

Symplectic resolutions, Coulomb branches and 3d mirror symmetry



- ① Symplectic resolutions
- ② Symplectic duality
- ③ BFN construction of the Coulomb branch
- ④ Generalized affine Grassmannian slices

⑤ Truncated shifted Yangians

Q/A sessions: Wed McBreen hypertoric
Thu Zhou physics

Symplectic resolutions

A conical symplectic resolution is

$$\pi: Y \rightarrow X \quad \text{birational, projective } / \mathbb{C}$$

Y smooth, symplectic

X affine, Poisson

$$\mathbb{C}[X] \text{ a Poisson algebra} \quad \{, \}_{\mathbb{C}} : \mathbb{C}[X] \otimes \mathbb{C}[X] \rightarrow \mathbb{C}[X]$$

- $\mathbb{C}^* \subset Y \rightarrow X$ scales ω symplectic form.

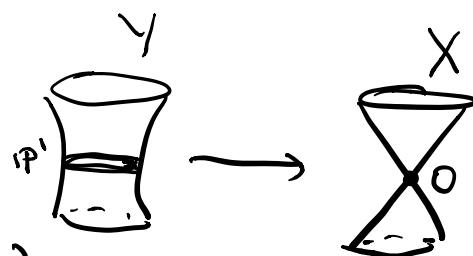
$$\mathbb{C}[X]_n = \begin{cases} 0 & \text{if } n < 0 \\ \mathbb{C} & n=0 \end{cases} \quad X^{\mathbb{C}} = \text{pt}$$

\mathbb{C}^* contracts X

- $T^* Y \rightarrow X$ preserves ω (actions commute)
 Y^T finite

Eg

$$\begin{aligned} \textcircled{1} \quad T^* \mathbb{P}^1 &\rightarrow \mathcal{N}_{SL_2} \\ \textcircled{2} \quad \{A \in M_2 \mathbb{C} \mid \text{tr } A = 0, \det A = 0\} \\ \{A, L\} : L \subset \mathbb{C}^2 & \quad A \in \mathbb{C}^2 \subset L \\ AL = 0 & \quad \left\{ \begin{bmatrix} w & u \\ v & -w \end{bmatrix} : w^2 + uv = 0 \right\} = \mathbb{C}^2 / \mathbb{Z}/2 \end{aligned}$$



conical \mathbb{C}^* scales matrix (cotangent) fibres
Hamiltonian $T = (\mathbb{C}^*)^2$ acts on \mathbb{C}^2

$$\textcircled{2} \quad T^* \mathbb{P}^{n-1} \xrightarrow{\gamma} \{ A \in M_n : A^2 = 0 \text{ or } \text{rk } A \leq 1 \} \xrightarrow{x}$$

$[T^* G/P \longrightarrow \text{nilpotent orbit closure}]$

G semisimple
 P parabolic group

$$T^*_{\parallel} \mathcal{F}_{\mathbb{A}_n} \longrightarrow \mathcal{N}_{\mathbb{A}_n} = \{ A \in M_n : A \text{ nilpotent} \}$$

$$\left\{ (\mathbf{A}, \mathbf{V}_\cdot) : V_1 \subset V_2 \subset \dots \subset \mathbb{C}^n \right\} \quad \begin{array}{l} \mathbb{C}^* \text{ conical scaling} \\ \mathbf{A} V_i \subset V_{i-1} \end{array}$$

$T = (\mathbb{C}^*)^n$ acts on \mathbb{C}^n .

$$\textcircled{3} \quad \Gamma \subset \text{SL}_2 \mathbb{C} \quad \text{finite}$$

$$\gamma = \widetilde{\mathbb{C}^2}/\Gamma \longrightarrow \mathbb{C}^2/\Gamma$$

$$\times \times \times \longrightarrow 0$$

\mathbb{P}^1 's arranged according to Dynkin diagram

To get $T^* \widetilde{\mathbb{C}^2}/\Gamma$
we need

$$\Gamma = \mathbb{Z}/n$$

$$\mathbb{C}^2/\Gamma := \text{Spec } \mathbb{C}[x, y]^\Gamma$$

④ Hypertoric varieties

torus

$$T \subset \mathbb{C}^n \quad \Phi : T^* \mathbb{C}^n = \mathbb{C}^n \oplus (\mathbb{C}^n)^* \xrightarrow{*} \mathbb{t}^*$$

of T wts

$$\Phi^{-1}(0) //_{+} \longrightarrow \Phi^{-1}(0) //$$

$$\begin{array}{ccc}
 \chi : T \rightarrow \mathbb{C}^* & \text{---} & \chi'' : \mathbb{C}^* \rightarrow \mathbb{C} \\
 \text{eg. } \mathbb{C}^* \subset \mathbb{C}^n & & T^* \mathbb{C}^n = \mathbb{C}^n \oplus \mathbb{C}^n \rightarrow \mathbb{C} \\
 \text{scaling} & & (a, b) \mapsto \sum a_i b_i \\
 \chi_{\text{id}} & \Phi'(0) // T & \rightarrow \Phi''(0) // T \\
 & \xrightarrow{\chi} & \xrightarrow{\circ} \\
 & T^* \mathbb{P}^{n-1} & \{A : \text{rk } A \leq 1, A^2 = 0\} \\
 \text{dual} & &
 \end{array}$$

⑤ Quiver varieties

$$\begin{array}{c}
 \text{Quiver: } \begin{array}{ccccc}
 & \square & & \square & \\
 & z & & 7 & \\
 & \downarrow & & \downarrow & \\
 3 & \xrightarrow{\quad A \quad} & 4 & \xrightarrow{\quad B \quad} & 5
 \end{array} \\
 G = \prod_i GL(V_i) \quad N = \bigoplus \text{Hom}(\mathbb{C}^{V_i}, \mathbb{C}^{V_j}) \oplus \text{Hom}(\mathbb{C}^{W_i}, \mathbb{C}^{V_i}) \\
 \text{circled vertices} \quad V_i \quad W_i
 \end{array}$$

$$\Phi : T^* N = N \oplus N^* \rightarrow \mathfrak{g} \\
 (A_{ij}, B_{ij}) \mapsto \sum_j A_{ij} B_{ji} + \sum_k B_{ik} A_{ki}$$

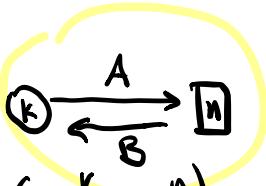
Nakajima quiver variety:

$$\frac{(\mathbb{C}^*)^{\sum w_i}}{T} \subset \frac{\Phi'(0)}{G} // G \rightarrow \frac{\Phi'(0)}{G} // G$$

$$X: G \rightarrow \mathbb{C}^*$$

$$g_i \mapsto \pi \det g_i$$

Eg ① $1 \rightarrow 2 \rightarrow 3 \rightarrow \dots \rightarrow n$ $\rightsquigarrow T^* Fl_n$

