Complex K-theory of dual Hitchin systems

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Higgs bundles and their moduli

- We fix a smooth proper curve X
- A Higgs bundle is a pair (E, θ) where E is an algebraic vector bundle of rank $n \neq 0$ and degree d and

$$\theta\colon E \to E \otimes \Omega^1_X$$

- The moduli space $\mathcal{M}(X)$ is a variety parametrising **semistable** Higgs bundles.
- We will assume that d and n are coprime, then $\mathcal{M}(X)$ is smooth.

Higgs bundles and their moduli

- $\mathcal{M}(X)$ is holomorphic symplectic
- there exists an integrable system $\mathcal{M}(X) \xrightarrow{\mathrm{Hit}} \mathcal{A} = \mathbb{C}^n$
- that is, the generic fibre of Hit is an abelian variety
- this map is known as the *Hitchin map*
- $\Rightarrow \mathcal{M}(X)$ is not compact

"SL_n"-Higgs bundles

- $\mathcal{M}_{\mathsf{SL}_n}(X) = \{(E,\theta) | \det E \simeq \mathcal{O}_X \text{ and } \mathrm{tr}(\theta) = 0\}$ is always singular for $n > 1$ (because $d = 0$)
- We therefore fix a line bundle L/X of degree d coprime to n
- We define $\mathcal{M}^L_{\mathsf{SL}_n}(X) = \{(E,\theta) | \det E \simeq L \text{ and } \mathrm{tr}(\theta) = 0\}$
- Henceforth we write $\hat{\mathcal{M}}(X) = \mathcal{M}^L_{\mathsf{SL}_n}$ as a shorthand

PGL_n-Higgs bundles

- $\mathcal{M}_{\mathrm{SL}_n}(X) = \{(E,\theta) | \det E \simeq \mathcal{O}_X \text{ and } \mathrm{tr}(\theta) = 0\}$ is always singular for $n > 1$ (because $d = 0$)
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- We define $\mathcal{M}^L_{\mathrm{SL}_n}(X) = \{(E,\theta) | \det E \simeq L \text{ and } \mathrm{tr}(\theta) = 0\}$
- Henceforth we write $\hat{\mathcal{M}}(X) = \mathcal{M}^L_{\mathrm{SL}_n}$ as a shorthand
- Similarly we write $\check{\mathcal{M}}$ for $[\mathcal{M}_{\mathrm{SL}_n}/\Gamma]$, where $\Gamma = \mathrm{Jac}_X[n]$
- $\check{\mathcal{M}}$ parametrises PGL_n-Higgs bundles of degree d
- We see that $\check{\mathcal{M}}$ is an orbifold (smooth Deligne-Mumford stack)

What is the Hitchin map?

- locally we may think of the Higgs field $\theta\colon E \to E \otimes \Omega^1_X$ as a matrix with 1-form coefficients
- the Hitchin map Hit is given by computing the characteristic polynomial of the Higgs field
- it is a proper map
- the generic fibre is an abelian variety

Duality of Hitchin fibres

- Langlands observed that reductive algebraic groups arise in pairs
- GL_n is **self-dual**
- SL_n is the **Langlands dual** of PGL_n
- The definition of Langlands dual groups uses the classification theory of reductive groups

Duality of Hitchin fibres

Theorem (Hausel–Thaddeus)

$$\hat{\mathcal{M}} \longrightarrow \mathcal{A}_{\mathrm{tr}=0} \longleftarrow \check{\mathcal{M}}$$

For a generic $a \in \mathcal{A}_{\mathrm{tr}=0}$ the **abelian varieties**, *arising as fibres, are* **dual**.

There is a similar result for $\mathcal{M}$: the generic Hitchin fibre is a **self-dual** abelian variety (a Jacobian of a smooth curve).

Duality of Hitchin fibres

The result from the previous slide can be generalised to G-Higgs bundles:

Theorem (Donagi–Pantev)

The Hitchin bases $\mathcal{A}_G$ and $\mathcal{A}_{^LG}$ are canonically isomorphic

$$\begin{array}{ccc} \mathcal{M}_G & & \mathcal{M}_{^LG} \\ & \searrow \quad \swarrow & \\ & \mathcal{A} & \end{array}$$

and for a generic $a \in \mathcal{A}$ the abelian varieties arising as fibres are dual.

