The skein algebra of the 4-punctured sphere from curve counting

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Enumerative Geometry, Physics and Representation Theory

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Talk based on arXiv:2009.02266

Introduction

Topics:

- Low-dimensional topology.
- Complex enumerative algebraic geometry.
- String theory realizations of supersymmetric gauge theories.

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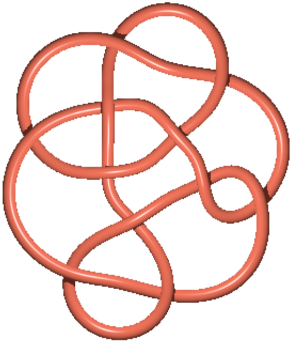
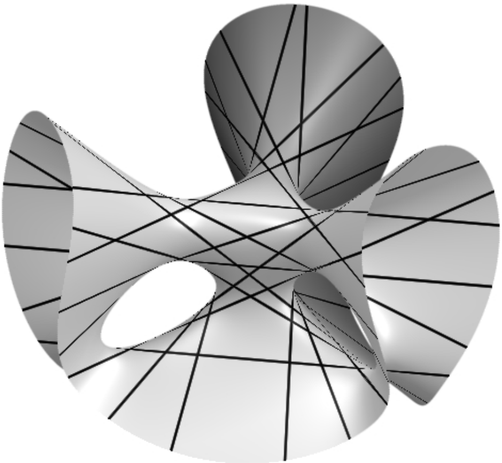
- Low-dimensional topology: knots, links...

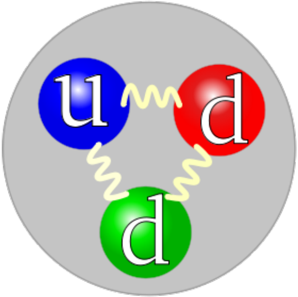


Introduction

- Complex (over $\mathbb{C}$) enumerative algebraic geometry: 27 complex lines on a complex cubic surface (Cayley, Salmon, 1849)



- String theory realizations of supersymmetric gauge theories



Today, focus on a specific example:

- Low-dimensional topology: skein algebra of $\mathbb{S}_{0,4}$, knots and links in $\mathbb{S}_{0,4} \times (0,1)$.
- Enumerative algebraic geometry: counting holomorphic curves in complex cubic surfaces.
- Physics: 4-dimensional $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory with 4 hypermultiplets in the fundamental representation ($N_f = 4$).

Non-trivial mathematical consequences: proof of positivity conjectures about the skein algebra of $\mathbb{S}_{0,4}$ (Thurston (2013), Bakshi, Mukherjee, Przytycki, Silvero and Wang (2018)) (so about curves drawn on a 4-punctured sphere) by counting Riemann surfaces in a complex cubic surface!

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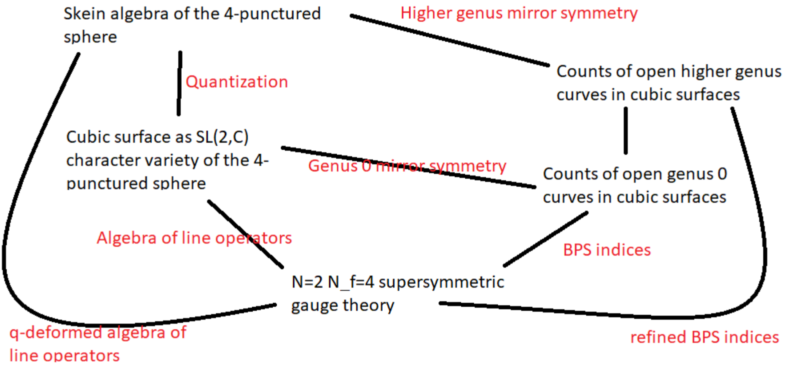
Summary



Skein algebra of the 4-punctured sphere
Higher genus mirror symmetry
Quantization
Counts of open higher genus curves in cubic surfaces
Cubic surface as SL(2,C) character variety of the 4-punctured sphere
Genus 0 mirror symmetry
Counts of open genus 0 curves in cubic surfaces
Algebra of line operators
BPS indices
N=2 N_f=4 supersymmetric gauge theory
q-deformed algebra of line operators
refined BPS indices

Plan

- The cubic surface as $SL_2(\mathbb{C})$ character variety and quantization via the skein algebra.
- Mirror symmetry and curve counting for the cubic surface. Quantization from higher genus curve counting.
- Comparison between the skein algebra and the higher genus mirror symmetry quantizations.