② 

$$v=k
w=n$$

$$N = \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$$

$$T^* N = \{(A, B)\}$$

$$T^* N \rightarrow gl_k$$

$$(A, B) \mapsto BA$$

$$\Phi'(0) = \{(A, B) : BA = 0\}$$

$$\frac{\Phi'(0)}{GL_k} = \{(A, B) : A \text{ injective}\} // GL_k$$

$$\cong T^* G(k, n)$$

$$= \{(W, C) : W \subset \mathbb{C}^n\}$$

$$C: \mathbb{C}^n \rightarrow \mathbb{C}^n \quad CW = 0$$

$$\mathbb{C} \mathbb{C}^n \subset W$$

$$W = \text{im } A$$

$$C = AB.$$

Eg $T^* \mathbb{P}^1 \rightsquigarrow \begin{cases} T^* Fl_n \\ \mathbb{C}^2 / \mathbb{Z} \\ \text{hypertoric} \end{cases}$

quiver varieties

$$T^* G/B$$

$$\left(\begin{array}{c} T^* \mathbb{C}^n / \mathbb{Z} \\ \text{finite} \end{array} \right)$$

affine Grassmannian slices

Theorem [Kaledin Namikawa]

① X has a partition $X = \bigsqcup_{\alpha \in I} X_\alpha$
 X_α symplectic, smooth

$$\pi: Y \rightarrow X,$$

② $Y \rightarrow X$ is semismall:

$$\dim Y \times_X Y = \dim Y = \dim X$$

$$\dim \pi^{-1}(X_\alpha) \leq \frac{1}{2} \operatorname{codim} X_\alpha$$

$$x_\alpha \in X_\alpha$$

$$\pi_* \mathbb{C}[dy] = \bigoplus_{\alpha \in I} IC_{x_\alpha} \otimes H_{\text{top}}(F_\alpha) \quad F_\alpha = \pi^{-1}(x_\alpha)$$

$dy = \dim Y$

(assuming X_α is simply-connected)

③ Existence of deformation:

$$Y \subset Y \xrightarrow{\int} H^2(Y) \quad \text{"period map"}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$X \subset X \rightarrow H^2(Y)/W$$

$W = \text{Namikawa Weyl group}$

$$\begin{aligned} T^* Fl_n &= \{(A, V_i) : AV_i \subset V_{i-1}\} \subset \tilde{g} = \{(A, V_i) : AV_i \subset V_i\} \rightarrow t = \mathbb{C}^n \\ &\downarrow \quad \downarrow \\ N &\subset gl_n \end{aligned}$$

$$\begin{aligned} &\downarrow \quad \downarrow \\ & \quad \quad \quad \downarrow (A|_{V_i/V_{i-1}}) \\ & \xrightarrow{\text{char Poly}} t/W \end{aligned}$$

$$\theta \in H^2(Y), \quad \xi^{-1}(\theta) = Y_\theta$$

$Y_\theta \cong Y$ as a smooth manifold (not as variety)
 $H^2(Y_\theta) = H^2(Y)$
 $[\omega_\theta] \mapsto [\omega_\theta] = \theta$

④ Existence of quantization

$\mathbb{C}[X]$ Poisson algebra

A quantization of X is a $\mathbb{C}[\hbar]$ -algebra A
 s.t. $A/\hbar A \cong \mathbb{C}[X]$ as a Poisson alg

$$\{\bar{a}, \bar{b}\} := \overline{\frac{i}{\hbar}(ab - ba)} \quad \text{note } \overline{ab} = \bar{b}\bar{a} \text{ in } Q[X]$$

$a, b \in A$

$\bar{a}, \bar{b} \in A/\hbar A$

A quantization of Y is a sheaf A of $\mathbb{C}[\hbar]$ -alg. on Y s.t. $A/\hbar A \cong \mathcal{O}_Y$

There is a universal quantization of X parametrized by $H^2(Y)/W$

$$A/\hbar A = \mathbb{C}[X]$$

$$\mathbb{C}[H^2(Y)]^W \subset A$$

centre of A

For any $\theta \in \text{Spec } (\mathbb{C}[H^2(Y)])^W = H^2(Y)/W$

$A_\theta = A/\hbar_\theta$ is a quantization of X
 every quantization of X arises this way

Eg

$$X = \mathbb{N} \quad Y = T^* \text{Fl}_n$$

$$g = \text{sl}_n$$

$$\mathcal{Z}(V_{\text{sl}_n}) \subset A = \bigcup_{t \in \mathbb{N}} \text{sl}_n = \text{Rees}(V_{\text{sl}_n}) = \langle \langle X : X \in \text{sl}_n \rangle \rangle$$

$\begin{aligned} & xy - yx \\ & = t \langle [x, y] \rangle \end{aligned}$

set $t=0$

$$\begin{aligned} \text{Spec}(A/t_0 A) &= \text{Spec}(\text{Sym } \text{sl}_n) \\ &= \text{sl}_n \end{aligned}$$

- $\mathbb{C}[\hbar]$ -algebra formal quantization

$\downarrow \hbar = 1$

$$A/A(\hbar-1) \quad \text{filtered quantization}$$

e.g.: $\bigcup_k g \rightsquigarrow \hbar = 1 \rightsquigarrow \bigcup g$

$$\pi: Y \rightarrow X \quad \text{symplectic resolution}$$

$$\begin{array}{ccc} \gamma_1 & \longrightarrow & X \\ \gamma_n & \longrightarrow & \text{symplectic singularity} \end{array}$$

different symplectic resolutions

$$Y = T^* N // G \rightarrow X = T^* N // G := \Phi^{-1}(0) // G$$

$\begin{array}{c} (\bullet, x) \\ \text{red} \end{array}$ $\begin{array}{c} (0, 0) \\ \text{symp. reduction} \end{array}$

$T^* N // \sim$

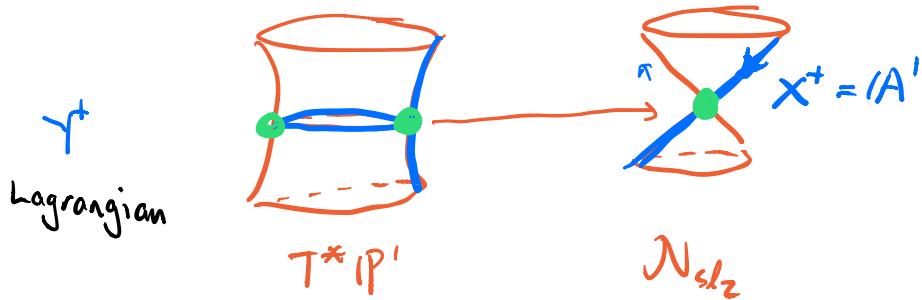
≠ \cup / G
 x
π def ?