Duality of Hitchin fibres

How do cohomologies of SL_n and PGL_n-moduli spaces compare? For moduli spaces of vector bundles we have the following:

Theorem (Harder–Narasimhan 1975)

The singular cohomology spaces $H^*(\hat{\mathcal{N}}, \mathbb{Q})$ and $H^*(\check{\mathcal{N}}, \mathbb{Q})$ are isomorphic.

- $\check{\mathcal{N}} = \hat{\mathcal{N}}/\Gamma$ is a quotient
- hence the finite group $\Gamma = \mathrm{Jac}(X)[n]$ acts trivially on $H^*(\hat{\mathcal{N}})$
- this isn't true for **Higgs bundles**

Duality of Hitchin fibres

The so-called *SYZ*-philosophy that two Calabi-Yau varieties are mirror partners if they admit dual torus fibrations led to the following conjecture:

Conjecture (Hausel–Thaddeus)

Using the right definition of Hodge numbers we have

$$h^{p,q}(\hat{\mathcal{M}}) = h^{p,q}_{st,\alpha}(\check{\mathcal{M}}).$$

The subscript "st" stands for **stringy** or **orbifold** cohomology. This equation is already interesting for the Betti numbers $b^i = \sum_{p+q=i} h^{p,q} = \operatorname{rk} H^i_{\text{sing}}(X, \mathbb{Q})$.

Orbifold cohomology

Untwisted orbifold cohomology:

- $=$ cohomology of the inertia stack: $H^*_{orb}(\check{\mathcal{M}}, \mathbb{Q}) = H^*(I\,\check{\mathcal{M}}, \mathbb{Q})$
- This amounts to $\bigoplus_{\gamma\in\Gamma} H^*([\hat{\mathcal{M}}^{\gamma}/\Gamma], \mathbb{Q})$
- Remark: $[\hat{\mathcal{M}}^{\gamma}/\Gamma]$ is itself a moduli space of Higgs bundles on a finite étale cover X_γ of X

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Twisted orbifold cohomology:

- a gerbe α gives rise to a morphism $\check{\mathcal{M}} \to B^2\mu_n$
- for the inertia stacks we therefore obtain $I\,\check{\mathcal{M}} \to IB^2\mu_n = B\mu_n \times B^2\mu_n \to B\mu_n$
- this yields a μ_n-torsor on $I\,\check{\mathcal{M}}$, to which we associate a unitary local system $\mathcal{L}^\alpha$ on $I\,\check{\mathcal{M}}$
- $H^*_{orb}(\check{\mathcal{M}}, \alpha) = H^*(I\,\check{\mathcal{M}}, \mathcal{L}^\alpha)$

Topological Mirror Symmetry

Theorem (G.–Wyss–Ziegler 2017)

We have

$$h^{p,q}(\mathcal{M}_{\mathrm{SL}_n}) = h^{p,q}_{st,\alpha}(\mathcal{M}_{\mathrm{PGL}_n}),$$

where α is the canonical gerbe on $\mathcal{M}_{\mathrm{PGL}_n}$.

This theorem is proven using Batyrev's p-adic integration

- Hausel–Thaddeus 2004 verified the case $n = 2, 3$ (Morse theory)
- Boccalini–Grandi 2014 verified the Hilbert scheme cases (Göttsche's formula)
- Gothen–Oliveira 2017 rank 2 and 3 for parabolic Higgs bundles (Morse theory)
- Shiyu Shen 2021 parabolic Higgs bundles and arbitrary rank (p-adic integration)

Topological Mirror Symmetry

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where α *is the canonical gerbe on* $\mathcal{M}_{\mathrm{PGL}_n}$.

New proofs/methods:

- Loeser & Wyss 2019 (motivic integration)
- Maulik & Junliang Shen 2020 (Ngô's correspondence, perverse sheaves, vanishing cycles)
- The latter article constructs an **explicit isomorphism** (well-defined up to a complex scalar)

$$H^*(\hat{\mathcal{M}}, \mathbb{C}) \simeq H^*_{orb,\alpha}(\check{\mathcal{M}}, \mathbb{C})$$

Integral Mirror Symmetry for Higgs bundles?

$$H^*(\hat{\mathcal{M}}, \mathbb{Z}) \simeq H^*_{orb,\alpha}(\check{\mathcal{M}}, \mathbb{Z})???$$

- RHS has **non-vanishing torsion**, e.g. in degree 2: $\Gamma \simeq H^2_{orb,\alpha}(\check{\mathcal{M}}, \mathbb{Z})_{tors})$
- LHS is **torsionfree**:

Theorem (M. G. & Shiyu Shen)

The singular cohomology groups $H^*(\hat{\mathcal{M}}, \mathbb{Z})$ (SL_n-space) and $H^*(\mathcal{M}, \mathbb{Z})$ (GL_n-space) are torsionfree.