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Character varieties

- Γ a finitely generated group, G a reductive algebraic group over $\mathbb{C}$ (e.g. $G = GL_n(\mathbb{C})$ or $SL_n(\mathbb{C})$)
- Affine variety $\mathrm{Hom}(\Gamma, G)$ of group morphisms from Γ to G.
- Natural action of G on $\mathrm{Hom}(\Gamma, G)$ by conjugation.
- Take the quotient in the sense of geometric invariant theory, get the character variety: $\mathrm{Ch}_G(\Gamma) = \mathrm{Spec}\,(\mathcal{O}(\mathrm{Hom}(\Gamma, G))^G)$. It is an affine variety of finite type over $\mathbb{C}$.
- Particularly interesting case: $\Gamma = \pi_1(\Sigma)$ for Σ a finite type topological space. Denote $\mathrm{Ch}_G(\Sigma) := \mathrm{Ch}_G(\pi_1(\Sigma))$.
- Take $\Sigma = \mathbb{S}_{g,\ell}$, a topological surface, complement of ℓ points in a genus g compact orientable surface.
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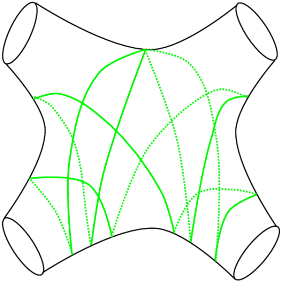
Example: $\ell = 0$, $G = GL_1(\mathbb{C}) = \mathbb{C}^*$, then $\mathrm{Ch}_{GL_1(\mathbb{C})}(\mathbb{S}_{g,0}) = (\mathbb{C}^*)^{2g}$. More precisely, taking monodromy around elements of a basis $(\gamma_j)_{1 \leq j \leq 2g}$ of $H_1(\mathbb{S}_{g,0}, \mathbb{Z})$, get monomials z^{γ_j} on $(\mathbb{C}^*)^{2g}$. Poisson bracket:

$$\{z^{\gamma_i}, z^{\gamma_j}\} = \langle \gamma_i, \gamma_j \rangle z^{\gamma_i} z^{\gamma_j}$$

where $\langle \gamma_i, \gamma_j \rangle$ is the intersection number of γ_i and γ_j.

$X = \mathrm{Ch}_{SL_2(\mathbb{C})}(\mathbb{S}_{0,4})$

Example: $g = 0$, $\ell = 4$, $G = SL_2(\mathbb{C})$, then $X = \mathrm{Ch}_{SL_2(\mathbb{C})}(\mathbb{S}_{0,4})$ is a 4-parameter family of affine cubic surfaces (Vogt 1889, Fricke, 1896).



Functions on $\mathrm{Ch}_{SL_2(\mathbb{C})}(\mathbb{S}_{0,4})$ are obtained by taking trace of the monodromy around loops on $\mathbb{S}_{0,4}$. Algebra generators: $a_1, a_2, a_3, a_4, \gamma_{v_1}, \gamma_{v_2}, \gamma_{v_3}$, where a_1, a_2, a_3, a_4 are traces around small loops around the punctures, and γ_{v_1}, γ_{v_2} and γ_{v_3} are traces around loops separating the set of the 4 punctures into two.

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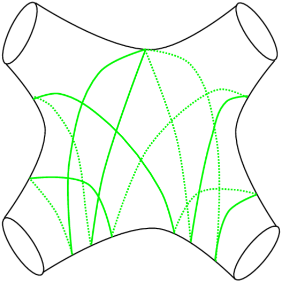

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$$\gamma_{v_1}\gamma_{v_2}\gamma_{v_3} = \gamma_{v_1}^2 + \gamma_{v_2}^2 + \gamma_{v_3}^2 + R_{1,0}\gamma_{v_1} + R_{0,1}\gamma_{v_2} + R_{1,1}\gamma_{v_3} + y - 4\,,$$

where

$$R_{1,0} := a_1 a_2 + a_3 a_4\,,\;\; R_{0,1} := a_1 a_3 + a_2 a_4\,,\;\; R_{1,1} := a_1 a_4 + a_2 a_3\,,$$
$$y := a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2$$

a_1, a_2, a_3, a_4 are in the center of the Poisson bracket, fixing them, get a cubic surface. Non-trivial Poisson brackets:

$$\{\gamma_{v_1}, \gamma_{v_2}\} = \gamma_{v_1}\gamma_{v_2} + 2\gamma_{v_3} - R_{1,1}\,,\; \{\gamma_{v_2}, \gamma_{v_3}\} = \gamma_{v_2}\gamma_{v_3} + 2\gamma_{v_1} - R_{1,0}\,,$$