Topology of symplectic resolutions

$$\begin{array}{ccccc} \mathbb{C}^* & \hookrightarrow & Y & \rightarrow & X \\ \text{conical} & & \text{smooth} & \text{affine} & \\ & & \text{symplectic} & \text{Poisson} & \end{array} \quad H^*(Y) \quad \text{almost}$$

$$\begin{array}{ccc} \mathbb{C}^* & \hookrightarrow & T \subset Y \rightarrow X \\ \text{torus} & & \end{array} \quad Y^T \quad \text{finite} \quad X^T = \{0\} = X^{\mathbb{C}^*}$$

$$Y^+ = \{y \in Y : \lim_{s \rightarrow 0} s \cdot y \text{ exists}\} \rightarrow X^+ = \dots$$



hyperbolic stalk: $\Phi: D_c(X) \rightarrow \text{Vect}$ $i: X^+ \hookrightarrow X$

$$F \mapsto H^*(X^+, i^! F)$$

- if F is perverse, then $\Phi(F)$ is concentrated in a single degree $2d$

$$2d = \dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X \quad d = \dim_{\mathbb{C}} Y^+ = \dim_{\mathbb{C}} X^+$$

On X we have finitely many symplectic leaves $\{X_\alpha\}_{\alpha \in I}$ give a stratification of X .

$$\pi_* \mathbb{G}_m [2d] = \bigoplus_{\alpha} IC_{X_\alpha} \otimes H_{\text{top}}(F_\alpha) \quad F_\alpha = \pi^{-1}(x_\alpha)$$

$$x_\alpha \in X_\alpha$$

apply \oplus :

$$H_*(Y^+) = \bigoplus_{\alpha} H_{top}(\bar{X}_{\alpha}^+) \otimes H_{top}(F_{\alpha})$$

always
BM
homology

$$\bar{X}_{\alpha}^+ = X^+ \cap \bar{X}_{\alpha}$$

$$H(\dots) = H_{top}(\dots)$$

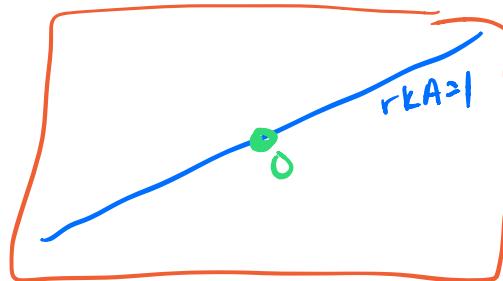
$$H(Y^+) = \bigoplus_{\alpha} H(\bar{X}_{\alpha}^+) \otimes H(F_{\alpha})$$

Eg

$$Y = T^* G(2,4) \longrightarrow \{ A \in M_4 : A^2 = 0 \} = X$$

$$\{ A \in M_4 : \begin{matrix} A^2 = 0 \\ \text{upper tri} \end{matrix} \} = X^+$$

$$X_0 = \{0\} \quad X_1 = \{A : A^2 = 0 \text{ rk } A = 1\} \quad X_2 = \{A : A^2 = 0 \text{ rk } A = 2\}$$



$$X_0^+ = \{0\} \quad \bar{X}_1^+ = \{A \text{ upper } A^2 = 0 \text{ rk } A \leq 1\} \quad \bar{X}_2^+ \text{ 2 components}$$

$$\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_0 = G(z, y) \quad F_1 = P$$

$$F_2 = pt$$

$$H(Y^+) = H(\bar{X}_0^+) \otimes H(F_0) \oplus H(\bar{X}_1^+) \otimes H(F_1) \oplus H(\bar{X}_2^+) \otimes H(F_2)$$

$H(Y^+)$ admits a categorification

A_θ , a quantization of X $\theta \in H^2(Y)$.

$C_c T C X \rightsquigarrow C_c T C A_\theta \rightarrow$ get a \mathbb{Z} -grading on A_θ

$$\text{eg: } Y = T^* \mathbb{P}^n$$

$A_\theta = V_{\mathbb{P}^n}/\mathbb{Z}_\theta$ grading is $\deg E_i = 1 \quad \deg F_i = -1$

$A_\theta^+ \subset A_\theta$ positive degree part.

Def

Category \mathcal{O} for A_θ = the category of f.g. A_θ -modules
on which A_θ^+ acts locally nilpotently

Theorem [BLPW]

There is a characteristic cycle map
(usually an isomorphism)

$$CC: K(\text{cat } \mathcal{O} \text{ for } A_\theta) \rightarrow H(Y^+)$$

supp $\Omega(M)$
 $\hat{\Xi}$

M an A_θ -module $\rightsquigarrow M$ an A_θ -module $\rightsquigarrow \text{gr}(M)$
in category \mathcal{O} choosing a good filtration on M coherent sheaf on Y

A_θ = sheaf of algebras on Y
quantizing \mathcal{O}_Y $\Gamma(A_\theta) = A_\theta$ supported on Y^+

Symplectic duality = 3d mirror symmetry

$Y \rightarrow X$ $Y^! \rightarrow X^!$
 two "dual" symplectic resolutions have matching
 properties $(Y^!)^! = Y$

Assume we have chosen $C^* \subset T^* Y$, Y^+ finite
 this is equivalent to choosing a resolution $Y^!$

Matching structures:

$$\textcircled{1} \quad t_C = H^2(Y^!; \mathbb{C}) \\ \cup \qquad \qquad \qquad \cup \\ \text{Hom}(C^*, T) = t_{\mathbb{Z}} = H^2(Y^!; \mathbb{Z}) = \text{Pic}(Y^!)$$

$$\{p' : p' \sim p\} = \text{ample cone}$$

$$C_p^* \subset T^* Y \rightarrow Y_p^+$$

$$p_1 \sim p_2 \quad \text{if} \quad Y_{p_1}^+ = Y_{p_2}^+$$

$$\textcircled{2} \quad \begin{aligned} &\text{An order reversing bijection } I \xrightarrow{\sim} I^! \\ &I = \text{strata of } X \quad I^! = \text{strata of } X^! \\ &H(F_\alpha) = H((X_\alpha^!)^+) \end{aligned}$$

in fact bijection on components

$$H(\bar{X}_\alpha^+) = H(F_\alpha^+)$$

$$H(Y^+) = \bigoplus_{\alpha} H(\bar{X}_\alpha^+) \otimes H(F_\alpha)$$

$$H(Y_!^+) = \bigoplus_{\alpha} H(\bar{X}_\alpha^{!+}) \otimes H(F_\alpha^!)$$

$$\begin{array}{l} Y = T^* \mathbb{P}^{n-1} \\ \downarrow \end{array}$$

$$X = \{A^2 = 0, \text{rk } A \leq 1\}$$

$$X_0 = \{0\} \quad X_1 = \{A : \text{rk } A = 1\}$$

$$Y_!^+ = \widetilde{\mathbb{C}^2}/\mathbb{Z}_{n!} \quad \text{XXX} \quad \text{($n-1$) IP's}$$

$$Y_!^+ = \mathbb{C}^2/\mathbb{Z}_{n!} \Rightarrow 0$$

$$X_0^! = \{0\} \quad X_1^! = \{A : \text{rk } A = 1\}$$

$$H((T^*\mathbb{P}^{n-1})^+) = H(X_0) \otimes H(\mathbb{P}^{n-1}) \oplus H(\bar{X}_1^+) \otimes H(F_1^+)$$

$$H(\widetilde{\mathbb{C}^2}/\mathbb{Z}_{n!}) = H(X_0) \otimes H(\mathbb{P}^{n-1}(0)) \oplus H(\bar{X}_1^{!+}) \otimes H(F_1^!)$$

