$SU(r)$ Vafa-Witten invariants and continued fractions

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We work over $\mathbb{C}$.
S smooth proj. surface. $p_g(S) = h^0(S, K_S) > 0$, $H^1(S, \mathbb{Z}) = 0$
H ample line bundle on S

$M_S^H(r, c_1, c_2) =$ moduli space of rank r H-semistable sheaves on S with Chern classes c_1, c_2

$\mathcal{E}$ semistable $\iff \frac{h^0(S,\mathcal{F}(n))}{\mathrm{rk}(\mathcal{F})} \leq \frac{h^0(S,\mathcal{E}(n))}{\mathrm{rk}(\mathcal{E})}$ for all $\mathcal{F} \subset \mathcal{E}$ for $n \gg 0$.
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$\mathcal{E}$ stable: $<$ for $\mathcal{F} \subsetneq \mathcal{E}$
Assume for simplicity stable=semistable)
$M_S^H(r, c_1, c_2)$ is projective, usually singular, expected dimension

$$\mathrm{vd} = \mathrm{vd}(r, c_1, c_2) = 2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(\mathcal{O}_S).$$

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Recently Tannaka-Thomas defined Vafa-Witten invariants in terms of moduli spaces of Higgs pairs on S.

(S, H) proj. surface with ample divisor
A Higgs pair on S is a pair $(\mathcal{E}, \phi)$ with

- $\mathcal{E}$ torsion free sheaf on S
- $\phi : \mathcal{E} \to \mathcal{E} \otimes K_S$ homomorphism with $\mathrm{tr}\phi = 0$

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Then Tanaka-Thomas define invariants by formal virtual localization

$$\int_{[N^{\mathrm{vir}}]} 1 := \int_{[N^{\mathbb{C}^*}]^{\mathrm{vir}}} \frac{1}{e^{\mathbb{C}^*}(\nu^{\mathrm{vir}})}$$

$\nu^{\mathrm{vir}} =$ moving part of $T_N^{\mathrm{vir}}|_{N^{\mathbb{C}^*}}$

Decomposition parametrized by partitions of r

$$N^{\mathbb{C}^*} = \coprod_\lambda N_\lambda^{\mathbb{C}^*}$$

$N_\lambda^{\mathbb{C}^*}$ parametrizes $(\mathcal{E}, \phi)$ with
$\mathcal{E} = \oplus_i \mathcal{E}_i$ weight decomposition, $\operatorname{rk} \mathcal{E}_i = \lambda_i$
with $\phi : \mathcal{E}_i \to \mathcal{E}_{i-1}, \quad (\phi|_{\mathcal{E}_{min}} = 0)$

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with $\phi : \mathcal{E}_i \to \mathcal{E}_{i-1}$, $\quad (\phi|_{\mathcal{E}_{min}} = 0)$
The horizontal component is $N_{(r)}^{\mathbb{C}^*}$ with $\phi = 0$
Thus $N_{(r)}^{\mathbb{C}^*} = M_S^H(r, c_1, c_2)$
The vertical component is $N_{(1^r)}^{\mathbb{C}^*}$

Vafa-Witten partition function:

$$Z^{SU(r)}_{S,H,c_1}(q) := r^{-1} q^{-\frac{\chi(\mathcal{O}_S)}{2r} - \frac{rK_S^2}{24}} \sum_{c_2} q^{\frac{\mathrm{vd}(r,c_1,c_2)}{2r}} (-1)^{\mathrm{vd}(r,c_1,c_2)} \int_{[N_S^H(r,c_1,c_2)]^{\mathrm{vir}}} 1$$

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By definition

$$Z^{SU(r)}_{S,H,c_1}(q) = r^{-1} \sum_{\lambda} Z^{\lambda}_{S,H,c_1}(q)$$

with Z^{λ}_{S,H,c_1} contribution of $N^{\mathbb{C}^*}_{\lambda}$

Moduli of sheaves $M_S^H(r, c_1, c_2)$ carries perfect obstruction theory defined by Mochizuki of virtual dimension $\mathrm{vd}(r, c_1, c_2)$
Virtual tangent bundle $T_M^{\mathrm{vir}} \in K^0(M)$, virtual fundamental class $[M]^{\mathrm{vir}} \in A_{\mathrm{vd}}(M)$

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Tanaka-Thomas:

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Thus

$$Z_{S,H,c_1}^{(r)}(q) =^{-\frac{\chi(\mathcal{O}_S)}{2r}+\frac{rK_S^2}{24}} \sum_{c_2} q^{\frac{\mathrm{vd}(r,c_1,c_2)}{2r}} e^{\mathrm{vir}}(M_S^H(r, c_1, c_2))$$

i.e. Vafa-Witten partition function contains generating function of virtual Euler numbers of moduli of stable sheaves

