Integrable QM operators from mirror curves

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- Typical example — the total space X of the canonical bundle on a toric del Pezzo surface S,

$$X = \omega_S \to S,$$

where $S = \mathbb{P}^2$ or (i) a Hirzerbruch surface $\mathbb{F}_n$, $n = 0, 1, 2$; (ii) $\mathbb{P}^2$ blown up at n points, $n = 0, 1, 2, 3$; (iii) non-smooth weighted projective spaces $\mathbb{P}(1, m, n)$, $m, n \in \mathbb{Z}_{>0}$.

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- Corresponding mirror CY is given by

$$zw - W_X(e^p, e^q) = 0,$$

  where the mirror curve is an elliptic curve

$$W_X = 0.$$

- Specifically, for $S = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$,

$$W_X = e^p + e^{-p} + e^q + \zeta e^{-q} + \varkappa,$$

where $\zeta > 0$ is a 'mass' parameter and $\varkappa$ is the modulus, and

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  - (i) Their spectral properties are encoded in the enumerative geometry of the toric CY threeford X.
  - (ii) Their spectral theory provides a non-perturbative definition of the topological string theory on X.

Weyl operators

- P and Q — QM momentum and position operators on the Hilbert space $L^2(\mathbb{R})$, satisfying

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- They have domains

$$D(U) = \{\psi \in L^2(\mathbb{R}) : e^{-2\pi b\xi}\, \widehat{\psi}(\xi) \in L^2(\mathbb{R})\},$$
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- Similarly,
$(U^{-1}\psi)(x) = \psi(x - ib), \quad (V^{-1}\psi)(x) = e^{-2\pi bx}\psi(x).$

The functional-difference operators

Corresponding operators have the form

- For $S = \mathbb{P}^1 \times \mathbb{P}^1$,

$$\boxed{H(\zeta) = U + U^{-1} + V + \zeta V^{-1}}$$

— a self-adjoint in $L^2(\mathbb{R})$ pseudo-differential operator of infinite order with the symbol $2\cosh(2\pi b\xi) + 2\cosh(2\pi bx)$ for $\zeta = 1$. Corresponding eigenvalue problem is

$$(H(\zeta)\psi)(x) = \psi(x+ib) + \psi(x-ib) + 2\cosh(2\pi bx)\psi(x).$$

The massless operator $H = H(0) = U + U^{-1} + V$ first appeared in the study of the quantum Liouville model on the lattice (L.D. Faddeev – L.T., 1982), and plays an important role in representation theory of the non-compact quantum group $\mathrm{SL}_q(2;\mathbb{R})$. In quantum Teichmüller theory it is the Dehn twist operator (R. Kashaev, 2000).

- For $S = \mathbb{P}(1, m, n)$,

$$\boxed{H_{m,n} = U + U^{-1} + V + q^{-mn} U^{-m} V^{-n}}$$

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- The operators $H(\zeta)$ for $\zeta > 0$ and H_{mn} have a pure discrete spectrum and $H(\zeta)^{-1}$ and H_{mn}^{-1} are of trace class. Moreover, for the eigenvalue counting function $N(\lambda)$ we have

$$N(\lambda) \sim \frac{\log^2 \lambda}{(\pi b)^2} \qquad \text{for the operator } H(\zeta)$$

$$N(\lambda) \sim c_{m,n} \frac{\log^2 \lambda}{(2\pi b)^2} \quad \text{for the operator } H_{m,n},$$

where $c_{m,n} = (m + n + 1)^2 / 2mn$. The proof of H. Weyl law (A. Laptev, L. Schimmer & L.T., Geometric and Functional Analysis (GAFA), **26**:1 (2016), 288–305) is based on the following ingredients.

- The coherent state transform $L^2(\mathbb{R}) \ni \psi \mapsto \tilde{\psi} \in L^2(\mathbb{R}^2)$

$$\widetilde{\psi}(x,\xi) = \frac{1}{\sqrt[4]{\pi}} \int_{-\infty}^{\infty} \psi(y) e^{-2\pi i y \xi} e^{-(x-y)^2/2} dy$$

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- Remarkable identity for the quadratic form (here $H = H(1)$)

$$(H\psi,\psi) = ((U+U^{-1})\psi,\psi) + ((V+V^{-1})\psi,\psi)$$
$$= 2 \iint\limits_{\mathbb{R}^2} (d_1 \cosh 2\pi b\xi + d_2 \cosh 2\pi b x) |\widetilde{\psi}(x,\xi)|^2 dx d\xi,$$

where $d_1 = e^{-b^2/4}$ and $d_2 = e^{-(\pi b)^2}$.