(3) There is an equivalence

$$D(\text{cat } \theta \text{ for } A_\theta) = D(\text{cat } \theta \text{ for } A_{\theta!}^!)$$

categories ②

[BLPW]

$\theta, \theta!$: generic integral

$$\theta \in H^2(Y)/W$$

Examples of symplectic dualities:

$$\begin{array}{ccc}
 \nearrow & T^*(\mathbb{P}^{n-1}) & \widetilde{\mathbb{C}/\mathbb{Z}_n} \\
 | & \text{hypertoric variety} & \\
 | & (\mathbb{C}^\times)^k \hookrightarrow (\mathbb{C}^\times)^n & (\mathbb{C}^\times)^{n-k} \hookrightarrow (\mathbb{C}^\times)^n \\
 | & \downarrow & \downarrow \\
 | & T^*\mathbb{C}^n // (\mathbb{C}^\times)^k & T^*\mathbb{C}^n // (\mathbb{C}^\times)^{n-k} \\
 | & \mathbb{C}^\times \hookrightarrow (\mathbb{C}^\times)^n & \{(t_1, \dots, t_n) : t_1 \cdots t_n = 1\} \hookrightarrow (\mathbb{C}^\times)^n
 \end{array}$$

Gale duality

$$T^* G/B \quad T^* G^\vee / B^\vee$$

quiver variety \rightsquigarrow affine Grassmannian slices

$$\begin{array}{c}
 \text{3d } N=4 \text{ supersymmetric field theory } T \xrightarrow{\text{S-duality}} \text{Higgs branch } M_H(T) \\
 \qquad \qquad \qquad \qquad \qquad \text{Coulomb branch } M_C(T) \\
 \text{another field theory } T' \xrightarrow{\text{ }} M_H(T') = M_C(T) \\
 \qquad \qquad \qquad \qquad \qquad M_C(T') = M_H(T)
 \end{array}$$

symplectic dual

$$\begin{array}{ccc}
 T^* G/P \xrightarrow{\text{ }} & \rightsquigarrow & \text{resolution of a Slodowy slice in } g^\vee
 \end{array}$$

~~T^{*}G/P~~
partial flag variety

nilpotent orbit closure
(may be minimal)

Shadowy slice in \mathbb{P}^m = \mathbb{C}^2/\mathbb{Z}

nilpotent orbit closures

shadowy slices

Quiver varieties and affine Grassmannian slices

$$T \subset Y \xrightarrow{\pi} X \quad T' \subset Y' \longrightarrow X'$$

$Y^+ \quad X^+$

$$H(Y^+) = \bigoplus_{\alpha} H(\bar{X}_{\alpha}^+) \otimes H(F_{\alpha}) \quad F_{\alpha} = \pi'^{-1}(x_{\alpha}) \quad x_{\alpha} \in X_{\alpha}$$

X crossed out

$$H(Y'^+) = \bigoplus_{\alpha} H(\bar{X}_{\alpha}^+) \otimes H(F'_{\alpha})$$

$H(Z) = \text{top Borel-Moore homology of } Z$

④ Hikita conjecture

⑤ Elliptic stable envelopes [Rimanyi]

⋮

\mathfrak{g} semisimple Lie alg. simply-laced (ADE type)

λ, μ dominant weights ($\lambda \geq \mu$)

$$\lambda = \sum w_i \omega_i \quad \lambda - \mu = \sum v_i \alpha_i$$

$w_i, v_i \in \mathbb{N} \quad i \in I = \text{vertices of the Dynkin diagram.}$

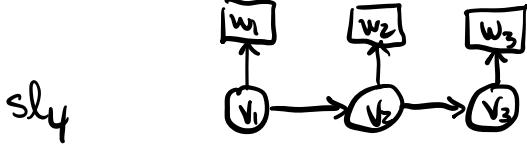
$\lambda = \lambda_1 + \dots + \lambda_n \quad \lambda_j$ fundamental weights

[Assume λ_j are all minuscule]

$$V(\lambda_1) \otimes \dots \otimes V(\lambda_n) = V(\underline{\lambda}) \quad V(\underline{\lambda})_{\mu} \quad \text{wt space.}$$

$$V(\underline{\lambda})_{\mu} = \bigoplus_{\nu} \text{Hom}(V(\nu), V(\underline{\lambda})) \otimes V(\nu)_{\mu}$$

choose an orientation of the Dynkin diagram.



$$G = \prod GL(v_i) \quad N = \bigoplus_{i \rightarrow j} \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \oplus \bigoplus_i \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})$$

$$M(\lambda, \mu) := T^*N \mathop{\!/\mkern-5mu/\limits}_{(0, \chi)} G \quad \chi: G \rightarrow \mathbb{C}^\times$$

$$g \mapsto \prod \det g_i$$

$$M_0(\lambda, \mu) := T^*N \mathop{\!/\mkern-5mu/\limits}_{(0, 0)} G \quad T = \prod (\mathbb{C}^\times)^{w_i} \subset M(\lambda, \mu)$$

Theorem [Nakajima] $\text{Spec } A_0 \leftarrow \text{Proj } A_0$

① $M(\lambda, \mu)$ is smooth

$M(\lambda, \mu)$ is a resolution of $M_0(\lambda, \mu)$ μ down

T acts on $M(\lambda, \mu)$ with finitely many fixed pts λ_i minus rule

② The symplectic leaves of $M_0(\lambda, \mu)$ are $M_0(\nu, \mu)^{\text{reg}}$ $\mu \leq \nu \leq \lambda$

③ There is an action of \mathfrak{g} on $H(M(\lambda, \mu)^+)$ s.t.

$$H(M(\lambda, \mu)^+) = \bigoplus_{\nu} H(M_0(\nu, \mu)^+) \otimes H(F_\nu)$$

$$\| \quad \| \quad \|$$

$$V(\lambda)_\mu = \bigoplus_{\nu} \text{Hom}(V(\nu), V(\lambda)) \otimes V(\nu)_\mu$$

$$\text{E.g. } g = sl_2 \quad \lambda = n\omega \quad \mu = \lambda - \alpha \quad w = n \quad v = 1$$

$$(\mathbb{C}^2)^{\otimes n} = \text{Hom}(V(n), (\mathbb{C}^2)^{\otimes n}) \otimes V(n)_{n-1} \oplus \text{Hom}(V(n-2), (\mathbb{C}^2)^{\otimes n}) \otimes V(n-2)_{n-1},$$

$$\begin{array}{ccccccc} \dots & & & & & & \dots \\ & \vdots & & & & & \\ & n & & & & & \\ \textcircled{1} & \xrightarrow{T^*(\mathbb{P}^{n-1})} & \{ A \in M_n : \text{rk } A \leq 1, A^2 = 0 \} & = X \end{array}$$

$$H((T^*\mathbb{P}^{n-1})^+) = H(X_0^+) \otimes H(F_0) \oplus \underset{\substack{n-1 \\ \text{components}}}{H(X^+)} \otimes H(F_1)$$