Modular form of weight k: holomorphic function $f : \mathbb{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\} \to \mathbb{C}$ satisfying

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbb{Z}),$$

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ gives $\tau \mapsto \tau + 1$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ gives $\tau \mapsto \frac{-1}{\tau}$

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$$f(\tau) = \sum_{n \geq 0} a_n q^n, \qquad a_n \in \mathbb{C}, \quad q = e^{2\pi i \tau}$$

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We also have modular forms for finite index subgroups of $Sl(2, \mathbb{Z})$, e.g.

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbb{Z}) \mid c \equiv 0 \mod N \right\}.$$

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Langlands dual partition function:

$$Z^{{}^L SU(r)}_{S,H,c_1}(q) := r \sum_{w \in H^2(S,Z)/(rH^2(S,Z))} \varepsilon_r^{wc_1} Z^{SU(r)}_{S,H,w}(q), \quad \varepsilon_r = e^{\frac{2\pi i}{r}}$$

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Conjecture (Vafa-Witten *S*-duality)

Let $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ *Then*

$$Z^{SU(r)}_{S,H,c_1}\left(\frac{-1}{\tau}\right) = (-1)^{(r-1)\chi(\mathcal{O}_S)} \left(\frac{r\tau}{i}\right)^{\frac{e(S)}{2}} Z^{^LSU(r)}_{S,H,c_1}(\tau)$$

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interchanges contributions of $N^{\mathbb{C}^*}_{(r)}$ and $N^{\mathbb{C}^*}_{(1^r)}$

If r is prime Thomas shows that $Z^{\lambda}_{S,H,c_1} = 0$ for $\lambda \neq (r),\ (1^r)$

Using S duality, if r prime, $Z^{(1^r)}_{S,H,c_1}(q)$ determines $Z^{SU(r)}_{S,H,c_1}(q)$

Laarakker: Structure formula for $Z^{(1^r)}_{S,H,c_1}(q)$

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$$\eta(q) = q^{\frac{1}{24}} \prod_{n>0}(1-q^n), \qquad \Delta(q) = \eta(q)^{24}$$

$$\Theta_{A_r,\ell}(q) = \sum_{v\in\mathbb{Z}^r} q^{\frac{1}{2}\langle v-\ell a, v-\ell a\rangle}, \quad \ell \in \mathbb{Z}, \quad a = \frac{1}{r+1}(r, r-1, \dots, 1)$$

eta-function, discriminant modular form and theta function for the A_r lattice

$\langle , \rangle$ is defined on standard basis $(e_i)_i$ of $\mathbb{Z}^r$ by the matrix

$$\big(\langle e_i, e_j\rangle\big)_{ij} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

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For $a, b \in H^2(S,\mathbb{Z})$ put $\delta_{a,b} = \begin{cases} 1 & a-b\in rH^2(S,\mathbb{Z}) \\ 0 & \text{otherwise} \end{cases}$

Deal only with proj. surf. S with $p_g(S) > 0$ and $H^1(S,\mathbb{Z}) = 0$

Theorem (Laarakker)

Fix $r > 0$. There are $C_0, (C_{ij})_{1\le i\le j\le r-1} \in \mathbb{Q}((q^{\frac{1}{2r}}))$ s.th for all (S,H) polarized surface as above, $c_1 \in H^2(S,\mathbb{Z})$ we have

$$Z^{(1^r)}_{S,H,c_1}(q) = \left(\frac{(-1)^{r-1}}{r\Delta(q^r)}\right)^{\chi(\mathcal{O}_S)} \left(\frac{\Theta_{A_{r-1},0}(q)}{\eta(q)^r}\right)^{-K_S^2} \times C_0(q)^{K_S^2} \sum_{\beta\in H^2(S,\mathbb{Z})^{r-1}} \delta_{c_1,\sum_i i\beta_i} \prod_i SW(\beta_i) \prod_{i\le j} C_{ij}(q)^{\beta_i\beta_j}$$