For $n = 2$ this result is due to Tom Baird.

Integral Topological Mirror Symmetry?

It is not surprising that there's no mirror symmetry relation for integral *singular cohomology*:

- Addington 2017: there are derived equivalent 3-folds X, X', such that $H^3(X,\mathbb{Z})_{tors} \neq H^3(X',\mathbb{Z})_{tors}$
- Treumann 2019: there is a T-dual pair of flat 3-folds with $\widehat{Z}$, $\check{Z}$ with $H^2(\widehat{Z},\mathbb{Z})_{tors} \neq H^3(\check{Z},\mathbb{Z})_{tors}$

Integral Topological Mirror Symmetry?

But surprisingly, things improve when replacing singular cohomology by **complex K-theory**:

- Addington 2017: there are derived equivalent 3-folds X, X' such that $H^3(X,\mathbb{Z})_{tors} \neq H^3(X',\mathbb{Z})_{tors}$
- $KU^*(X) \simeq KU^*(X') \leftarrow$ always the case for FM partners
- Treumann 2019: there is a T-dual pair of flat 3-folds $\hat{Z}$, $\check{Z}$ with $H^2(\hat{Z},\mathbb{Z})_{tors} \neq H^3(\check{Z},\mathbb{Z})_{tors}$

  Treumann proved: $KU^0(\hat{Z}) \simeq KU^1(\check{Z})$ and $KU^1(\hat{Z}) \simeq KU^0(\check{Z})$

Integral Topological Mirror Symmetry!

Theorem (M. G. & Shiyu Shen)

There is an isomorphism $KU^*(\hat{\mathcal{M}}) \simeq KU^*(\check{\mathcal{M}}, \alpha)$.

- RHS $=$ α-twisted Jac-equivariant complex K-theory of $\mathcal{M}$

Integral Topological Mirror Symmetry!

Theorem (M. G. & Shiyu Shen)

There is an isomorphism $KU^*(\hat{\mathcal{M}}) \simeq KU^*(\check{\mathcal{M}}, \alpha)$*.*

- RHS $=$ α-twisted Jac-equivariant complex K-theory of $\mathcal{M}$
- We actually construct an equivalence of Ω-spectra:

$$KU(\hat{\mathcal{M}}) \simeq KU(\check{\mathcal{M}}, \alpha),$$

- where for a space X, $KU(X)$ denotes an Ω-spectrum such that $\pi_*(KU(X)) = KU^*(X)$
- Spectra are indispensable for our argument.

Integral Topological Mirror Symmetry!

Theorem (M. G. & Shiyu Shen)

There is an isomorphism $KU^*(\hat{\mathcal{M}}) \simeq KU^*(\check{\mathcal{M}}, \alpha)$.

- The LHS is **torsionfree** by our first theorem
- The RHS is also torsionfree (in fact, this needs to be shown independently)
- $\Rightarrow$ This confirms the often observed phenomenon that passing to complex K-theory **annihilates just the right amount of torsion**

A short detour: Kirwan surjectivity

- Let G be a compact Lie group and M a Hamiltonian G-space
- with a proper moment map μ
- 0 is a regular value of μ

Theorem (Kirwan 1984)

$H^*_G(M, \mathbb{Q}) \twoheadrightarrow H^*(M//G, \mathbb{Q})$.

- Kirwan surjectivity fails for integral coefficients,
- unless one imposes certain assumptions (Tolman–Weitsman 1998)

A short detour: Kirwan surjectivity

- Let G be a compact Lie group and M a Hamiltonian G-space
- with a proper moment map μ
- 0 is a regular value of μ

Theorem (Harada–Landweber 2005)

Kirwan surjectivity holds for **complex K-theory**: $KU_G(M) \twoheadrightarrow KU(M//G)$

This is another instance where passing to complex K-theory annihilates just the right amount of torsion.