\overline{G} Langlands dual group to \mathfrak{g} ($\mathfrak{g} \cong \mathfrak{g}^\vee$)

$$Gr_G = G(\mathbb{C}(t))/G(\mathbb{C}[t])$$

$$\lambda: \mathbb{C}^* \rightarrow T \subset G \quad t^\lambda \in Gr_G$$

e.g.: $SL_n \quad \lambda = (\lambda_1, \dots, \lambda_n) \quad \lambda_1 + \dots + \lambda_n = 0 \quad t^\lambda = \begin{bmatrix} t^{\lambda_1} & & & \\ & \ddots & & 0 \\ & & \ddots & \\ 0 & & & t^{\lambda_n} \end{bmatrix}$

$$Gr^\lambda = G(\mathbb{C}[t]) t^\lambda \quad \text{"spherical Schubert cell"} \\ \text{finite-dimensional}$$

$$W_\mu = G_1[t^{-1}] t^\mu \quad G_1[t] \rightarrow G(\mathbb{C}[t]) \rightarrow G$$

transverse orbits $t^{-1} \mapsto 0$

infinite-dimensional

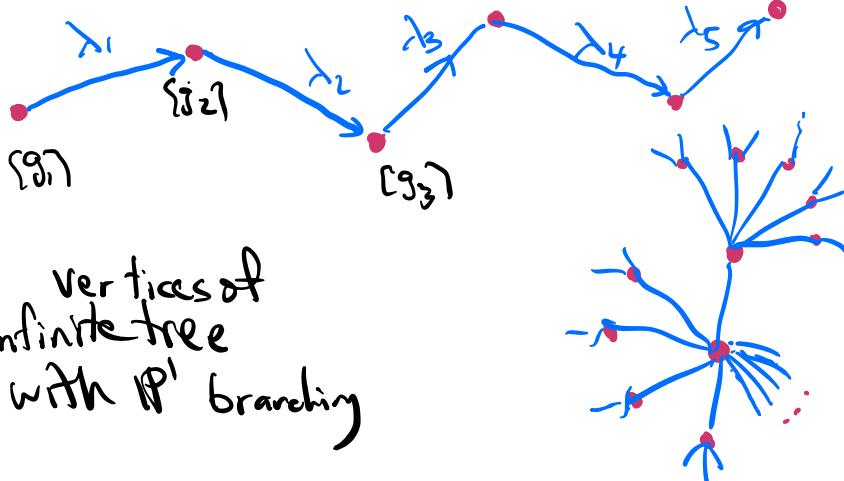
$$W_\mu^\lambda = Gr^\lambda \cap W_\mu \quad \overline{W}_\mu^\lambda = \overline{Gr^\lambda} \cap W_\mu \quad \begin{array}{l} \text{Poisson} \\ \text{affine variety} \end{array}$$

$\dim \overline{W}_\mu^\lambda = \langle 2\rho, \lambda - \mu \rangle$

$$\begin{aligned} Gr^\Delta &= Gr^\lambda_1 \times \dots \times Gr^{\lambda_n} \xrightarrow{m_\Delta} Gr \\ &= \{ [g_1], \dots, [g_n] \in Gr : [g_i^\top g_i] \in Gr^{\lambda_i} \text{ for all } i \} \end{aligned}$$

$$g_j \in G((t))$$

$$\frac{w_{i+1}}{\alpha(g_{i+1}, g_i)}$$



$G = \mathrm{PGL}_2$ vertices of
 Gr_G = infinite tree
 with \mathbb{P}^1 branching

$$\tilde{W}_\mu^\lambda := m_\lambda^{-1}(W_\mu)$$

λ minuscule.

Theorem [K Webster Weekes Jacobi]

- ① \tilde{W}_μ^λ is a symplectic resolution of \overline{W}_μ^λ
- ② T acts on \tilde{W}_μ^λ with finitely many fixed pts
- ③ The leaves of \overline{W}_μ^λ are W_μ^ν for $\mu \leq \nu \leq \lambda$

[Mirkovic-Vilonen]

④ There is an isomorphism

$$H((\tilde{W}_\mu^\lambda)^+) = \bigoplus_v H(m_\lambda^{-1}(t^\nu)) \otimes H((\overline{W}_\mu^\nu)^+)$$

$$\text{with } V(\lambda)_\mu = \bigoplus_v \mathrm{Hom}(V(\nu), V(\lambda)) \otimes V(\nu)_\mu$$

Mirkovic-Vilonen cycles

$$\text{Ex } \lambda = n\omega \quad \mu = n\omega - \alpha$$

$$\overline{W}_\mu^\lambda = \mathbb{C}/\mathbb{Z}_{\ell^n}$$

$$\tilde{W}_\mu^\lambda = \widetilde{\mathbb{C}/\mathbb{Z}_{\ell^n}}$$

$$\langle \gamma_\beta, \alpha \rangle = 2$$

Claim

$M(\lambda, \mu)$ and \tilde{W}_μ^λ are symplectic dual

Recall $T = (\mathbb{C}^*)^{\sum w_i} \subset M(\lambda, \mu)$

$$t = \mathbb{C}^{\sum w_i}$$

$$\lambda = \sum w_i \omega_i$$

$$\lambda = \lambda_1 + \dots + \lambda_n$$

$$\tilde{W}_\mu^\lambda \subset Gr^n$$

$$H^2(\tilde{W}_\mu^\lambda) \leftarrow H^2(Gr^n) = \frac{H^2(Gr)}{\mathbb{C}^n} \quad n = \sum w_i$$

$$\rightsquigarrow t = H^2(\tilde{W}_\mu^\lambda)$$

$$H^2(M(\lambda, \mu)) = \mathbb{C}^I$$

[line bundles are determinants of tautological vector bundles]

$$(\mathbb{C}^*)^I = T \subset \tilde{W}_\mu^\lambda \text{ in } Gr_G$$

$$\text{get } H^2(M(\lambda, \mu)) \cong \mathbb{C}^I = \text{Lie}((\mathbb{C}^*)^I)$$

Can produce a bijection

$$\text{Irr } F_\nu = \text{Irr } (\tilde{W}_\mu^\nu)^+$$

$$F_\nu \subset M(\lambda, \mu)$$

Symplectic/Poisson structures on W_m come from
the Manin triple

$$(g \otimes \mathbb{C}((t)), g \otimes \mathbb{C}[t], g \otimes t^{\mathbb{C}}[t^{-1}])$$

BFN Construction

hyperbolic varieties] $T^*N //_{(0,\infty)} G$
 quiver varieties T^*G/P
 \mathbb{C}^*
 affine G slices

G reductive group N rep of G .