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$SW(\beta_i)$ is Seiberg-Witten inv. of $\beta_i \in H^2(S,\mathbb{Z})$. If S is minimal of gen. type $SW(0) = 1$, $SW(K_S) = (-1)^{\chi(\mathcal{O}_S)}$, rest vanish
Have $\Delta(q)$ in the formulas because of the invariants of $K3$ surface, and theta functions for A_r-lattice because of blowup formulas relating the invariants of a surface and its blowup in a point
Put

$$\phi_{r,S,c_1}(q) := C_0(q)^{K_S^2} \sum_{\beta\in H^2(S,\mathbb{Z})^{r-1}} \delta_{c_1,\sum_i i\beta_i} \prod_i SW(\beta_i) \prod_{i\le j} C_{ij}(q)^{\beta_i\beta_j}$$

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We have conjectural formulas for the $\phi_{r,S,c_1}(q)$ for $r \leq 5$
For simplicity assume S is minimal of general type

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Put $t_{A_r,\ell}(q) := \frac{\Theta_{A_r,0}}{\Theta_{A_r,\ell}}(q)$, with

$$\Theta_{A_r,\ell}(q) = \sum_{v \in \mathbb{Z}^r} q^{\frac{1}{2}\langle v-\ell a, v-\ell a\rangle}, \quad \ell \in \mathbb{Z}, \quad a = \frac{1}{r+1}(r, r-1, \ldots, 1)$$

Recall $\delta_{a,b} = \begin{cases} 1 & a-b \in rH^2(S,\mathbb{Z}) \\ 0 & \text{otherwise} \end{cases}$

r=2

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Conjecture (Vafa-Witten)

$$\phi_{2,S,c_1} = \delta_{c_1,0} + \delta_{c_1,K_S}(-1)^{\chi(\mathcal{O}_S)} t_{A_1,1}(q)^{K_S^2}$$

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r=3

Conjecture (G-K)

$$\phi_{3,S,c_1} = \delta_{c_1,0} t_{A_2,1}^{K_S^2}\left(X_+^{K_S^2} + X_-^{K_S^2}\right) + \left(\delta_{c_1,K_S} + \delta_{c_1,-K_S}\right)(-1)^{\chi(\mathcal{O}_S)} t_{A_2,1}^{K_S^2}$$

With $X_\pm$ *the roots of* $X^2 - 4t_{A_2,1}^2 X + 4t_{A_2,1}$

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With $X_\pm$ the roots of $X^2 - 4t_{A_2,1}^2 X + 4t_{A_2,1}$

Theta functions come from blowup formula, but already in case $r = 3$ formula is more subtle.

rank 4**:** Ramanujan's octic continued fraction

$$u(q) = \cfrac{\sqrt{2}q^{\frac{1}{8}}}{1 + \cfrac{q}{1+q+\cfrac{q^2}{1+q^2+\cfrac{q^3}{1+q^3+\cdots}}}} = \frac{\sqrt{2}\,\eta(q)\eta(q^4)^2}{\eta(q^2)^3}.$$

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Conjecture

$$\Phi_{4,S,c_1} = \delta_{c_1,0}\Big\{\Big(\frac{Z-Z^{-1}}{t_{A_3,2}^{-1}u(q^2)^{-4}-Z^{-1}}\Big)^{K_S^2} + \Big(\frac{Z^{-1}-Z}{t_{A_3,2}^{-1}u(q^2)^{-4}-Z}\Big)^{K_S^2}\Big\}$$

$$+\,\delta_{c_1,2K_S}(-1)^{\chi(\mathcal{O}_S)}\Big\{\Big(\frac{Z-Z^{-1}}{t_{A_3,2}^{-1}Z-u(q^2)^4}\Big)^{K_S^2} + \Big(\frac{Z^{-1}-Z}{t_{A_3,2}^{-1}Z^{-1}-u(q^2)^4}\Big)^{K_S^2}\Big\}$$

$$+\,(\delta_{c_1,K_S}+\delta_{c_1,-K_S})\,t_{A_3,1}^{K_S^2}\Big\{\big(1+u(q^2)^{-4}\big)^{K_S^2} + (-1)^{\chi(\mathcal{O}_S)}\big(u(q^2)^4+1\big)^{K_S^2}\Big\},$$

where Z, Z^{-1} roots of $Z - 6u^{-4} + Z^{-1} = 0$.

rank 5 *Rogers-Ramanujan continued fraction*

$$r(q) = \frac{q^{\frac{1}{5}}}{1+\frac{q}{1+\frac{q^2}{1+\cdots}}} = q^{\frac{1}{5}} \prod_{n>0} \frac{(1-q^{5n-1})(1-q^{5n-4})}{(1-q^{5n-2})(1-q^{5n-3})}$$