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- Upper and lower bounds for the Riesz means

$$\mathcal{R}_j(\lambda) = \sum_{j\geq 1} (\lambda - \lambda_j)_+ = \sum_{\lambda_j < \lambda} (\lambda - \lambda_j)$$

obtained using Jensen's inequality and representations

$$\iint\limits_{\mathbb{R}^2} \sum_j (\lambda - \lambda_j)_+ |\widetilde{\psi}_j(x,\xi)|^2 dx d\xi = \mathcal{R}_j(\lambda) = \sum_j (\lambda - (H\psi_j,\psi_j))_+$$

The massless ($\zeta = 0$) operator

$$H = U + U^{-1} + V$$

is a self-adjoint operator in $L^2(\mathbb{R})$ with a simple absolutely continuous spectrum filling $[2, \infty)$, and the eigenfunction expansion theorem for H generalizes Kontorovich-Lebedev transform in the theory of Bessel functions (L.D. Faddeev – L.T., Izvestiya: Mathematics, **79**:2 (2015), 388–410). Specifically,

- The resolvent $R^0_\lambda = (H_0 - \lambda I)^{-1}$ of the free operator $H_0 = U + U^{-1}$ is an integral operator in $L^2(\mathbb{R})$,

$$(R^0_\lambda \psi)(x) = \int_{-\infty}^{\infty} R^0_\lambda(x-y)\psi(y)dy,$$

where

$$R^0_\lambda(x) = \frac{i}{2b\sinh(2\pi bk)}\left(\frac{e^{-2\pi ikx}}{1-e^{2\pi x/b}} + \frac{e^{2\pi ikx}}{1-e^{-2\pi x/b}}\right)$$

and $\lambda = 2\cosh(2\pi bk)$.

- Equation $(H_0 - \lambda I)R_\lambda^0 = I$ follows from Sokhotski-Plemelj formula

$$\theta_b(x - i0) - \theta_b(x + i0) = ib\delta(x),$$

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- The equation
$$\psi(x+ib-i0)+\psi(x-ib+i0)+e^{2\pi bx}\psi(x) = 2\cosh(2\pi bk)\psi(x)$$
admits a scattering solution $\varphi(x,k)$ — Fourier transform of the product of two Faddeev's quantum dilogarithm functions $\Phi_b(z)$ — and Jost solutions $f_\pm(x,k)$ with the asymptotics
$$f_\pm(x,k) = e^{\pm 2\pi ikx} + o(1) \quad \text{as} \quad x \to -\infty,$$
satisfying
$$\varphi(x,k) = M(k)f_+(x,k) + M(-k)f_-(x,k),$$
where
$$|M(k)|^{-2} = 4\sinh(2\pi bk)\sinh(2\pi b^{-1}k).$$

- The resolvent $R_\lambda = (H - \lambda I)^{-1}$ is an integral operator in $L^2(\mathbb{R})$ with the kernel

$$R_\lambda(x,y) = \frac{i}{2b\sinh(2\pi bk)M(k)}$$
$$\times \left[f_-(x,k)\varphi(y,k)\theta_b(y-x) + f_-(y,k)\varphi(x,k)\theta_b(x-y) \right],$$

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- The operator $\mathscr{U} : L^2(\mathbb{R}) \to \mathscr{H}_0 = L^2([0,\infty), |M(k)|^{-2}dk)$,

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  is an isometry.
- The operator $\mathscr{U}H\mathscr{U}^{-1}$ is a multiplication by the function $2\cosh(2\pi bk)$ operator in $\mathscr{H}_0$, i.e., H has simple absolutely continuous spectrum $[2,\infty)$. The eigenfunction expansion theorem for the operator H is a q-analogue of the classical Kontorovich-Lebedev transform in the theory of special functions.

The trace identities

Expressing asymptotic expansion of

$$\log\det(H - \lambda I) \quad \text{as} \quad \lambda \to \infty$$

for a self-adjoint operator H in terms of its coefficients.

- Basic example:

$$H = -\frac{d^2}{dx^2} + v(x), \quad x \in I \subseteq \mathbb{R},$$

  with appropriate boundary conditions.

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Pure discrete spectrum

Suppose the operator $H > 0$ in the Hilbert space $L^2(I)$ has a pure discrete spectrum with eigenvalues $\lambda_n \to \infty$. Two cases:

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- $H^{-1-\varepsilon}$ is of trace class for some $0 < \varepsilon < 1$,

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- Using

$$\log \frac{\det(H - \lambda I)}{\det H} = \frac{1}{2i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\zeta_H(s)}{s \sin \pi s} (-\lambda)^s ds,$$

where

$$\zeta_H(s) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^s},$$

one gets asymptotic expansion as $\lambda \to -\infty$.

- Coefficients of this expansion in terms of $v(x)$ are obtained by the Liouville-Green method — asymptotics as $\lambda \to -\infty$, uniform in x, for the decaying as $x \to +\infty$ solution of the eigenvalue equation

$$H\psi = \lambda\psi$$

for $\lambda < 0$. Converting it to the Riccati equation gives these coefficients recursively.

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- The case $v(x) = x^2$. Solutions of the eigenvalue equation are Weber functions (parabolic cylinder functions),

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  trace identities give Stirling formula for the gamma function.

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- The case $v(x) = e^{2x}$ on the half-line, $\psi(0) = 0$. Solutions — modified Bessel functions of the second kind.

- $$\frac{\det(H-\lambda I)}{\det H} = \frac{K_{i\sqrt{\lambda}}(1)}{K_0(1)},$$

  used by G. Pólya as a simple model for Riemann zeros. Trace identities — large order asymptotics of modified Bessel functions — regularized sums of positive integer powers of zeros of $K_{i\sqrt{\lambda}}(1) = 0$.

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- The case $v(x) = 2\cosh 2x$ as a semiclassical limit of

  $$H = U + U^{-1} + V + V^{-1}$$

  — trace identities should express the mirror symmetry and possible relation to the Gopakumar-Vafa invariants. One needs to develop Liouville-Green method for the functional-difference operators.

Happy Birthday, Samson!