$(G, N \text{ define } \alpha)$ \rightsquigarrow Higgs branch $T^*N // G$
 3d gauge theory \rightsquigarrow Coulomb branch $M_c(G, N)$ $\stackrel{\text{; symplectic dual}}{\sim}$

$$B = \overline{D} = D \cup D^* \quad \text{non-separated} \quad \simeq \mathbb{P}^1$$

$$\mathbb{P}^1 = A' \overset{t \mapsto t^2}{\cup} A' \quad A' \cup A' \quad \overbrace{(O)}^{A' \text{ with doubled origin}}$$

$$M_c(G, N) \parallel \text{Spec} \left(H_*(\text{Maps}(B, [N/G])) \right)$$

carries a convolution
algebra structure

\overline{D} disk with tripled origin

BFN Construction

G reductive group N rep of G
 $m \rightarrow n$ $G = GL_m$ $N = \text{Hom}(\mathbb{C}^m, \mathbb{C}^n)$

$G, N \rightsquigarrow$ Higgs branch $T^*N // G$ $\xleftarrow{\text{symp.}} T^*G(m, n)$
 \rightsquigarrow Coulomb branch $M_c(G, N) \xleftarrow{\text{dual}}$

$M_c(G, N) = \text{Spec } H_*(\text{Maps}(\mathbb{B}, [N/G]))$ $\mathbb{B} = D \cup_{D^\times} D$ \circlearrowleft
 $\text{Maps}(\mathbb{B}, [N/G]) = \left\{ (P, s) : P \text{ principal } G\text{-bundle on } \mathbb{B}$
 $s \text{ section of } N_P = P \times N/G \right\}$
 $= \left\{ (P_1, P_2, \varphi, s) : P_i \text{ principal bundle on } D_i;$
 $\varphi \text{ iso } P_1|_{D^\times} \rightarrow P_2|_{D^\times}, s \in \Gamma(D, N_{P_1}) \right.$
 $\left. \text{s.t. } \varphi \cdot s \in \Gamma(D, N_{P_2}) \right\}$

$T_{G,N} = \left\{ (P, \varphi, s) : P \text{ principal } G\text{-bundle on } D \right.$
 $\left. \varphi: P \rightarrow P_0 \text{ on } D^\times, s \in \Gamma(D, N_P) \right\}$
 $R_{G,N} = \left\{ (P, \varphi, s) : \text{as before } \varphi \circ s \in \Gamma(D, N \otimes \mathcal{O}_P) \right\} \quad \mathfrak{g}_{G(0)}$
 $\mathcal{O} = \mathbb{C}[[t]] \quad K = \mathbb{C}((t)) \quad \text{Spec } \mathcal{O} = D \quad \text{Spec } K = D^\times$
 $R_{G,N}/G(0) = \text{Maps}(\mathbb{B}, [N/G])$

$T_{G,N} \longrightarrow T_{G,0} = \{(P, \varphi)\} = G_G := G(K)/G(0)$
 $\parallel \text{Vector bundle of infinite rank}$
 $\{(Lg), s\} : Lg \in G_G, s \in gN(0)\}$ generalized affine

\cup — ↓ Springer fibres
 $R_{G,N} = \{([g], s) : s \in N(\mathbb{O}) \cap N(\mathbb{O})\} \longrightarrow Gr$
 over each Gr , it is a vector bundle
 of finite codimension in $N(\mathbb{O})$.

$$\begin{aligned}
 G &= GL_m \quad N = \text{Hom}(\mathbb{C}^m, \mathbb{C}^n) \\
 T_{G,N} &= \{ (L, A) : A \in M_{mn}(K) \} \longrightarrow Gr = \{ L \subset K^m : L \text{ is} \\
 &\quad A L \subset \mathbb{O}^n \} \quad \text{an } \mathbb{O}\text{-lattice} \} \\
 R_{G,N} &= \{ (L, A) : A \in M_{mn}(\mathbb{O}) \} \\
 &\quad A L \subset \mathbb{O}^n
 \end{aligned}$$

Theorem [BFN]

$A(G, N) = H_*^{G(\mathbb{O})}(R_{G,N})$ carries a commutative algebra structure
 work with cycles which are f.d. along Gr
 and finite codimensional in the f.brs

$$\begin{aligned}
 T_{G,N} &\rightarrow N(K) \\
 \text{analog of} \\
 T^*G/B &\rightarrow N
 \end{aligned}$$

$$\begin{aligned}
 T_{G,N} \times_{N(K)} T_{G,N} &=: Z_{G,N} \\
 \text{analog of} \\
 T^*G/B \times_{N} T^*G/B
 \end{aligned}$$

$$Z_{G,N}/G(K) = R_{G,N}/G(\mathbb{O})$$

$$\begin{aligned}
 &\{ (g_1, [g_2], s) : s \in N(\mathbb{O}), s \in g_2 N(\mathbb{O}), g_1 s \in N(\mathbb{O}) \} \\
 &\quad \downarrow \quad \downarrow \quad \downarrow \\
 R_{G,N}([g_2], s) &\quad R_{G,N}([g_1], g_1 s) && R_{G,N}([g_1 g_2], g_1 s)
 \end{aligned}$$

Eg

$$\textcircled{1} \longrightarrow \boxed{n} \quad G = \mathbb{C}^\times \subset \text{Hom}(\mathbb{Q}, \mathbb{C}^n) = \mathbb{C}^n \quad \text{inverse scaling}$$

$$Gr_G = \mathbb{Z} = \{ t^p : p \in \mathbb{Z} \}$$

$$\bigsqcup_p \{t^p\} \times (t^{-p}\theta \cap \theta)^n$$

$$u = [\{t^0\} \times \theta^n] \quad v = [\{t^{-1}\} \times (t\theta)^n] \quad w \text{ generator}$$

$$u, v, w \in H_{\mathbb{C}^\times}^{G(\mathbb{R})}(R_{G,N}) = A(G, N) \quad \text{of } H_{\mathbb{C}^\times}^*(Pt)$$

To compute uv use the correspondence diagram

$$[\{t^0\} \times (t\theta)^n] \quad I = [\{t^0\} \times \theta^n]$$

$$w' = [\theta^n / t\theta^n]$$

$$\stackrel{\theta}{\circ} \circ \quad uv = w' \quad \text{in } A(G, N)$$

$$A(G, N) = \mathbb{C}[u, v, w] /_{uv - w'}$$

$$M_C(G, N) = \mathbb{C}^2 / \mathbb{Z}_n$$

Theorem

If G is a torus, then $M_C(G, N)$ is the affine hyperbolic variety

Symplectic / 'cak' dual to $T^* \dot{N} // G$.

