Generalized Beltrami-Bernoulli Flow Model for Astrophysical Disk-Jet Structure Formation

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Based on:

1. N.L. Shatashvili & Z. Yoshida. *AIP Conf. Proc*. 1392, 73 (2011); *ArXiv:*1105.5281v1 [astro-ph.GA] (2011)
2. Z. Yoshida & N.L. Shatashvili. *ArXiv:*1210.3558v1 [physics.flu-dyn] (2012)
3. E. Arshilava, M. Gogilashvili, V. Loladze, I. Jokhadze, B. Modrekiladze, N.L. Shatashvili, A.G. Tevzadze Hydrodynamic jets from protostellar accretion disks with turbulent viscosity. *JHEAP* 23, 6 (2019).. *ArXiv:* 1906.09420v1 [astro-ph.HE]

Outline

- **The macroscopic disk-jet geometry -** ***a marked similarity***
- **The basic Properties of Disk-Jet systems**
- **Role of Magnetic field**
- **Momentum Equation –** ***Simplest MHD Model***
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- **Beltrami model of Disk-Jet system**
- ***Simplest MHD Model with turbulent viscosity***
- **Model Equations for Disk-Jet Structure of YSOs**
- **Stationary Solutions in Similarity Variables** ***– Generalized Beltrami Flow***
- **Properties of Disk-Jet Solutions**
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The macroscopic disk-jet geometry - a marked similarity

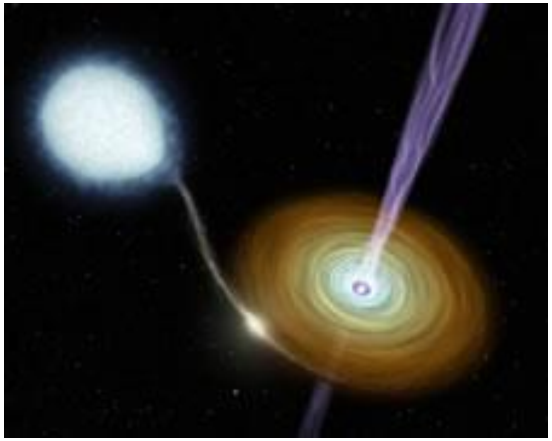
An accretion disk (AD) often combines with spindle-like jet of ejecting gas

&

constitutes *a typical structure* that accompanies a massive object of various scales, ranging from young stars to AGN.

The mechanism that rules each part of different systems - not universal.

Elliptical Galaxy M87 Emiiting Relativistic Jet



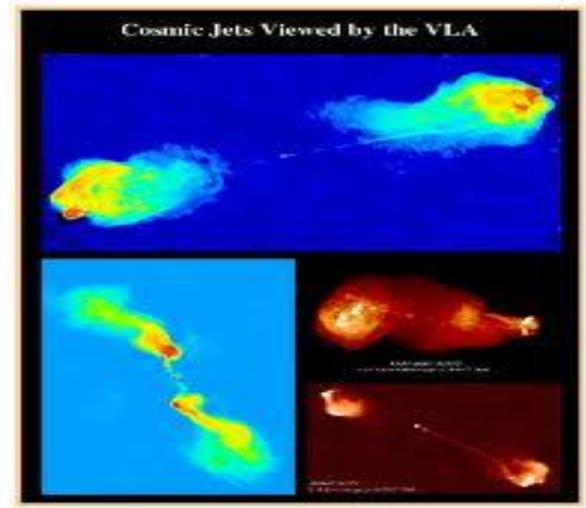
Neutron –Star Jet

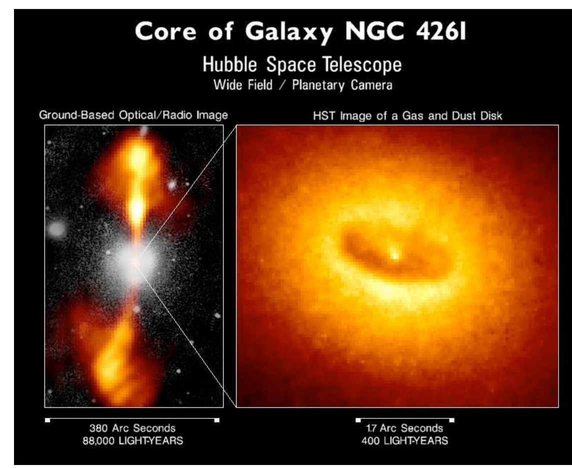
The macroscopic disk-jet geometry - a marked similarity - 2

Since early 70s (the discovery of radio galaxies & quasars) the main evidence from detecting **jets in different classes of astrophysical systems** observed to produce collimated jets near the massive central object - **the direct association with an AD** ***(reflecting different accretion regimes***).