It satisfies $r^{-5} - 11 - r^5 = \left(\frac{\eta(q)}{\eta(q^5)}\right)^6$

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It satisfies $r^{-5} - 11 - r^5 = \left(\frac{\eta(q)}{\eta(q^5)}\right)^6$ Put

$$\beta_1 := \frac{1}{25} t_{A_4,1}\left(3r^{-5} + 2 - 8r^5\right), \quad \beta_2 := \frac{1}{25} t_{A_4,2}\left(8r^{-5} + 2 - 3r^5\right).$$

Conjecture

$$\begin{aligned}\Phi_{5,S,c_1} =& \delta_{c_1,0}\left\{\left(\frac{ZX_+^2Y_+^2}{\beta_1\beta_2}\right)^{\frac{K_S^2}{2}} + \left(\frac{X_+^2Y_-^2}{Z\beta_1\beta_2}\right)^{\frac{K_S^2}{2}} + \left(\frac{X_-^2Y_+^2}{Z\beta_1\beta_2}\right)^{\frac{K_S^2}{2}} + \left(\frac{ZX_-^2Y_-^2}{\beta_1\beta_2}\right)^{\frac{K_S^2}{2}}\right\} \\ &+ \left(\delta_{c_1,K_S} + \delta_{c_1,-K_S}\right)\left\{\beta_1^{K_S^2} + (-1)^{\chi(\mathcal{O}_S)}\left(X_+^{K_S^2} + X_-^{K_S^2}\right)\right\} \\ &+ \left(\delta_{c_1,2K_S} + \delta_{c_1,-2K_S}\right)\left\{\beta_2^{K_S^2} + (-1)^{\chi(\mathcal{O}_S)}\left(Y_+^{K_S^2} + Y_-^{K_S^2}\right)\right\},\end{aligned}$$

rank 5 *Rogers-Ramanujan continued fraction*

$$r(q) = \frac{q^{\frac{1}{5}}}{1+\frac{q}{1+\frac{q^2}{1+\cdots}}} = q^{\frac{1}{5}} \prod_{n>0} \frac{(1-q^{5n-1})(1-q^{5n-4})}{(1-q^{5n-2})(1-q^{5n-3})}$$

It satisfies $r^{-5} - 11 - r^5 = \left(\frac{\eta(q)}{\eta(q^5)}\right)^6$ Put

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$X_\pm, Y_\pm, Z, Z^{-1}$ *roots of*

$$X^2 - \frac{4}{5}\beta_1\left(\beta_1 t_{A_4,1}^{-1} - 1\right)\left(3r^{-5} + 1\right)X + \frac{4}{5}\beta_1^2\left(3r^{-5} + 1\right) = 0,$$

$$Y^2 - \frac{4}{5}\beta_2\left(\beta_2 t_{A_4,2}^{-1} - 1\right)\left(1 - 3r^5\right)Y + \frac{4}{5}\beta_2^2\left(1 - 3r^5\right) = 0,$$

$$Z - \frac{6}{25}\left(8r^{-5} - 13 - 8r^5\right) + Z^{-1} = 0.$$

In all these cases C_0 and the C_{ij} are modular functions, in a finite algebraic extension of the field of modular functions on $\Gamma_0(r)$

We also have partial results in ranks $6, 7$, expressing (some of) the C_0, C_{ij} in terms of the $t_{A_{r-1},\ell}$ and the Hauptmodul (generator) for $\Gamma_0(r)$ by algebraic equations.

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We conjecture a similar structure formula to Laarakker's formula also for the horizontal Vafa-Witten invariants (virtual Euler numbers of moduli spaces of sheaves)

The $A_r^\vee$ lattice is defined as $\mathbb{Z}^r$ with bilinear pairing $\langle\cdot,\cdot\rangle^\vee$ determined by the inverse of the matrix defining $\langle\cdot,\cdot\rangle$

$$\Theta_{A_r^\vee,\ell}(q) = \sum_{v\in\mathbb{Z}^r} e^{2\pi i\langle v,\ell(1,0,\ldots,0)\rangle^\vee} q^{\frac{1}{2}\langle v,v\rangle^\vee}, \quad \ell\in\mathbb{Z},$$

$$t_{A_r^\vee,\ell}(q) = \frac{\Theta_{A_r^\vee,0}(q)}{\Theta_{A_r^\vee,\ell}(q)}$$