Properties of the Coulomb branch

$$\textcircled{1} \quad R_{G,N} \hookrightarrow Gr_G \times N(\theta)$$

$$A(G, N) \xleftarrow{\cong} A(G, 0) = H_+^{G(0)}(Gr)$$

$$M_c(G, N) \xleftarrow{\text{birational}} M_c(G, 0) \stackrel{[BFN]}{\cong} \left\{ (g, x) \in G^\vee \times (G^\vee)^{reg} \mid Ad_g(x) = x \right\} / G^\vee$$

↓
Spec $H_+^{G(0)}(Pt) = \mathfrak{t}^*/W = \mathfrak{g}^\vee/G^\vee$ is T^\vee
generic fibre

The generic fibre of $M_c(G, N) \rightarrow \mathfrak{t}^*/W$
is also T^\vee

$$\dim M_c(G, N) = 2\dim T^\vee - 2 \operatorname{rank} G$$

\textcircled{2} Non commutative deformation / Poisson structure

$$A_\hbar(G, N) = H_+^{G(0) \times \mathbb{C}^*}(R_{G, N}) \quad \begin{matrix} \mathbb{C}^* \curvearrowright R_{G, N} \text{ by loop rotation} \\ \mathbb{C}^* \curvearrowright \mathcal{O} \text{ by scaling } t \end{matrix}$$

$$uv = [(t\theta)^n]$$

$$\underline{[0^n / (t\theta)^n]} = w^n$$

$$vu = \underline{[(t^{-1}\theta)^n / 0]} = (w + \hbar)^n$$

$$\frac{A_\hbar(G, N)}{\hbar A_\hbar(G, N)} = A(G, N)$$

$$\boxed{\begin{array}{l} A(G, N) \hookrightarrow A(T, N) \\ M(G, N) \rightarrow M(T, N) \end{array}}$$

$\therefore A(G, N)$ carries a Poisson structure
 $\{u, v\} = nw^{n-1}$?

③ Torus action on $M_c(G, N)$

$$\pi_0(\text{Gr}_G) = \pi_1(G) \quad G = \prod_{i \in I} GL(v_i)$$

$$\pi_1(G) = \mathbb{Z}^I$$

let T' be the torus whose wt lattice is $\pi_1(G)$.

$$\begin{aligned} \pi_0(R_{G, N}) &= \pi_1(G) \rightsquigarrow A(G, N) \text{ is } \pi_1(G)\text{-graded} \\ &\Rightarrow T' \subset A(G, N) \\ &T' \subset M_c(G, N) \end{aligned}$$

$$\text{Hom}(G, \mathbb{C}^\times) \rightarrow \text{Pic}(T^*N //_{\mathbb{Z}} G)$$

$t'^{\parallel}_{\mathbb{Z}}$ = const lattice
of T'

$$t' \rightarrow H^2(T^*N // G)$$

$A(G, N)$ also carries a homological grading
 $\rightsquigarrow \mathbb{C}^\times \subset M_c(G, N)$
This is the (conical) \mathbb{C}^\times -action.

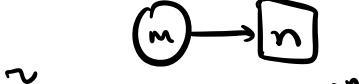
$$\mathbb{C}^{\times} \rightarrow \mathbb{C}^{\times} \times T^! \cap M_c(G, N)$$

↓ ↓
 homological \$\pi_0(b)\$
 grading

④ Deformation / resolution

$G \subset \tilde{G} \cap N$ $\tilde{G}/G =: F$ is a torus

$$F = (\mathbb{C}^{\times})^{\sum w_i}$$


 $\tilde{G} = GL_m \times (\mathbb{C}^{\times})^n$

$\tilde{G}(0) \cap R_{G, N}$ G F

$$(a) \quad \tilde{A}(G, N) = H_*^{\tilde{G}(0)}(R_{G, N})$$

$$\text{Spec } \tilde{A}(G, N) =: \tilde{M}(G, N)$$

\downarrow \downarrow $\cancel{\downarrow}$
 $\text{Spec } H_F^*(pt) = f$ $H^*(Y) = f$

$$(b) \quad \tilde{A}_{\cancel{f}}(G, N) = H_*^{\tilde{G}(0) \times \mathbb{C}^{\times}}(R_{G, N})$$

universal quantization

$$(c) \quad \text{Choose } j: \mathbb{C}^{\times} \rightarrow F$$

$R_{\tilde{G}, N} \rightarrow Gr_F = \text{cont lattice of } F$

\cup
 $R_{\tilde{G}, N}^{n\lambda} \rightarrow \{n\lambda\}$

$$R_{\tilde{G}, N} = \bigsqcup_{n \in \mathbb{N}} R_{\tilde{G}, N}^{n\lambda} \quad R_{\tilde{G}, N}^0 = R_{G, N}$$

$\bigoplus_{n \in \mathbb{N}} H_0^{G(0)}(R_{\tilde{G}, N}^{n\lambda})$ a graded ring
 0-degree part $\cong A(G, N)$

$$\text{Proj} \left(\bigoplus_{n \in \mathbb{N}} H_0^{G(0)}(R_{\tilde{G}, N}^{n\lambda}) \right) \longrightarrow \text{Spec } A(G, N) = M_c(G, N)$$

Candidate to be a resolution.

$$M_c(G_1 \times G_2, N_1 \oplus N_2) = M_c(G_1, N_1) \times M_c(G_2, N_2)$$

$$M_c(G, N_1 \oplus N_2) \xleftarrow{\text{birational}} M_c(G, N_1)$$

$$M_c(G, N^*) \cong M_c(G, N)$$

$$\mathbb{C}^2/\mathbb{Z}_{m+k} \xleftarrow{\quad} \mathbb{C}^2/\mathbb{Z}_m$$

$$(u, w^k v, w) \xleftarrow{\quad} (u, v, w)$$

$$u w^k v = u v w^k = w^{n+1} \quad uv = w^n$$

$$① \rightarrow \boxed{n} \quad ① \rightarrow \boxed{m+k}$$

Generalized affine Grassmannian slices

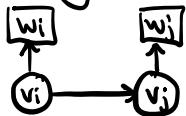
$$G, N \rightsquigarrow R_{G,N} \rightsquigarrow A(G,N) = H_*^{G(\mathbb{Q})}(R_{G,N})$$

$$A_+(G,N) = H_*^{G(\mathbb{Q}) \times \mathbb{C}^*}(R_{G,N})$$

$$M_C(G,N) = \text{Spec } A(G,N)$$

Quiver gauge theories

Γ directed graph , $v, w \in \mathbb{N}^I$ $I = \text{Vertices of } \Gamma$



$$G = \prod GL(v_i) \quad N = \bigoplus_{i \neq j} \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \oplus \bigoplus_i \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})$$

\mathfrak{g}_Γ Kac-Moody Lie algebra

λ dominant wt μ wt

$$\lambda = \sum w_i \omega_i \quad \lambda - \mu = \sum v_i \alpha_i$$

$$M_H(\lambda, \mu) = T^* N \mathbin{\!/\mkern-5mu/\!}_{(0, \infty)} G$$

$$M_C(\lambda, \mu) := M_C(G, N)$$

\mathfrak{g}_Γ finite-type (ADE) G_Γ group

$$G_{G_\Gamma} = G_\Gamma(\mathbb{K}) / G_\Gamma(\mathbb{O}) \quad G^\lambda = G_\Gamma(\mathbb{O}) t^\lambda$$

$$W_\mu = G_{\Gamma, \mathbb{I}}[t^{-1}] t^\mu$$

$$\bar{W}_\mu^\lambda = G^\lambda \cap W_\mu \quad \text{affine scheme.}$$

Theorem [BFN]

G_r finite type μ dominant
then $M_C(\lambda, \mu) = \overline{W}_\mu^\lambda$

$$W_\mu = U_+ [t^\pm] \ t^\mu T_1 [t^\pm] U_- [t^\pm] \subset G_r (K)$$

$U_+, U_- \subset G_r$ opposite unipotent subgroup

$$\ker \left(G[t^\pm] \xrightarrow{\quad t^\pm \mapsto 1 \quad} G \right) = G_1 [t^\pm] \quad \begin{bmatrix} 1 & t^\pm t^{-2} \\ 0 & \ddots \end{bmatrix} \quad U_+ [t^\pm]$$

(i) if μ is dominant $W_\mu \rightarrow G_r$ gives an isomorphism $W_\mu = W_\mu$.