The opposite is not true in some objects for which AD-s do not require collimated jets *(viscous transport/disk winds play the similar role in the energy balance)*.

The macroscopic disk-jet geometry - a marked similarity despite the huge variety of the scaling parameters (*Lorentz factor, Reynolds number, Lundquist number, ionization fractions, etc.)*



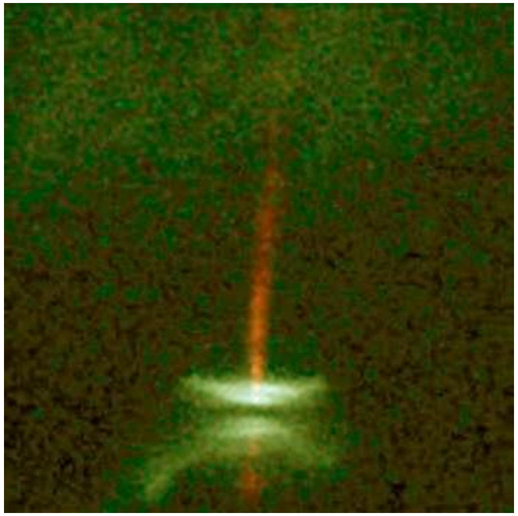


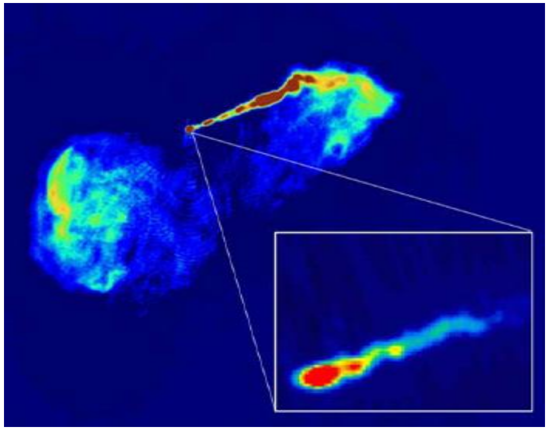
Jets in AGN

The basic properties of disk-jet systems

In the disk region:

- **transport processes of mass, momentum & energy depend strongly on the scaling parameters.**
- the **classical *(collisional)* processes are evidently insufficient to account for the accretion rate**, thus **turbulent transports *(involving magnetic perturbations)* must be invoked**.
- **winds may also remove the angular momentum from Disk.**



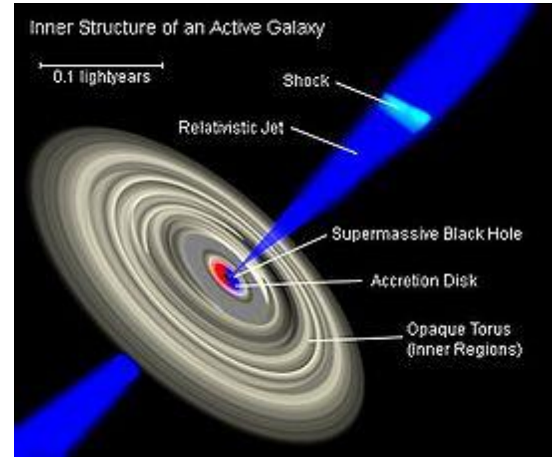


The connection of the disk & the jet is more complicated:

- **mass & energy of jet are fed by the accreting flow**, the mechanism & process of transfer are still not clear.
- **the major constituent of jets is the material of an AD surrounding the central object**.

The basic properties of disk-jet systems - 2

For the fastest outflows the contributions to the total mass flux may come from outer regions as well. In AGN one may think of taking some energy from the central black hole.

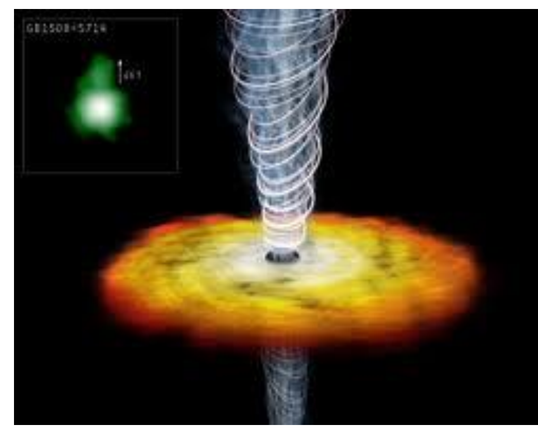


Livio (1997):

(i) **powerful jets are produced by systems in which on top of an AD** threaded by a vertical field, **there exists an additional source of energy/wind,** possibly **associated with the central object;**

(ii) **launching of an outflow from an AD requires a hot corona *or some additional source of energy;***

(iii) **extensive hot atmosphere around the compact object can provide additional acceleration.**



Role of Magnetic Field

Magnetic fields are considered to play an important role in defining the local accretion.