Complex K-theory fact sheet

- KU^* is a 2-periodic or $\mathbb{Z}/2\mathbb{Z}$-graded cohomology theory
- A vector bundle on X gives rise to an element $[E]$ of $KU^0(X)$ (where X is a finite CW-complex)
- $[E \oplus F] = [E] + [F]$

Complex K-theory fact sheet

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- $[E \oplus F] = [E] + [F]$
- The Atiyah–Hirzebruch spectral sequence relates complex K-theory with singular cohomology:

$$E_2^{p,q} = H^p(X, KU^q(pt)) \Rightarrow KU^{p+q}(X),$$

  where $KU^q(pt) = \mathbb{Z}$ if and only if q is even
- The AH spectral sequence degenerates on E_2 after tensoring with $\mathbb{Q}$
- $\Rightarrow$ differentials can only be non-zero on torsion classes

Complex K-theory fact sheet

The properties of AH spectral sequence imply the following:

- $KU^*(X) \otimes \mathbb{Q} \simeq H^*(X, \mathbb{Q}[\beta, \beta^{-1}])$, where $\deg \beta = -2$
- $\mathrm{rk}\, KU^j(X) = \sum_{i-j \text{ even}} b_i(X)$
- $H^*(X, \mathbb{Z})$ torsionfree $\Rightarrow$ $KU^*(X)$ torsionfree
- It is possible that $KU^*(X)$ is torsionfree when $H^*(X, \mathbb{Z})$ isn't

Complex K-theory fact sheet

Let G be a group acting on X:

- replacing vector bundles on X by G-equivariant vector bundles we obtain equivariant K-theory $KU_G^*(X)$
- Informally, we may write $KU^*([X/G]) = KU_G^*(X)$
- For a G-equivariant gerbe α, we can define twisted equivariant complex K-theory $KU_G^*(X, \alpha)$ using α-twisted equivariant vector bundles

Theorem (Freed–Hopkins–Teleman 2002)

$KU_G^*(X, \alpha) \otimes \mathbb{C} \simeq H_{orb}^*([X/G], \alpha) \otimes \mathbb{C}[\beta, \beta^{-1}]$, *where* β *has degree* -2.

K-theoretic mirror symmetry for Higgs bundles

Theorem (Freed–Hopkins–Teleman 2002)

$KU^*_G(X, \alpha) \otimes \mathbb{C} \simeq H^*_{orb}([X/G], \alpha) \otimes \mathbb{C}[\beta, \beta^{-1}]$, where β has degree -2.

In light of the theorem above, our result (below) can really be thought of as an integral analogue of the Hausel–Thaddeus conjecture.

Theorem (M. G. & Shiyu Shen)

There is an isomorphism $KU^*(\hat{\mathcal{M}}) \simeq KU^*(\check{\mathcal{M}}, \alpha)$.

Strategy: Fourier-Mukai

- Let X and Y be smooth schemes, which are proper over a common smooth base S

$$X \xrightarrow{f} S \xleftarrow{g} Y$$

- A Fourier-Mukai kernel $\mathcal{F} \in D^b_{perf}(X \times Y)$ gives rise to a morphism $\mathrm{fm}_{\mathcal{F}}\colon KU(X) \to KU(Y)$
- $\mathrm{fm}_{\mathcal{F}}([E]) = (pr_2)_!(pr_1^*[E] \otimes [\mathcal{F}]])$
- This also works equivariantly and with respect to twists

Strategy: Fourier-Mukai

- In fact, $\mathcal{F}$ induces a map of sheaves of spectra with respect to the standard topology on S^{an}

$$\underline{\mathrm{fm}}_F \colon Rf_*\underline{KU}_X \to Rg_*\underline{KU}_Y$$

- That this is possible, isn't obvious (at least to me) since a sheaf of spectra is a delicate object to manipulate
- We require the following theory to do this:

Topological K-theory of DG categories after Blanc and Moulinos

DG category = $\mathbb{C}$-linear stable ∞-category

Theorem (Blanc 2012)

There exists a functor from DG categories to spectra

$$K^{top}\colon \mathrm{DGCat} \to \mathrm{Spectra}$$

sending $\mathrm{Perf}(X)$ *to* $KU(X)$ *for a complex scheme* X.