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- As a character variety: Riemann-Hilbert analytic isomorphism with moduli space of flat connections with regular singularities, non-abelian Hodge correspondence: homeomorphic to a moduli space of parabolic Higgs bundles. Hitchin elliptic fibration, Seiberg-Witten geometry of $\mathcal{N} = 2$ $N_f = 4$ $SU(2)$ gauge theory.
- Smooth fibers of $\nu \colon X = \mathrm{Ch}_{SL_2(\mathbb{C})}(\mathbb{S}_{0,4}) \to \mathbb{A}^4_{a_1,a_2,a_3,a_4}$ admits complete hyperkähler metrics.
- Specific to $\mathbb{S}_{0,4}$: phase space of the Painlevé VI non-linear differential equation (isomonodromy condition for $SL_2(\mathbb{C})$-connections on $\mathbb{S}_{0,4}$).
- Rich dynamics of the mapping class group action.

In this talk, focus on the question of quantizing X. Two approaches: one is well-known (via 3-dimensional topology and the skein algebra), the other is new (higher-genus version of mirror symmetry). Non-trivial results when comparing the two.

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Deformation quantization

Definition

Let $(A, \{-,-\})$ be a Poisson algebra. A *deformation quantization* of A is a flat formal 1-parameter family of associative algebras $A_\hbar$ such that

- $A_{\hbar=0} = A$
- if we lift elements $f, g \in A$ to $\tilde{f}, \tilde{g} \in A_\hbar$, then

$$\tilde{f}\tilde{g} - \tilde{g}\tilde{f} = \{f, g\}\hbar + O(\hbar^2)\,.$$

General questions: given a Poisson algebra, can we find a deformation quantization? Can we find a “nice" deformation quantizations?

One can ask these questions for the algebra of regular functions of the character varieties $\mathrm{Ch}_G(\mathbb{S}_{g,\ell})$, and in particular for $X = \mathrm{Ch}_{SL_2(\mathbb{C})}(\mathbb{S}_{0,4})$.

Deformation quantization

- Example: $\ell = 0$, $G = GL_1(\mathbb{C}) = \mathbb{C}^*$, then $\mathrm{Ch}_{GL_1(\mathbb{C})}(\mathbb{S}_{g,0}) = (\mathbb{C}^*)^{2g}$.

$$\{z^{\gamma_i}, z^{\gamma_j}\} = \langle \gamma_i, \gamma_j \rangle z^{\gamma_i} z^{\gamma_j}$$

  A "nice" deformation quantization is provided by the quantum torus: $\hat{z}^{\gamma_i}\hat{z}^{\gamma_j} = q^{\langle \gamma_i, \gamma_j \rangle} \hat{z}^{\gamma_j}\hat{z}^{\gamma_i}$, where $q = e^{\hbar}$.

- For $X = \mathrm{Ch}_{SL_2(\mathbb{C})}(\mathbb{S}_{0,4})$, it is non-trivial

$$\{\gamma_{v_1}, \gamma_{v_2}\} = \gamma_{v_1}\gamma_{v_2} + 2\gamma_{v_3} - R_{1,1}\,,\; \{\gamma_{v_2}, \gamma_{v_3}\} = \gamma_{v_2}\gamma_{v_3} + 2\gamma_{v_1} - R_{1,0}\,,$$

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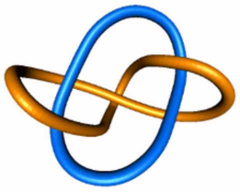
Knots, links and framing



- Knot in a manifold: a connected compact embedded 1-dimensional submanifold.
- Link in a manifold: the disjoint union of finitely many knots.
- Framing of a link: a choice of nowhere vanishing section of its normal bundle, that is the choice of realization of the link as the union of boundary components of some annuli.