Conjecture

Given $r > 0 \;\exists\; D_0, (D_{ij})_{1\le i\le j\le r-1} \in \mathbb{C}[[q^{\frac{1}{2r}}]]$ *s.th. for all* (S, H, c_1) $e^{\mathrm{vir}}(M_S^H(r, c_1, c_2))$ *is coefficient of* $q^{\frac{1}{2r}\mathrm{vd}(r,c_1,c_2)-\frac{r}{2}\chi(\mathcal{O}_S)+\frac{r}{24}K_S^2}$ *of*

$$r^{2+K_S^2-\chi(\mathcal{O}_S)}\left(\frac{1}{\Delta(q^{\frac{1}{r}})^{\frac{1}{2}}}\right)^{\chi(\mathcal{O}_S)}\left(\frac{\Theta_{A_{r-1}^\vee,0}(q)}{\eta(q)^r}\right)^{-K_S^2}$$

$$\times D_0(q)^{K_S^2}\sum_{\beta\in H^2(S,\mathbb{Z})^{r-1}}\prod_i \varepsilon_r^{i\beta_i c_1} SW(\beta_i)\prod_{i\le j} D_{ij}^{\beta_i\beta_j}$$

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We put

$$\psi_{r,S,c_1} := D_0(q)^{K_S^2}\sum_{\beta\in H^2(S,\mathbb{Z})^{r-1}}\prod_i \varepsilon_r^{i\beta_i c_1} SW(\beta_i)\prod_{i\le j} D_{ij}^{\beta_i\beta_j}$$

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Proposition

Assume the above conjecture holds. Then $\psi_{r,S,c_1}(q)$ is obtained from $\phi_{r,S,c_1}(q)$ by replacing

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4. *for* $r = 5$ *replacing* $r(q)$ *by* $\frac{1-\varphi r(q)}{\varphi + r(q)}$ *with* $\varphi = \frac{1+\sqrt{5}}{2}$

$N^{\mathbb{C}^*}_{(1^r)}$ param. (E, ϕ) with

$$E = \oplus_i E_i, \quad \mathrm{rk}(E_i) = 1, \quad \phi : E_i \to E_{i-1} \otimes K_S$$

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$$S^{[\overrightarrow{n}]} = \prod_{i=0}^{r-1} S^{[n_i]}, \quad |\overrightarrow{\beta}| = \prod_{i=1}^{r-1} |\beta_i|$$

and the nested Hilbert scheme

$$S^{[\overrightarrow{n}]}_{\overrightarrow{\beta}} := \left\{ (Z_0, \ldots, Z_{r-1}, C_1, \ldots, C_{r-1}) \;\middle|\; I_{Z_i}(-C_i)) \subset I_{Z_{i-1}} \right\} \subset S^{[\overrightarrow{n}]} \times |\overrightarrow{\beta}|$$

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$\bigcup_{c_2} N^H_S(r, c_1, c_2)^{\mathbb{C}^*}_{(1^r)}$ is isomorphic to a union of certain $S^{[\overrightarrow{n}]}_{\overrightarrow{\beta}}$

Putting

$$Q(a_1, \ldots, a_{r-1}) := -\sum_{i<j} \frac{i(r-j)}{r} a_i a_j + \sum_{i=1}^{r-1} \frac{i(r-i)}{2r} a_i^2$$

$N_S^H(r, c_1, c_2)^{\mathbb{C}^*}_{(r)}$ is isomorphic to the union of the stable $S^{[\overrightarrow{n}]}_{\overrightarrow{\beta}}$ such that

$$c_1 \equiv \sum_{i=1}^{r-1} i(K_S - \beta_i) \mod rH^2(S, \mathbb{Z})$$

$$c_2 = |\overrightarrow{n}| + \frac{r-1}{2r} c_1^2 + Q(K_S - \beta_1, \ldots, K_S - \beta_{r-1})$$

Gholampour-Thomas: define virtual fund. class on $S^{[\vec{n}]}_{\vec{\beta}}$,
coinciding with that on $N^{\mathbb{C}^*}_{1^r}$ from virtual localization
Write $\mathcal{I}_i$ pullback of i^{th} universal ideal sheaf to $S \times S^{[\vec{n}]} \times |\vec{\beta}|$,
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Then the pushforward of $[S^{[\vec{n}]}_{\vec{\beta}}]^{\mathrm{vir}}$ to $S^{[\vec{n}]} \times |\vec{\beta}|$ is

$$\prod_{i=1}^{r-1} SW(\beta_i) e(R^*\Gamma(S, \mathcal{O}_S(\beta_i)) \otimes \mathcal{O} - R\mathcal{H}om_\pi(\mathcal{I}_{i-1}, \mathcal{I}_i(\beta_i)))$$
$$\cap [S^{[\vec{n}]} \times pt \times \ldots \times pt].$$