(ii) W_μ has a moduli interpretation [BFN]

(iii) moduli space of singular monopoles
[Bullimore - Dimofte - Gaioffo]

$$\overline{W}_\mu^\lambda = W_\mu \cap \overline{G_r(0) t^\lambda G_r(0)}$$

generalized affine Grassmannian slice.

Not conical unless $\mu + \rho$ is dominant.

Eg ① $G_r = SL_2$ $\begin{array}{c} m \\ \text{---} \rightarrow \\ n \end{array}$ $\lambda = n\omega$ $\mu = n\omega - m\alpha$
 $M_H(\lambda, \mu) = T^* G(m, n)$ $= n$ $= n - 2m$

$$\bar{W}_n^\lambda = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C}[t]) : \begin{array}{l} a \text{ monic of degree } n \\ b, c \text{ of degree } < n \\ ad - bc = t^n \end{array} \right\}$$

$$n-2m \geq 0$$

$\cong (m, n-m)$ Slodowy slice

② If finite or affine type A

$M_C(\lambda, \mu)$ is a bow variety [Nakajima-Takayama]
(Rimanyi)

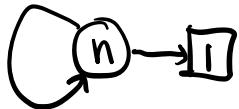
$$\textcircled{3} \quad \lambda = \omega_m + \omega_{n-m} \quad \mu = 0$$

$$G_r = SL_n$$



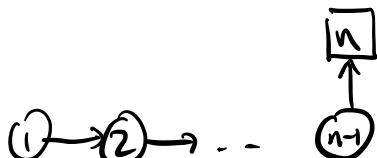
$$\bar{W}_n^\lambda = T^* G(m, n)$$

④



M_C Hilbert scheme
 $Sym^1 \mathbb{C}^2$

⑤



$$\begin{aligned} M_H &= T^* \mathbb{P}^{n-1} \\ M_C &= T^* \mathbb{P}^{n-1} \\ \text{"}\bar{W}_0^{\lambda, \omega_m}\text"} \end{aligned}$$

Properties

① gr finite type $\overline{W}_\mu^\lambda \xrightarrow{\sim} \overline{Gr}$ $\dim \widehat{W}_\mu^\lambda = 2\rho(\lambda - \mu)$
 $\dim \widehat{Gr} = 2\rho(\lambda)$
 if we restrict to $(\overline{W}_\mu^\lambda)^+$ this
 gives an $\xrightarrow{\sim}$
 $(\overline{W}_\mu^\lambda)^+ \xrightarrow{\sim} \overline{Gr} \cap S^{\mu_-} \quad S^{\mu_-} = U_-(\lambda) \cdot t^{\mu_-}$
 [Krylov]

$$H((\overline{W}_\mu^\lambda)^+) = H(\overline{Gr} \cap S^{\mu_-}) \cong V(\lambda)_\mu$$

[Mirkovic-Vilonen]

Conjecture [BFN]

- For any Γ , (not finite type)

$$H(\underline{M}_c(\lambda, \mu)^+) \cong V(\lambda)_\mu \cong H(F_\mu^+)$$

Known in affine type A.

- $T = (\mathbb{C}^\times)^I$ has a fixed point in $M_c(\lambda, \mu)$
 iff $V(\lambda)_\mu \neq 0$.

- ② $\lambda = 0$ gr finite type

$$W_\mu^0 = B\text{Maps}^{-\mu}(P^*, G/B) \xrightarrow{\Psi} \mathbb{C}^\mu = \text{Spec } H_G^*(pt)$$

$$H(\psi'(0)) = M_\mu \quad M \text{ Verma module of h.w. } 0.$$

③ G_F not of simply-laced type

For this there is a modification of Coulomb branches [Nakajima-Weekes]

④ G_F finite type simply-laced

The symplectic leaves of \overline{W}_μ^λ are W_μ^ν
for $\mu \leq \nu \leq \lambda$ & dominant.
[Muthiah-Weekes]

$$\begin{array}{ccc} X^+ \subset \psi'(0) \subset X & & \text{symplectic singularity with } T\text{-action} \\ \downarrow \psi & \text{integrable system} & \\ \mathbb{C}^n & & \\ M_C(G, N) \longrightarrow \text{Spec } H_G^*(\text{pt}) & & \end{array}$$

$$\begin{array}{ccc} \pi \subset \psi'(0) \subset N_{\text{sl}_n} & & A \\ \downarrow \psi & \text{GT integrable system} & \downarrow J \\ O \in \mathbb{C}^{\binom{n}{2}} & & (\text{char poly of } A_{ii}, i=1, \dots, n-1) \\ & & A_i = \text{upper } i \times i \text{ minor of } A. \end{array}$$

$$\begin{array}{ccc} \mathbb{C}[x_i] \subset A & \text{quantization of } X & \\ (\text{cat } \mathcal{O} \text{ for } A) & \text{categorifies } H(X^+) & \end{array}$$

(GT-modules
for A) categorifies \mathcal{Y}'/δ)

GT-modules
for $U\text{sl}_n/\mathbb{Z}$

Quantizations of $M_c(\lambda, \mu)$

gr finite type

Y_μ shifted Yangian

it acts by difference operators on
the image of Y_μ is called Y_μ^λ

Y_0 usual Yangian

μ dominant $Y_\mu \subset Y_0$

$\mathbb{C}(x_{i,r} : r=1,..,v_i)$

Kodera

Theorem [BFKKNNW]

$$A_{\text{fr}}(G, N) \cong Y_\mu^\lambda$$

$$A_{\text{fr}}(G, N) \longrightarrow$$

$$A_{\text{fr}}(T, 0)[\dots]^\sim$$

using
localization in
equivariant homology

"ring of difference
operators"

e.g. $0 \rightarrow \dots \rightarrow \overset{\text{fr}}{\square} \rightarrow \square$ $Y_\mu^\lambda = U\text{sl}_n/\mathbb{Z} = A_{\text{fr}}(G, N)$

Theorem [KTWWY]

① There is a categorical gr action on

$\bigoplus_{\mu} Y_{\mu}^{\lambda} - \emptyset$ categorifies $V(\lambda)$. (usually)

② $Y_{\mu}^{\lambda} - \emptyset$ is symplectic dual to
category \mathcal{D} for quantized $M_H(\lambda, \mu)$

To prove both results we relate
 $Y_{\mu}^{\lambda} - \emptyset$ to modules for KLRW-modules