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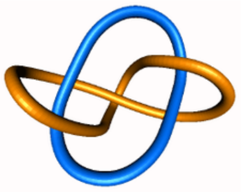

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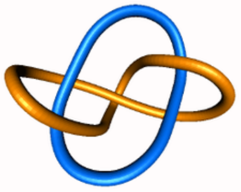

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Skein modules of 3-manifolds

- The Kauffman bracket skein module (Przytycki, Turaev, 1988) of an oriented 3-manifold $\mathbb{M}$ is the $\mathbb{Z}[A^{\pm}]$-module generated by isotopy classes of framed links in $\mathbb{M}$ satisfying the skein relations

- The diagrams in each relation indicate framed links that can be isotoped to identical embeddings except within the neighborhood shown, where the framing is vertical.
- The skein module of $\mathbb{M} = \mathbb{R}^3$ is $\mathbb{Z}[A^{\pm}]$ (generated by the empty link). The class of a framed link $L \subset \mathbb{R}^3$ in $\mathbb{Z}[A^{\pm}]$ is the Kauffman bracket polynomial of L (equivalent to the Jones polynomial).

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 $= A$  $+ A^{-1}$  and $L \cup$  $= -(A^2 + A^{-2})\, L\,.$

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- Given an oriented 2-manifold $\mathbb{S}$, one can define a natural algebra structure on the Kauffmann bracket skein module of the 3-manifold $\mathbb{M} := \mathbb{S} \times (-1,1)$: given two framed links L_1 and L_2 in $\mathbb{S} \times (-1,1)$, and viewing the interval $(-1,1)$ as a vertical direction, the product $L_1 L_2$ is defined by placing L_1 on top of L_2.
- We denote by $\mathrm{Sk}_A(\mathbb{S})$ the resulting associative $\mathbb{Z}[A^{\pm}]$-algebra with unit. The skein algebra $\mathrm{Sk}_A(\mathbb{S})$ is in general non-commutative.

Skein algebras of surfaces

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- We consider the case where $\mathbb{S}$ is the complement $\mathbb{S}_{g,\ell}$ of a finite number ℓ of points in a compact oriented 2-manifold of genus g.
- A multicurve on $\mathbb{S}_{g,\ell}$ is the union of finitely many disjoint compact connected embedded 1-dimensional submanifolds of $\mathbb{S}_{g,\ell}$ such that none of them bounds a disc in $\mathbb{S}_{g,\ell}$. Identifying $\mathbb{S}_{g,\ell}$ with $\mathbb{S}_{g,\ell} \times \{0\} \subset \mathbb{S}_{g,\ell} \times (-1,1)$, a multicurve on $\mathbb{S}_{g,\ell}$ endowed with the vertical framing naturally defined a framed link in $\mathbb{S}_{g,\ell} \times (-1,1)$.

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The skein algebra quantization of the SL_2 character variety

- For every γ multicurve on $\mathbb{S}_{g,\ell}$ with connected components $\gamma_1, \cdots, \gamma_r$, the map sending a representation $\rho\colon \pi_1(\mathbb{S}_{g,\ell}) \to SL_2(\mathbb{C})$ to $\prod_{j=1}^r (-\operatorname{tr}(\rho(\gamma_j)))$ defines a regular function f_γ on $\mathrm{Ch}_{SL_2(\mathbb{C})}(\mathbb{S}_{g,\ell})$.

Theorem (Bullock, Przytycki-Sikora, Charles-Marché)

The skein algebra $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ with $A = -e^{\frac{\hbar}{4}}$ is a deformation quantization of the algebra of regular functions on $X = \mathrm{Ch}_{SL_2(\mathbb{C})}(\mathbb{S}_{g,\ell})$. The isomorphism at $A = -1$ is given by $\gamma \mapsto f_\gamma$.

Classical limit of the skein relation: for every $M, N \in SL_2(\mathbb{C})$,

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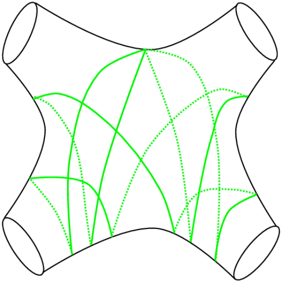
The skein algebra of the 4-punctured sphere

- Focus on the case of the 4-punctured sphere $\mathbb{S}_{0,4}$.
- Peripheral curves a_1, a_2, a_3, a_4, in the center of $\mathrm{Sk}_A(\mathbb{S}_{0,4})$, so we can view $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ as a $\mathbb{Z}[A^{\pm}][a_1, a_2, a_3, a_4]$-module.
- Isotopy classes of multicurves in $\mathbb{S}_{0,4}$ without peripheral connected components are in bjection with

$$B(\mathbb{Z}) := \mathbb{Z}^2/\langle \pm id \rangle \simeq \{(m,n) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \mid m \geq 0 \text{ if } n = 0\} .$$

- $\{\gamma_p\}_{p \in B(\mathbb{Z})}$ is a basis of $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ as $\mathbb{Z}[A^{\pm}][a_1, a_2, a_3, a_4]$-module.



