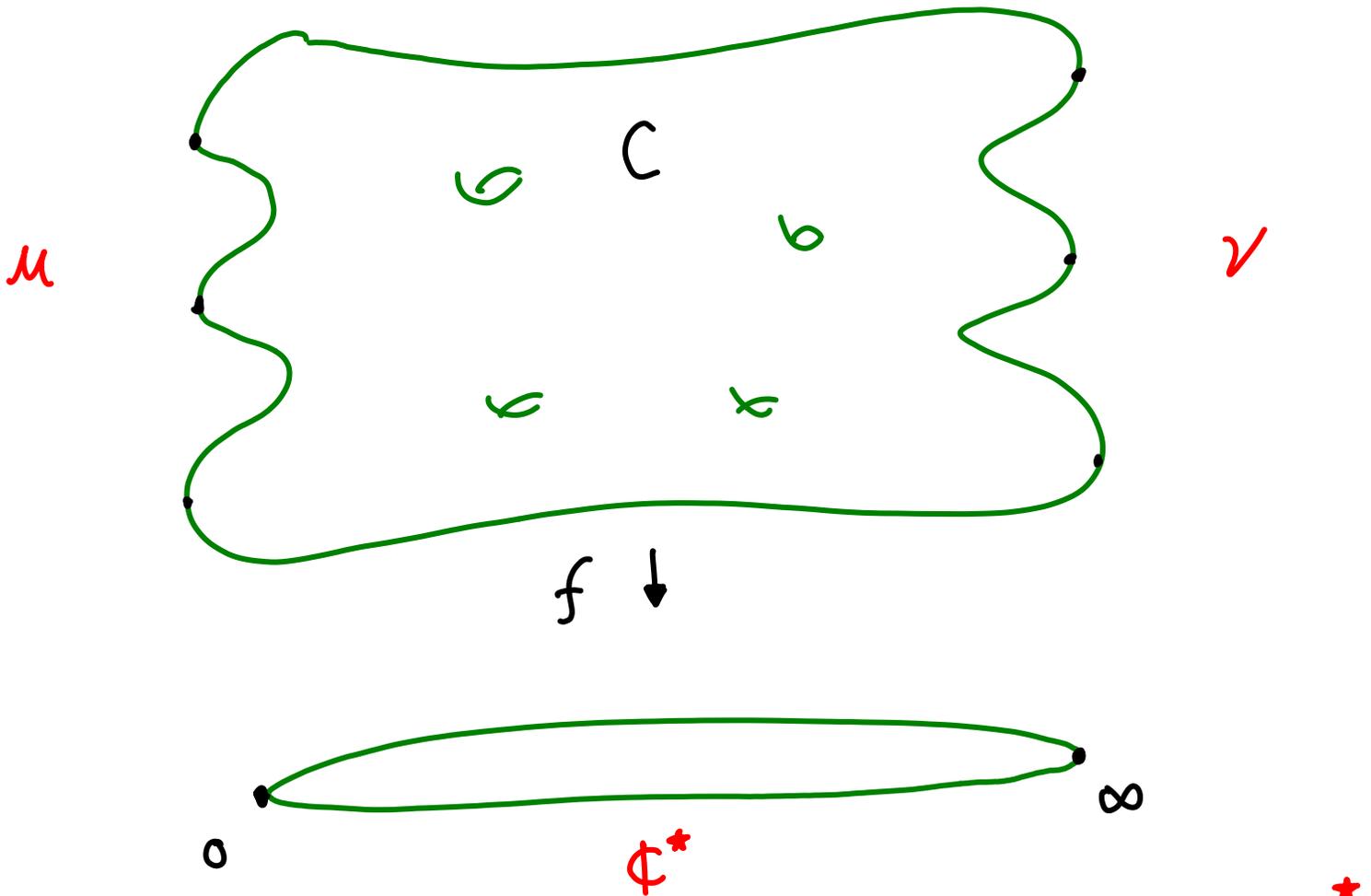
Third step is the relative case

which is technically more complicated.

$$[\bar{M}_g(\mathbb{P}^1/0,\infty)_{\mu,\nu}]^{\text{vir}}$$

← what is the expected dimension?