For effective $\overrightarrow{\beta}$ consider line bundles $L_0, \ldots, L_{r-1}$ with $K_S - \beta_i = c_1(L_{i-1}) - c_1(L_i)$ and $\sum_{i=0}^{r-1} c_1(L_i) = c_1$
On $S \times S^{[\overrightarrow{n}]} \times |\overrightarrow{\beta}|$ have line bundles $\mathcal{L}_i = L_i \otimes \boxtimes_{j=1}^{r} \mathcal{O}_{|\beta_j|}(1)$
Put $\mathcal{E} = \sum_{i=0}^{r-1} \mathcal{I}_i \otimes \mathcal{L}_i \times \mathfrak{t}^{-1}$
Get tautological equivariant Higgs pair $\phi : \mathcal{E} \to: \mathcal{E} \otimes K_S$

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Contribution of $S^{[\overrightarrow{n}]}_{\overrightarrow{\beta}} \subset N^{\mathbb{C}^*}_{(1^r)}$ is

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Laarakker shows

$$T_N^{\mathrm{vir}}|_{S^{[\overrightarrow{n}]}_{\overrightarrow{\beta}}} = R^*\mathcal{H}om_\pi(\mathcal{E}, \mathcal{E} \otimes K_S \otimes \mathfrak{t}) - R^*\mathcal{H}om_\pi(\mathcal{E}, \mathcal{E}).$$

Alltogether

$$Z^{(1^r)}_{S,H,c_1}(q) = q^{-\frac{r\chi(\mathcal{O}_S)}{2}+\frac{rK_S^2}{24}} \sum_{\overrightarrow{n},\overrightarrow{a}} \delta_{c_1,\sum ia_i} q^{Q(\overrightarrow{a})+|\overrightarrow{n}|} \prod_{i=1}^{r-1} SW(a_i) \int_{S^{[\overrightarrow{n}]}} \Upsilon(\overrightarrow{a},\overrightarrow{n},t)$$

$\Upsilon(\overrightarrow{a},\overrightarrow{n},t)$ explicit equivariant class on $S^{[\overrightarrow{n}]}$ with constant term
$\Upsilon(\overrightarrow{a}) = \left(\frac{(-1)^{r-1}}{r}\right)^{\chi(\mathcal{O}_S)} \prod_{i=1}^{r-1} \binom{r}{i}^{-a_i^2} \prod_{1\leq i<j\leq r-1} \left(\frac{j(r-i)}{(j-i)r}\right)^{a_i a_j}$

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Normalized generating series

$$G_{S,\vec{a}}(q) = \frac{1}{\Upsilon(\vec{a})} \sum_{\vec{n}} q^{|\vec{n}|} \int_{S^{[\vec{n}]}} \Upsilon(\vec{a},\vec{n},t)$$

$\Upsilon(\vec{a},\vec{n},t)$ expressed in terms of Chern classes of tautological sheaves, universal ideal sheaves and a_i. By cobordism invariance, $\int_{S^{[\vec{n}]}} \Upsilon(\vec{a},\vec{n},t)$ is universal polynomial in $\chi(\mathcal{O}_S)$, K_S^2, a_iK_S, a_ia_j

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This gives there are universal power series s.th.

$$G_{S,\vec{a}}(q) = A(q)^{\chi(\mathcal{O}_S)} B(q)^{K_S^2} \prod_i E_i(q)^{a_i K_S} \prod_{i \leq j} E_{ij}(q)^{a_i a_j}$$

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Compute $G_{S, \vec{a}}(q)$ for $\binom{r+1}{2} + 1$ tuples $(S_i, \vec{a}_i)$ of toric surfaces and toric line bundles such that vectors of numbers $\chi(\mathcal{O}_S)$, K_S^2, $a_i K_S$, $a_i a_j$ are linearly independent

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Contribution of each fixpoint given by the combinatorics of partitions

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With this use a Pari/GP program to compute normalized C_0, C_{ij} for $r \leq 7$ modulo $O(q^{13})$

$$(S_1 \sqcup S_2)^{[\vec{n}]} = \coprod_{\vec{n}_1+\vec{n}_2=\vec{n}} S_1^{[\vec{n}_1]} \times S_2^{[\vec{n}_1]} \Longrightarrow G_{S_1 \sqcup S_2, \vec{a}_1 \sqcup \vec{a}_2} = G_{S_1, \vec{a}_1} G_{S_2, \vec{a}_2}$$

This gives there are universal power series s.th.

$$G_{S,\vec{a}}(q) = A(q)^{\chi(\mathcal{O}_S)} B(q)^{K_S^2} \prod_i E_i(q)^{a_i K_S} \prod_{i \le j} E_{ij}(q)^{a_i a_j}$$