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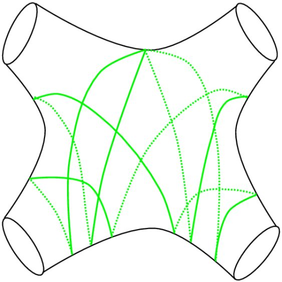

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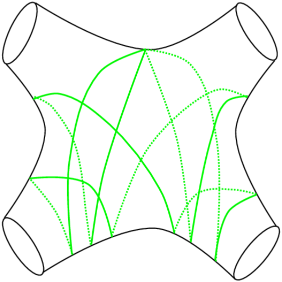

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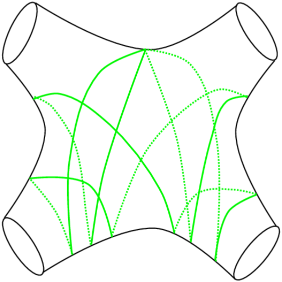

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Theorem (Bullock-Przytycki, 2000)

$\mathrm{Sk}_A(\mathbb{S}_{0,4})$ is generated as $\mathbb{Z}[A^{\pm}][a_1, a_2, a_3, a_4]$-algebra by $\gamma_{v_1} := \gamma_{(1,0)}$, $\gamma_{v_2} := \gamma_{(0,1)}$, $\gamma_{v_3} = \gamma_{(-1,1)}$, with the relations

$$A^{-2}\gamma_{v_1}\gamma_{v_2} - A^2\gamma_{v_2}\gamma_{v_1} = (A^{-4} - A^4)\gamma_{v_3} - (A^2 - A^{-2})R_{1,1}\,,$$

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$$A^{-2}\gamma_{v_1}\gamma_{v_2}\gamma_{v_3} = A^{-4}\gamma_{v_1}^2 + A^4\gamma_{v_2}^2 + A^{-4}\gamma_{v_3}^2 + A^{-2}R_{1,0}\gamma_{v_1} + A^2R_{0,1}\gamma_{v_2}$$

$$+A^{-2}R_{1,1}\gamma_{v_3} + y - 2(A^4 + A^{-4})\,.$$

Same algebra from quantum Liouville theory (Teschner, Vartanov, 2013).

Mirror symmetry for log Calabi-Yau surfaces

Mirror symmetry between two Calabi-Yau varieties: exchanges symplectic geometry and complex geometry.

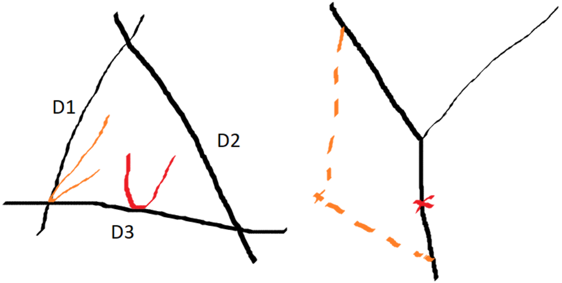
- Non-compact Calabi-Yau varieties.
- Log Calabi-Yau variety: (Y, D), Y compact, D anticanonical divisor, $V = Y - D$ is non-compact Calabi-Yau.
- Mirror symmetry as a way to construct algebraic varieties.
- In dimension 2, mirror symmetry construction for log Calabi-Yau surfaces (Gross-Hacking-Keel, 2001).
- Enumerative geometry: counts rational curves in (Y, D) (morally holomorphic curves in $V = Y - D$).
- Construction of the mirror family $\mathcal{V} \to \text{Spec}\ \mathbb{C}[NE(Y)]$.
- Claim: one recovers $X = \text{Ch}_{SL_2(\mathbb{C})}(\mathbb{S}_{0,4}) \to \mathbb{A}^4$ for Y: smooth projective cubic surface, and D: triangle of lines on Y.

Enumerative geometry of log Calabi-Yau surfaces

- Y: smooth projective surface, $D = D_1 + ... + D_n$ anticanonical cycle of rational curves.
- Fix a curve class $\beta \in NE(Y) \subset H_2(Y, \mathbb{Z})$.
- Want to count rational curves (genus 0) C in Y of class β such that $C \cap D$ is a single point.
- Tangency condition at the intersection point?