When the domain and the target are fixed, the theory is similar:



$$\text{vir dim } \overline{M}_g(\mathbb{P}^1/0, \infty)_{\mu, \nu} \stackrel{\sim}{=} 2d + 1 - g + 3g - 3 - 1 - (|\mu| - l(\mu)) - (|\nu| - l(\nu))$$

$d = |\mu| = |\nu|$

$$\text{Log geometry} = l(\mu) + l(\nu) + 2g - 3$$

(II) The double ramification cycle

For the open set $M_g(\mathbb{P}^1/0, \infty)_{\mu, \nu}^{\sim} \subset \bar{M}_g(\mathbb{P}^1/0, \infty)_{\mu, \nu}^{\sim}$

we have

$$M_g(\mathbb{P}^1/0, \infty)_{\mu, \nu}^{\sim} \cong \mathbb{Z} \subset M_{g, n}$$

Abel-Jacobi solutions

$$A = (a_1, \dots, a_n) \quad a_i \neq 0$$

$$\mu = (\mu_1, \dots) \quad \nu = (\nu_1, \dots)$$

↑
positive
parts of A

↑
absolute value of
the negative parts of A

We have $\varepsilon: \bar{M}_g(\mathbb{P}^1/0, \infty)_{\mu, \nu} \rightarrow \bar{M}_{g, n}$

We can define

$$\varepsilon \left(\bar{M}_g (\mathbb{P}^1 / 0, \infty)_{\mu, \nu}^{\sim} \right) \stackrel{\text{def}}{=} \mathcal{Z}_{g, A} \subset \bar{M}_{g, n}$$

Many components
of different dimensions

defined as the
image of ε

To define a cycle class for universal
Abel-Jacobi theory, we use the virtual class:

$$DR_{g, A} = \varepsilon_* \left[\bar{M}_g (\mathbb{P}^1 / 0, \infty)_{\mu, \nu}^{\sim} \right]^{\text{vir}}$$

$$DR_{g, A} \in H^{2g}(\bar{M}_{g, n})$$

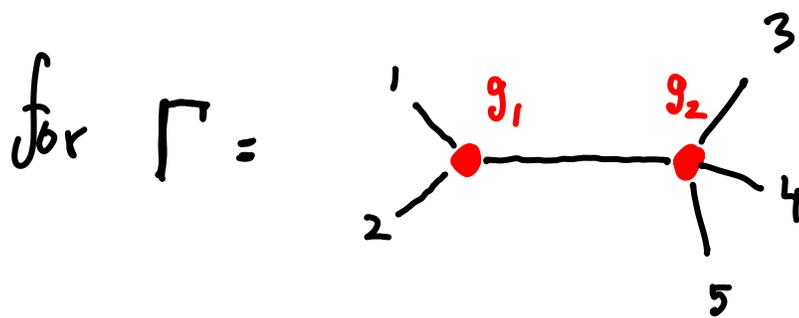
(12) Pixton's Formula

Let $G_{g,n}$ = set of stable graphs
of genus g with n markings

finite set

$$\text{for } \Gamma \in G_{g,n} \rightsquigarrow \overline{M}_{\Gamma} \xrightarrow{\exists \Gamma} \overline{M}_{g,n}$$

product of moduli spaces
determined by the vertices



$$\overline{M}_{\Gamma} = \overline{M}_{g_1, 3} \times \overline{M}_{g_2, 4}$$

Tautological classes are given by

$$\sum_{\Gamma} \left(\begin{array}{ccc} \prod \psi_i^{m_i} & \prod \psi_j^{n_j} & \prod \kappa_{\text{vertices}} \\ \text{markings} & \text{halves} & \text{vertices} \\ & \text{of edges} & \end{array} \right) \in R\mathcal{H}^*(\bar{\mathcal{M}}_{g,n})$$

$$\bar{\mathcal{M}}_{\Gamma} \xrightarrow{\sum_{\Gamma}} \bar{\mathcal{M}}_{g,n}$$


The linear span of all such classes defines the tautological ring

$$R\mathcal{H}^*(\bar{\mathcal{M}}_{g,n}) \subset \mathcal{H}^*(\bar{\mathcal{M}}_{g,n})$$

We can also define the tautological ring

$$\mathcal{R}^*(\bar{\mathcal{M}}_{g,n}) \subset \text{Ch}^*(\bar{\mathcal{M}}_{g,n}).$$

Let $\Gamma \in G_{g,n}$ be a stable graph.