Topological K-theory of DG categories after Blanc and Moulinos

This construction was subsequently lifted to a relative setting:

Theorem (Moulinos)

Let S be a complex scheme. There exists a functor from S-linear DG categories to sheaves of spectra on S^{an}:

$$K_S^{top} : \mathrm{DGCat}_S \to \mathsf{Sh}_{\mathrm{Spectra}}(S^{an})$$

such that $\mathsf{Perf}(X, \alpha)$ *is sent to the twisted K-theory sheaf* $\underline{KU}_X^{\alpha}$.

Furthermore, Moulinos proves that for a proper morphism $S \xrightarrow{f} S'$ one has

$$K_{S'}^{top}(\mathsf{Perf}(S)) \simeq Rf_* K_S^{top}(\mathsf{perf}(S)) \simeq Rf_* \underline{KU}_S.$$

Strategy: Fourier-Mukai

- Applying K_S^{top} to the actual Fourier-Mukai transform (between S-linear DG categories) $\mathrm{FM}_{\mathcal{F}}\colon \mathrm{Perf}(X) \to \mathrm{Perf}(Y)$
- (Warning: not every Fourier-Mukai transform preserves perfect complexes!)
- we obtain a morphism of sheaves on S^{an}

$$\underline{\mathrm{fm}}_{\mathcal{F}}\colon Rf_*\underline{KU}_X \to Rg_*\underline{KU}_Y$$

- This also works with twists and Γ-equivariantly (since Γ is finite abelian).

Strategy: Fourier-Mukai

Tensoring (= smash product) the sheaves of spectra on S^{an} with $\mathbb{Q}$ we obtain:

$$\underline{\mathrm{fm}}_{\mathcal{F}} \otimes \mathbb{Q} \colon Rf_* \underline{KU}_X \otimes \mathbb{Q} \to Rg_* \underline{KU}_Y \otimes \mathbb{Q}$$

$$\|$$

$$\underline{\mathrm{fm}}_{\mathcal{F}} \otimes \mathbb{Q} \colon Rf_* \mathbb{Q}_X[\beta, \beta^{-1}] \to Rg_* \mathbb{Q}_Y[\beta, \beta^{-1}]$$

Strategy: Fourier-Mukai

$$\underline{\mathrm{fm}}_{\mathcal{F}} \otimes \mathbb{Q} \colon Rf_* \mathbb{Q}_X[\beta, \beta^{-1}] \to Rg_* \mathbb{Q}_Y[\beta, \beta^{-1}].$$

- According to the **decomposition theorem** (by Beilinson–Bernstein–Deligne–Gabber), the derived pushforwards above are sums of shifted semisimple perverse sheaves.
- A semisimple perverse sheaf is the **middle extension** of a semisimple local systems on a locally closed subvariety S_i (a so-called "support")
- Assume that $\mathrm{FM}_{\mathcal{F}}$ is an equivalence over every supp S_i
- Then, $\mathrm{fm}_{\mathcal{F}} \otimes \mathbb{Q}$ is an equivalence of sheaves of rationalised spectra (over the full base S^{an})

Strategy: Fourier-Mukai

$$\underline{\mathrm{fm}}_{\mathcal{F}} \otimes \mathbb{Q} \colon Rf_*\mathbb{Q}_X[\beta, \beta^{-1}] \to Rg_*\mathbb{Q}_Y[\beta, \beta^{-1}].$$

- This implies $\mathrm{fm}_{\mathcal{F}} \colon KU^*(X) \to KU^*(Y)$ is an isomorphism **after rationalisation**
- Furthermore, one can show that $\mathrm{fm}_{\mathcal{F}}$ induces an isomorphism of the non-torsion parts of $KU^*(X)$ and $KU^*(Y)$
- So, if $KU^*(X)$ and $KU^*(Y)$ are torsionfree, then $\mathrm{fm}_{\mathcal{F}} \colon KU^*(X) \to KU^*(Y)$ is an isomorphism, and hence
- $KU(X)$ and $KU(Y)$ are equivalent spectra (Whitehead lemma)

Strategy: Fourier-Mukai

$$\underline{\mathrm{fm}}_{\mathcal{F}} \otimes \mathbb{Q} \colon Rf_* \mathbb{Q}_X[\beta, \beta^{-1}] \to Rg_* \mathbb{Q}_Y[\beta, \beta^{-1}].$$

Overall this is a straight-forward construction, except for the following steps:

- choosing a FM kernel which induces a derived equivalence over a sufficiently big open subset $S^\heartsuit \subset S$
- proving that the perverse supports of $Rf_*\mathbb{Q}$ are contained in $S^\heartsuit$
- establishing vanishing of torsion in $KU^*(X)$ and $KU^*(Y)$

The case of the Hitchin fibration

Can those problems be solved for the Hitchin fibration? To a certain extent, yes:

- Arinkin constructs a **derived equivalence** over $A_{ell} \subset A$, the locus of integral spectral curves (a.k.a. elliptic locus)
- Use an **arbitrary** coherent extension of Arinin's **Poincaré sheaf** to the full Hitchin base
- Working with Higgs fields $\theta\colon E \to E \otimes D$ where D is a fixed divisor of **even degree** $\geq$ **2g**, the perverse supports are known to lie in A_{ell} (de Cataldo 2016)
- **Problem:** the perverse supports are still unknown if $D = \omega_X$ is the **canonical divisor**.

Vanishing cycle method

- As a work-around we borrow the **vanishing cycle method** from work of Davesh Maulik & Junliang Shen 2020
- For a divisor D we denote by $\hat{\mathcal{M}}^D$ and $\check{\mathcal{M}}^D$ the moduli spaces of Higgs bundles on X where the Higgs field is given by a map $\theta \colon E \to E \otimes \mathcal{O}(D)$
- We write A^D for the corresponding Hitchin base
- Let $p \in X$ be a closed point and assume that D is the canonical divisor or has degree $> 2g - 2$

Theorem (D. Maulik & J. Shen)

There exists a regular function $f \colon \mathrm{A}^{D+p}_{\mathrm{tr}=0} \to \mathbb{A}^1$, such that the sheaf of vanishing cycles $\phi_f \mathrm{RHit}_* \underline{\mathbb{Q}}_{\hat{\mathcal{M}}^{D+p}}$ agrees (up to shift) with $\mathrm{RHit}_* \underline{\mathbb{Q}}_{\hat{\mathcal{M}}^D}$ (and similarly for $\check{\mathcal{M}}$).

Vanishing cycle method

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There exists a regular function $f\colon \mathrm{A}^{D+p}_{\mathrm{tr}=0} \to \mathbb{A}^1$, such that the sheaf of vanishing cycles $\phi_f \mathrm{RHit}_* \underline{\mathbb{Q}}_{\hat{\mathcal{M}}^{D+p}}$ agrees (up to shift) with $\mathrm{RHit}_* \underline{\mathbb{Q}}_{\hat{\mathcal{M}}^D}$ (and similarly for $\check{\mathcal{M}}$).

We prove an analogous result for **sheaves of spectra**:

Theorem (M. G. & S. Shen)

The sheaf of vanishing cycles $\phi_f \mathrm{RHit}_* \underline{KU}_{\hat{\mathcal{M}}^{D+p}}$ agrees (up to shift) with $\mathrm{RHit}_* \underline{KU}_{\hat{\mathcal{M}}^D}$, and $\phi_f \mathrm{RHit}_* \underline{KU}^{\alpha}_{\check{\mathcal{M}}^{D+p}}$ agrees (up to shift) with $\mathrm{RHit}_* \underline{KU}^{\alpha}_{\hat{\mathcal{M}}^D}$.

The red part isn't straightforward.

Conclusion

$$
\begin{array}{ccc}
R\mathrm{Hit}_* \underline{KU}^{\hat{\alpha}}_{\hat{\mathcal{M}}^{D+p}} & \xrightarrow{\mathrm{fm}_{\bar{\mathcal{P}}}} & R\mathrm{Hit}_* \underline{KU}^{\alpha}_{\check{\mathcal{M}}^{D+p}} \\
& \rightsquigarrow & \\
\phi_f R\mathrm{Hit}_* \underline{KU}^{\hat{\alpha}}_{\hat{\mathcal{M}}^{D+p}} & \xrightarrow{\phi_f(\mathrm{fm}_{\bar{\mathcal{P}}})} & \phi_f R\mathrm{Hit}_* \underline{KU}^{\alpha}_{\check{\mathcal{M}}^{D+p}} \\
\| & & \| \\
R\mathrm{Hit}_* \underline{KU}^{\hat{\alpha}}_{\hat{\mathcal{M}}^{D}} & \longrightarrow & R\mathrm{Hit}_* \underline{KU}^{\alpha}_{\check{\mathcal{M}}^{D}}
\end{array}
$$