Enumerative geometry of log Calabi-Yau surfaces

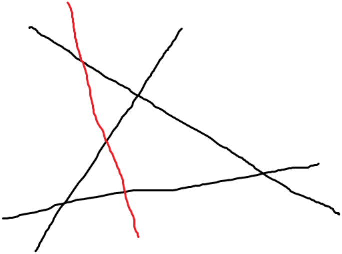
- B: dual intersection complex of (Y, D), cone $\mathbb{R}^2_{\geq 0}$ for each intersection $D_i \cap D_{i+1}$.
- Contact order of a curve with D: $B(\mathbb{Z})$.



Enumerative geometry of log Calabi-Yau surfaces

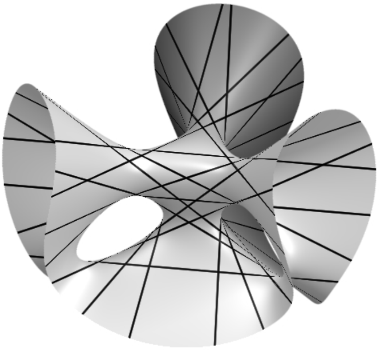
- $v \in B(\mathbb{Z})$, $\beta \in H_2(Y, \mathbb{Z})$
- $N_{0,v}^{\beta}$: number of rational curves (genus 0) in Y of class β intersecting D at a single point with contact order v.
- Virtual dimension 0.
- Precise definition of $N_{0,v}^{\beta}$: log Gromov-Witten invariants of (Y, D) (Abrmovich-Chen, Gross-Siebert, 2011), $N_{0,v}^{\beta} \in \mathbb{Q}$ in general.

Example: $Y=\mathbb{P}^2$, D: triangle of lines, $N_{0,v}^{\beta}=0$ for every v and β (a curve of degree $d>0$ in $\mathbb{P}^2$ always intersect D_1, D_2, D_3).

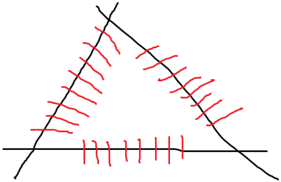


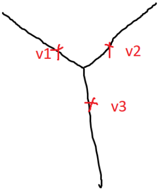
Enumerative geometry of log Calabi-Yau surfaces

Example: Y: cubic surface in $\mathbb{P}^3$, D: triangle of lines. Y contains 27 lines. Each of the 24 lines not-contained in D intersects D in a single point.



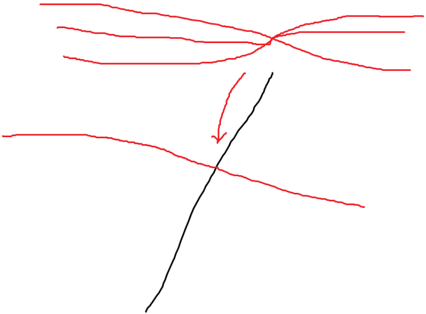
$24 = 3 \times 8$, $N_{0,v_i}^{\beta_{ij}} = 1$, ($v_i$: transverse intersection with D_i, β_j: class of the line L_{ij} intersecting D_i, $1 \leq j \leq 8$).





Enumerative geometry of log Calabi-Yau surfaces

In Gromov-Witten theory, one considers maps $f : C \to Y$ and not just embedded curves $C \subset Y$. If $C = L_{ij}$ is the line contributing to $N_{0,v_i}^{\beta_{ij}} = 1$, then every genus 0 cover of L_{ij} totally ramified over $L_{ij} \cap D_i$ contributes to $N_{0,kv_i}^{k\beta_{ij}}$. Non-trivial moduli space, virtual computation (Bryan-Pandharipande, 2001), $N_{0,kv_i}^{k\beta_{ij}} = \frac{(-1)^{k-1}}{k^2}$.



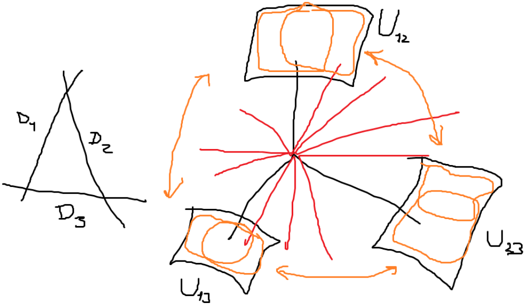
Mirror symmetry for log Calabi-Yau surfaces

Gross-Hacking-Keel (2011)