Let r be a positive integer

A **weighting mod r** of Γ is

$$w : H(\Gamma) \rightarrow \{0, 1, \dots, r-1\}$$

↑
half edges

Remember

$$A = (a_1, \dots, a_n)$$

$$\sum a_i = 0$$

$$(I) \quad i \in \text{Marking}, \quad w(i) = a_i \pmod{r}$$

$$(II) \quad e = (h, h') \in \text{Edge}, \quad w(h) + w(h') = 0 \pmod{r}$$

$$(III) \quad v \in \text{Vertex}, \quad \sum_{h \vdash v} w(h) = 0 \pmod{r}$$

$W_{\Gamma, r}$ is set of **weightings mod r** of Γ

$$|W_{\Gamma, r}| \\ \stackrel{||}{=} r^{h(\Gamma)}$$

Definition (Pixton)

Let $P_g^{d,r}(A) \in \mathcal{R}^d(\overline{\mathcal{M}}_{g,n})$

be the degree d component of

$$\sum_{\Gamma \in \mathcal{G}_{g,n}} \sum_{W \in \mathcal{W}_{\Gamma,r}} \frac{1}{|\text{Aut}(\Gamma)|} \frac{1}{r^{h'(\Gamma)}} \cdot$$

$$\sum_{\Gamma \star} \left[\prod_i^n \exp\left(\frac{a_i^2}{2} \psi_i\right) \cdot \prod_{e=(h,h')} \frac{1 - \exp\left(-\frac{w(h)w(h')}{2} \cdot (\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}} \right]$$

Various motivations: Compact type restriction,

Givental - Teleman theory

Claim (Pixton):

$P_g^{d,r}(A) \in R^d(\bar{\mathcal{M}}_{g,n})$ is
polynomial in r for all $r \gg 0$.

Definition (Pixton):

$P_g^d(A) \in R^d(\bar{\mathcal{M}}_{g,n})$ is
the **constant term** of $P_g^{d,r}(A)$
 \uparrow
 $r=0$

Theorem (Conjectured by Pixton, proven in JPPZ)

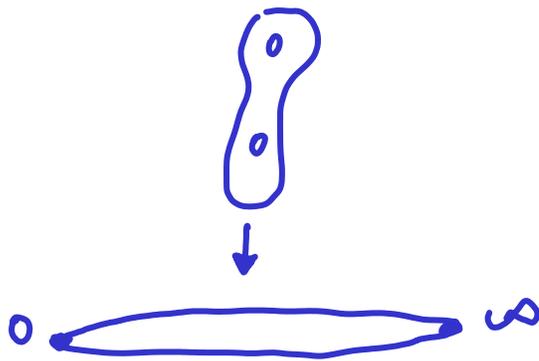
$$DR_{g,A} = P_g^g(A) \in R^g(\bar{\mathcal{M}}_{g,n}).$$

(13) formula for $\lambda_g = c_g(\mathbb{E}_g)$
 supported on $\Delta_0 \subset \bar{\mathcal{M}}_g$

Answer: We view $\bar{\mathcal{M}}_g = \bar{\mathcal{M}}_{g,0}$ ↖ n=0

Let $\lambda = \phi$

Geometry $\Rightarrow \bar{\mathcal{M}}_g(\mathbb{P}^1, \phi)^\sim \cong \bar{\mathcal{M}}_g$



Moreover $[\bar{\mathcal{M}}_g(\mathbb{P}^1, \phi)^\sim]^{\text{vir}} = (-1)^g \lambda_g$

$$\mathbb{D}R_g(\phi) = (-1)^g \lambda_g$$

So we can apply the DR cycle Formula:

Genus 1.

$$\lambda_1 = \frac{1}{24} \text{Diagram 1}$$

Diagrams
from JPPZ

Genus 2.

$$\lambda_2 = \frac{1}{240} \text{Diagram 2} + \frac{1}{1152} \text{Diagram 3}$$

Genus 3.