$$\rightsquigarrow$$

$$KU^{\hat{\alpha}}_{\hat{\mathcal{M}}^D} \to KU^{\alpha}_{\check{\mathcal{M}}^D}$$

$$\rightsquigarrow$$

$$KU^{\hat{\alpha}}_{\hat{\mathcal{M}}} \to KU^{\alpha}_{\check{\mathcal{M}}}$$

Conclusion

- The morphism $\phi_f(\mathrm{fm}_{\bar{\mathcal{P}}})\colon KU_{\hat{\mathcal{M}}} \otimes \mathbb{Q} \to KU^{\alpha}_{\check{\mathcal{M}}} \otimes \mathbb{Q}$ is an equivalence
- **Vanishing of torsion** in $KU^*(\hat{\mathcal{M}})$ and $KU^*(\check{\mathcal{M}}, \alpha)$ is crucial for showing that $\phi_f(\mathrm{fm}_{\bar{\mathcal{P}}})$ is an equivalence

Conclusion

- The morphism $\phi_f(\mathrm{fm}_{\bar{\mathcal{P}}})\colon KU_{\hat{\mathcal{M}}} \otimes \mathbb{Q} \to KU^{\alpha}_{\check{\mathcal{M}}} \otimes \mathbb{Q}$ is an equivalence
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- **Remark:** Both SL_n and PGL_n-side need to be **twisted** when working with **sheaves** of spectra
- That's because $d \neq 0$ and thus the Fourier-Mukai transform relates two derived categories of **twisted sheaves**
- The twist is **invisible on the SL_n-side**, after having taken global sections

Vanishing of torsion

SL_n-case: we use work by Heinloth–García-Prada–Schmitt 2011 and Heinloth–García-Prada 2012

Theorem (M. G. & Shiyu Shen)

The singular cohomology $H^*(\hat{\mathcal{M}}, \mathbb{Z})$ is torsionfree. Thus, $KU^*(\hat{\mathcal{M}})$ is also torsionfree.

PGL_n-case: follows from the GL_n-case + invariance of KU w.r.t. derived equivalences

Theorem (M. G. & Shiyu Shen)

Twisted complex K-theory $KU^*(\check{\mathcal{M}}, \alpha^i)$ is torsionfree for every $i \geq 0$.

Vanishing of p-torsion: a direct proof if $n = p$ is a prime

- We have a smooth fibration

$$\hat{\mathcal{M}} \to \mathcal{M} \xrightarrow{\det} T^*\mathrm{Jac}^d$$

- $R\det_* \underline{\mathbb{Z}}$ is a locally constant sheaf with fibre $H^*(\hat{\mathcal{M}}, \mathbb{Z})$
- The monodromy is given by a homomorphism

$$\rho\colon \pi_1(\mathrm{Jac}^d) \twoheadrightarrow \Gamma \to \mathrm{Aut}(H^*(\hat{\mathcal{M}}, \mathbb{Z}))$$

- Assume by contradiction that $H^*(\hat{\mathcal{M}}, \mathbb{Z})[p] \neq 0$
- Since $\Gamma = (\mathbb{Z}/p\mathbb{Z})^{2g}$, linear algebra implies the existence of a p-torsion class fixed by Γ
- $\Rightarrow H^*(\mathcal{M}, \mathbb{Z})[p] \neq 0$. **Contradiction!**

Vanishing of p-torsion: a direct proof if $n = p$ is a prime

- The quotient map $\mathcal{M}_{\mathrm{tr}=0} \to \check{\mathcal{M}}$ is a Jac-torsor
- Let β be the **Barsotti–Weil gerbe**, then

$$D^b_{perf}(\mathcal{M}_{\mathrm{tr}=0}) \cong D^b_{perf}(\check{\mathcal{M}} \times \mathrm{Jac}, \beta)$$

- $\Rightarrow KU(\mathcal{M}_{\mathrm{tr}=0}) \simeq KU(\check{\mathcal{M}} \times \mathrm{Jac}, \beta)$
- RHS agrees with $R(pr_2)_* \underline{KU}^{\beta}_{\check{\mathcal{M}} \times \mathrm{Jac}}$, where

$$pr_2 \colon \check{\mathcal{M}} \times \mathrm{Jac} \to \mathrm{Jac}$$

- same observation as before $\Rightarrow$ vanishing of p-torsion

Thanks!

Thanks for your attention!