- Starting point, log Calabi-Yau surfaces (Y, D).
- Enumerative geometry: counts rational curves in (Y, D), invariants $N_{0,\beta}$
- Construction of the mirror family $\mathcal{V} \to \mathrm{Spec}\ \mathbb{C}[NE(Y)]$. For each intersection point $D_i \cap D_{i+1}$, local models $U_{i,i+1} \to \mathrm{Spec}\ \mathbb{C}[NE(Y)]$. Glue open sets $T_i \subset U_{i-1,i}$ and $T_i' \subset U_{i,i+1}$ isomorphic $(\mathbb{C}^*)^2 \times \mathrm{Spec}\ \mathbb{C}[NE(Y)]$ using birational transformations $\exp\{H_v, -\}$ generated by the Hamiltonians

$$H_v = \sum_{k \geq 0} \sum_{\beta} N_{0,kv}^{\beta} z^{kv} t^{\beta} \,,$$

  where $v \in B(\mathbb{Z})$ is primitve in the cone dual to $D_i \cap D_{i+1}$. Order according to the slope of v.



- For $H = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k^2} x^k$ (dilogarithm), $\exp\{H, -\}$ is a cluster birational transformation $x \mapsto x$, $y \mapsto y(1+x)$.
- Consistent gluing construction? The collection of numbers $(N_{0,v}^{\beta})_{v,\beta}$ needs to satisfy a non-trivial constraint, proved using tropical geometry (Gross-Pandharipande-Siebert, 2009).

- (Y, D)
- Counts of genus 0 curves: $N_{0,\beta}$
- Mirror family $\mathcal{V} \to \mathrm{Spec}\, \mathbb{C}[NE(Y)]$, Poisson variety, family of holomorphic symplectic (Calabi-Yau) surfaces.
- Consequence of the construction: “canonical basis of theta functions" $(\vartheta_v)_{v \in B(\mathbb{Z})}$ pour $\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{V}})$.
- Question: can we deform this mirror construction to produce a deformation quantization of $\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{V}})$?

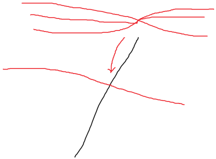
- Problem: find a 1-parameter deformation of the enumerative question.
- Naive idea: replace genus 0 curves by curves of arbitrary genus g.
- Problem: virtual dimension wrong.
- Solution: replace the non-compact Calabi-Yau surface $V = Y - D$ by the non-compact Calabi-Yau 3-fold $V \times \mathbb{C}^*$ and count higher genus curves here. Get a definition of $N_{g,v}^{\beta} \in \mathbb{Q}$.

Quantum mirror symmetry for log Calabi-Yau surfaces

Construct a deformation quantization of the mirror family $\mathcal{V} \to \mathrm{Spec}\ \mathbb{C}[NE(Y)]$. For each intersection point $D_i \cap D_{i+1}$, local $\hat{U}_{i,i+1} \to \mathrm{Spec}\ \mathbb{C}[NE(Y)]$. Glue "non-commutative open sets" $\hat{T}_i \subset U_{i-1,i}$ and $\hat{T}_i' \subset U_{i,i+1}$ isomorphic to quantum tori $\widehat{(\mathbb{C}^*)^2} \times \mathrm{Spec}\ \mathbb{C}[NE(Y)]$ using the quantum transformation $\exp[\hat{H}_v, -]$ defined by the quantum Hamiltonian

$$\hat{H}_v = \sum_{g \geq 0} \sum_{k \geq 0} \sum_{\beta} N_{g,kv}^{\beta} z^{kv} t^{\beta} \hbar^{2g-1} \,.$$

If $C = L_{ij}$ is the line contributing to $N_{0,v_i}^{\beta_{ij}} = 1$, every genus g cover of L_{ij} entirely ramified above $L_{ij} \cap D_i$ contributes to $N_{g,kv_i}^{k\beta_{ij}}$. Non-trivial moduli space, virtual computation (Bryan-Pandharipande, 2001), $\sum_{g \geq 0} N_{g,kv_i}^{k\beta_{ij}} \hbar^{2g-1} = \frac{(-1)^{k-1}}{k} \frac{1}{2\sin\left(\frac{k\hbar}{2}\right)} = i \frac{(-1)^{k-1}}{k} \frac{1}{q^{\frac{k}{2}} - q^{-\frac{k}{2}}}$ (Quantum dilogarithm).



Consistency of the gluing? The collection of invariants $(N_{g,v}^{\beta})_{g,v,\beta}$ needs to satisfy a non-trivial constraint, proof using tropical geometry (B, 1806.11495).