Compute $G_{S,\vec{a}}(q)$ for $\binom{r+1}{2} + 1$ tuples $(S_i, \vec{a}_i)$ of toric surfaces and toric line bundles such that vectors of numbers $\chi(\mathcal{O}_S)$, K_S^2, $a_i K_S$, $a_i a_j$ are linearly independent

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Find relations between these power series until they are determined

Let (S, H) be a smooth polarized surface with $H_1(S, \mathbb{Z}) = 0$, and let $r > 0$, $c_1 \in H^2(S, \mathbb{Z})$, $c_2 \in H^4(S, \mathbb{Z})$. $N := N_S^H(r, c_1, c_2)$ has virtual structure sheaf $\mathcal{O}_N^{\mathrm{vir}} \in K_0^{\mathbb{C}^*}(N)$. We put

$$\widehat{\mathcal{O}}_N^{\mathrm{vir}} = \mathcal{O}_N^{\mathrm{vir}} \otimes (K_N^{\mathrm{vir}})^{\frac{1}{2}},$$

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$$\chi(N, \widehat{\mathcal{O}}_N^{\mathrm{vir}}) = \chi\Big(N^{\mathbb{C}^*}, \frac{\mathcal{O}_{N^{\mathbb{C}^*}}^{\mathrm{vir}} \otimes (K_N^{\mathrm{vir}})^{\frac{1}{2}}|_{N^{\mathbb{C}^*}}}{\Lambda_{-1}(\nu^{\mathrm{vir}})^\vee}\Big) \in \mathbb{Q}(y^{\frac{1}{2}}).$$

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The contribution from the Gieseker-Maruyama moduli space $M:=M_S^H(r,c_1,c_2)$ is

$$(-1)^{\mathrm{vd}}y^{-\frac{\mathrm{vd}}{2}}\chi_{-y}^{\mathrm{vir}}(M),\quad \mathrm{vd}=\mathrm{vd}(M).$$

K-theoretic $SU(r)$ Vafa-Witten partition function

$$Z^{SU(r)}_{S,H,c_1}(q,y)=r^{-1}q^{-\frac{\chi(\mathcal{O}_S)}{2r}+\frac{rK_S^2}{24}}\sum_{c_2\in\mathbb{Z}}q^{\frac{1}{2r}\mathrm{vd}(r,c_1,c_2)}(-1)^{\mathrm{vd}(r,c_1,c_2)}\chi(N^H_S(r,c_1,c_2),\widehat{\mathcal{O}}^{\mathrm{vir}}_{N^H_S}).$$

Again we have

$$Z^{SU(r)}_{S,H,c_1}(q,y)=r^{-1}\sum_{\lambda}\mathbb{Z}^{\lambda}_{S,H,c_1}(q,y).$$

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Theta functions:

$$\Theta_{A_r,\ell}(q,y)=\sum_{v\in\mathbb{Z}^r}q^{\frac{1}{2}\langle v-\ell\lambda,v-\ell\lambda\rangle}y^{\langle v,M_{A_r}^{-1}(1,\ldots,1)\rangle},\quad \lambda=\frac{1}{r+1}(r,r-1,\ldots,1)$$

$$t_{A_r,\ell}(q,y)=\frac{\Theta_{A_r,0}(q,y)}{\Theta_{A_r,\ell}(q,y)},\quad \ell\in\mathbb{Z},$$

where $\langle\cdot,\cdot\rangle$ is the symmetric bilinear form of the A_r lattice and M_{A_r} is the corresponding symmetric matrix

$$\phi_{-2,1}(q,y) = (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^2 \prod_{n=1}^{\infty} \frac{(1-yq^n)^2(1-y^{-1}q^n)^2}{(1-q^n)^4},$$

Theorem (Laarakker)

For any $r > 1$, there exist C_0, $\{C_{ij}\}_{1\le i\le j\le r-1} \in \mathbb{Q}(y^{\frac{1}{2}})((q^{\frac{1}{2r}}))$ with the following property. For any smooth polarized surface (S,H) satisfying $H_1(S,\mathbb{Z}) = 0$, $p_g(S) > 0$, and $c_1 \in H^2(S,\mathbb{Z})$, we have

$$\frac{Z^{(1^r)}_{S,H,c_1}(q,y)}{(y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{\chi(\mathcal{O}_S)}} = \left(\frac{(-1)^{r-1}}{\phi_{-2,1}(q^r,y^r)^{\frac{1}{2}}\Delta(q^r)^{\frac{1}{2}}}\right)^{\chi(\mathcal{O}_S)} \left(\frac{\Theta_{A_{r-1},0}(q,y)}{\eta(q)^r}\right)^{-K_S^2}$$
$$\times\, C_0(q,y)^{K_S^2} \sum_{\beta} \in H^2(S,\mathbb{Z})^{r-1} \delta_{c_1,\sum_i i\beta_i} \prod_i SW(\beta_i) \prod_{i\le j} C_{ij}(q,y)^{\beta_i\beta_j}.$$