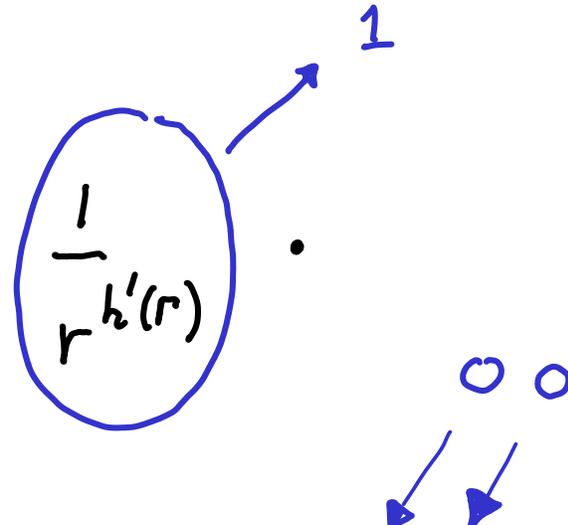
$$\lambda_3 = \frac{1}{2016} \text{Diagram 4} + \frac{1}{2016} \text{Diagram 5} - \frac{1}{672} \text{Diagram 6} + \frac{1}{5760} \text{Diagram 7} \\ - \frac{13}{30240} \text{Diagram 8} - \frac{1}{5760} \text{Diagram 9} + \frac{1}{82944} \text{Diagram 10}$$

Genus 4.

$$\lambda_4 = \frac{1}{11520} \text{Diagram 11} + \frac{1}{3840} \text{Diagram 12} - \frac{1}{2880} \text{Diagram 13} - \frac{1}{3840} \text{Diagram 14} - \frac{1}{1440} \text{Diagram 15} \\ - \frac{1}{1920} \text{Diagram 16} - \frac{1}{2880} \text{Diagram 17} - \frac{1}{3840} \text{Diagram 18} + \frac{1}{48384} \text{Diagram 19} + \frac{1}{48384} \text{Diagram 20} \\ + \frac{1}{115200} \text{Diagram 21} + \frac{1}{960} \text{Diagram 22} - \frac{23}{100800} \text{Diagram 23} - \frac{1}{57600} \text{Diagram 24} \\ - \frac{1}{16128} \text{Diagram 25} - \frac{1}{16128} \text{Diagram 26} - \frac{1}{57600} \text{Diagram 27} - \frac{1}{16128} \text{Diagram 28} \\ - \frac{1}{16128} \text{Diagram 29} - \frac{23}{100800} \text{Diagram 30} + \frac{23}{100800} \text{Diagram 31} + \frac{23}{50400} \text{Diagram 32} + \frac{1}{16128} \text{Diagram 33} \\ + \frac{1}{115200} \text{Diagram 34} + \frac{1}{276480} \text{Diagram 35} - \frac{13}{725760} \text{Diagram 36} - \frac{1}{138240} \text{Diagram 37} \\ - \frac{43}{1612800} \text{Diagram 38} - \frac{13}{725760} \text{Diagram 39} - \frac{1}{276480} \text{Diagram 40} + \frac{1}{7962624} \text{Diagram 41}$$

Why only graphs with loops?

If Γ is an unpointed tree,
then $W = 0$.

$$\sum_{\Gamma \in G_{g,n}} \sum_{W \in W_{\Gamma,r}} \frac{1}{\text{Aut}(\Gamma)} \left(\frac{1}{r^{h'(\Gamma)}} \right) \cdot$$


The diagram shows a root node (a circle) with an arrow pointing to the number '1'. Below the root node are two child nodes (circles), each with an arrow pointing downwards. The root node is circled in blue.

$$\sum_{\Gamma \neq \emptyset} \left[\prod_{e=(h,h')} \frac{1 - \exp\left(-\frac{w(h)w(h')}{2} \cdot (\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}} \right]$$

The formula
has no contributions
in degree ≥ 1 !

Can have
no nodes

For $g > 0 \Rightarrow$ Tree contributions vanish

(14) Map:

Pixton's Formula

limit $r \rightarrow \infty$
set $r=0$

Φ^* action/Localization

Chiodo's formula

Rubber geometry

Orbi-GW theory

$$\bar{M}_{g,\mu}(\mathbb{P}^1[0/r], \infty)_v$$

relative

Orbifold
Conditions

$$\bar{M}_{g,\mu}(\mathbb{P}^1/0, \infty)_v$$

orbifold
point

def

$$DR_{g,A}$$

Double ramification
cycle