Conclusion (B, 2018):

- (Y, D)
- Genus g log Gromov-Witten invariants $N_{g,v}^{\beta}$.
- Deformation quantization $\hat{\mathcal{V}} \to \mathrm{Spec}\, \mathbb{C}[NE(Y)]$ of the mirror family.
- $\hat{A}$ non-commutative algebra deforming $\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{V}})$
- Consequence of the construction: canonical basis of quantum theta functions for $(\hat{\vartheta}_v)_{v \in B(\mathbb{Z})}$ for $\hat{A}$.

Comparison of quantizations

- $SL_2(\mathbb{C})$ character variety $X = \mathrm{Ch}_{SL_2(\mathbb{C})}(\mathbb{S}_{0,4}) \to \mathbb{A}^4$, quantization given by the skein algebra $\mathrm{Sk}(\mathbb{S}_{0,4})$
- (Gross-Hacking-Keel-Siebert, 2019), $X = \mathrm{Ch}_{SL_2(\mathbb{C})}(\mathbb{S}_{0,4}) \to \mathbb{A}^4$ is the result of the classical mirror construction applied to Y: smooth projecitve cubic surface, D: triangle of lines.
- Quantum mirror symmetry gives another deformation quantization $\hat{A}$ of $X = \mathrm{Ch}_{SL_2(\mathbb{C})}(\mathbb{S}_{0,4}) \to \mathbb{A}^4$.

Theorem (B, 2020)

The skein quantization and the mirror symmetry quantization agree:

$$\mathrm{Sk}(\mathbb{S}_{0,4}) \simeq \hat{A}\,.$$

Gauge theories from string/M-theory

- $\mathcal{T}$: $\mathcal{N}=2$ $N_f=4$ $SU(2)$ gauge theory.
- Realization of $\mathcal{T}$ as a class S theory: $\mathcal{N}=(2,0)$ 6d SCFT of class A_1 compactified on $\mathbb{S}_{0,4}$. Physical realization of the skein algebra $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ as an algebra of supersymmetric line operators.
- V: complement of a triangle of lines D in Y, hyperkähler manifold, D_4 elliptic fibration in rotated complex structure, Σ: elliptic fiber.
- Realization of $\mathcal{T}$ from M-theory on $\mathbb{R}^{1,3} \times V \times \mathbb{R}^3$ with a $M5$-brane on $\mathbb{R}^{1,3} \times \Sigma$. Physical realization of holomorphic curves in (Y,D) as $M2$-branes determining the BPS spectrum of $\mathcal{T}$ (uses Ooguri-Vafa relation between open $M2$-branes and higher genus open topological string).

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Gauge theories from string/M-theory

- Gaiotto-Moore-Neitzke: IR expansions of line operators in terms of framed BPS states. Wall-crossing of these IR expansions in terms of (unframed) BPS states.
- BPS states of charges $(m, 0)$: 1 vector multiplet of charge $(2, 0)$, and 8 hypermultiplets of charge $(1, 0)$. The 8 hypermultiplets correspond to the 8 lines of Y intersecting in a single point intersecting one component of D $(27 = 3 \times 8 + 3)$.
- $SL_2(\mathbb{Z})$ S-duality and triality realized geometrically from the point of view of enumerative geometry.

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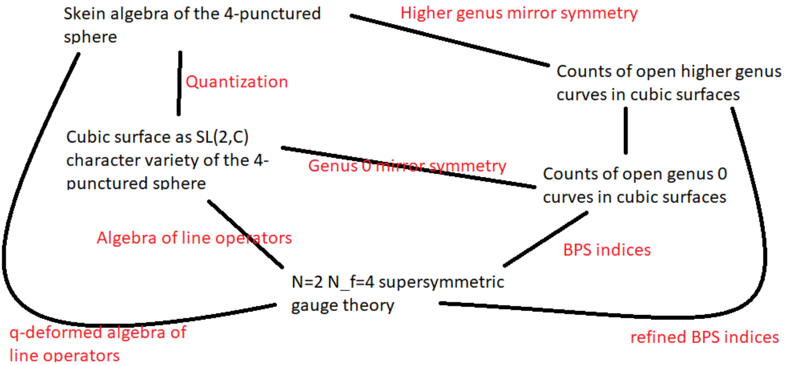
Summary


Skein algebra of the 4-punctured sphere
Higher genus mirror symmetry
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Counts of open higher genus curves in cubic surfaces
Cubic surface as SL(2,C) character variety of the 4-punctured sphere
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Thank you for your attention!