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Put

$$\phi_{r,S,c_1}(q,y) := C_0(q,y)^{K_S^2} \sum_{\beta\in H^2(S,Z)^{r-1}} \delta_{c_1,\sum_i i\beta_i} \prod_i SW(\beta_i) \prod_{i\leq j} C_{ij}(q,y)^{\beta_i\beta_j}$$

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r=2

Conjecture

$$\phi_{2,S,c_1} = \delta_{c_1,0} + \delta_{c_1,K_S}(-1)^{\chi(\mathcal{O}_S)} t_{A_1,1}(q,y)^{K_S^2}$$

$$\phi_{r,S,c_1}(q,y) := C_0(q,y)^{K_S^2} \sum_{\beta\in H^2(S,Z)^{r-1}} \delta_{c_1,\sum_i i\beta_i} \prod_i SW(\beta_i) \prod_{i\leq j} C_{ij}(q,y)^{\beta_i\beta_j}$$

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r=3

Conjecture

$$\phi_{3,S,c_1} = \delta_{c_1,0} t_{A_2,1}(q,y)^{K_S^2}(X_+^{K_S^2}+X_-^{K_S^2})+(\delta_{c_1,K_S}+\delta_{c_1,-K_S})(-1)^{\chi(\mathcal{O}_S)} t_{A_2,1}(q,y)^{K_S^2}$$

With $X_\pm$ the roots of

$$X^2 - t_{A_2,1}(q,y)(t_{A_2,1}(q,y)+3t_{A_2,1}(q,1))X + t_{A_2,1}(q,y)+3t_{A_2,1}(q,1) = 0$$

$$\phi_{-2,1}(q,y) = (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^2 \prod_{n=1}^{\infty} \frac{(1-yq^n)^2(1-y^{-1}q^n)^2}{(1-q^n)^4},$$

$$J(q,y) = \frac{u(q^2)^4}{4} \frac{\phi_{-2,1}(q^2,y^2)\phi_{-2,1}(q^8,y^4)^2}{\phi_{-2,1}(q^4,y^2)^2\phi_{-2,1}(q^4,y^4)}.$$

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Conjecture

$$\Phi_{4,S,c_1} = \delta_{c_1,0}\Big\{\Big(\frac{Z-Z^{-1}}{t_{A_3,2}^{-1}J^{-1}-Z^{-1}}\Big)^{K_S^2} + \Big(\frac{Z^{-1}-Z}{t_{A_3,2}^{-1}J^{-1}-Z},\Big)^{K_S^2}\Big\}$$
$$+\delta_{c_1,2K_S}(-1)^{\chi(\mathcal{O}_S)}\Big\{\Big(\frac{Z-Z^{-1}}{t_{A_3,2}^{-1}Z-J},\Big)^{K_S^2} + \Big(\frac{Z^{-1}-Z}{t_{A_3,2}^{-1}Z^{-1}-J}\Big)^{K_S^2}\Big\}$$
$$+(\delta_{c_1,K_S}+\delta_{c_1,-K_S})\,t_{A_3,1}(1+J^{-1}),\Big)^{K_S^2} + (-1)^{\chi(\mathcal{O}_S)}\big(t_{A_3,1}(J+1)\big)^{K_S^2}\Big\},$$

where Z, Z^{-1} *roots of*

$$Z^2 - (J^{\frac{1}{2}} + J^{-\frac{1}{2}} + 2J(q,1)^{\frac{1}{2}} + 2J(q,1)^{-\frac{1}{2}})Z + 1 = 0$$

Analoguously to the nonrefined case the horizontal refined Vafa-Witten invariants are again determined from the vertical ones via S-duality. With notations similar to the nonrefined case we have

Conjecture

With $q = e^{2\pi i\tau}$, $y = e^{2\pi iz}$ *with* $(\tau, z) \in \mathfrak{H} \times \mathbb{C}$ *we have*

$$D_0(\tau, z) = C_0(-1/\tau, z/\tau), \quad D_{ij}(\tau, z) = C_{ij}(-1/\tau, z/\tau).$$