- **When magnetic field is advected inwards by accreting material or/and generated locally** by some mechanism, the centrifugal force due to rotation may boost **jet along the magnetic field lines** up to a super-Alfvénic speed.

- **AGN - there is an alternative idea** suggested by *Blandford & Znajek (1977)* **based on electro-dynamical processes extracting energy from a rotating black hole.**

- **Extra-galactic radio jets might be accelerated b highly disorganized magnetic fields that are strong enough to dominate the dynamics until the terminal Lorentz factor is reached.**

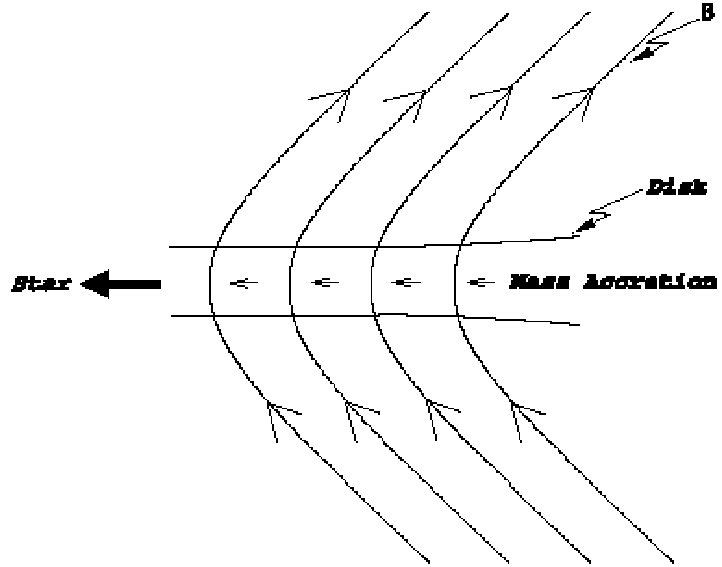


FIG. 1.—Schematic diagram of accretion flow in a disk threaded by magnetic flux accumulated by the process of star formation.

In addition to the energetics to account for the acceleration of ejecting flow, we have to explain how the streamlines/magnetic-field lines change the topology through the disk-jet connection.

Disk-Jet system is a "generalized" Beltrami vortex

The combination of a thin disk and collimated jet is a common structure that is created in the vicinity of a massive object [Blandford & Payne 1982, Begelman 1993].

Beneath a large variety of scales, constituents, and local processes of such systems, there must be a simple and universal principle that dictates the remarkably similar geometry.

We show that the collimated structure of jet is a natural consequence of the alignment of the flow velocity and the vorticity, i.e. so-called *Beltrami condition determines the structure.*

On a Keplerian thin disk, the vorticity becomes a vertical vector with a magnitude $\propto r^{-3/2}$ (r is the radius from the center of the disk), which appears as a spindle of the disk.

Then, the alignment is the unique solution for avoiding singularity of Coriolis force near the center.

However, **we need to generalize appropriately the "vorticity" to deal with the strong** heterogeneity of disk-jet system, as well as to account for dissipation that causes accretion.

Momentum Equation – Simplest HD Model

Despite the diversity & complexity of holistic processes, there must yet be a simple and universal principle that determines the geometric similarity of disk-jet compositions!

Let's invoke a simple model for neutral fluid ($\boldsymbol{P} = \rho \boldsymbol{V}$ momentum density);
Momentum equation reads as:

$$\partial_t \boldsymbol{P} + \nabla \cdot (\boldsymbol{V}\,\boldsymbol{P}) = -\rho \nabla \phi - \nabla p - \nabla \cdot \boldsymbol{\Pi}, \qquad (1)$$

$\boldsymbol{\rho}$ - the mass density, $\boldsymbol{V}$ - flow velocity,

$\boldsymbol{\Pi}$ - the effective viscosity tensor, $\boldsymbol{\phi}$ - the gravity potential.

Energy densities are normalized by unit kinetic energy $\mathcal{E}_0 = \frac{\rho_0 V_0^2}{2}$.

Scale parameter is $\epsilon = \frac{\delta_i}{L_0}$, $\boldsymbol{L_0}$ - system size, $\delta_i = mc/\sqrt{4\pi e^2 \rho_0}$

$\boldsymbol{V_0}$ and $\boldsymbol{\rho_0}$ are the representative flow velocity and mass density in the disk.

Multiplying on (ρ_2 / ρ_1) both sides of (2), **assuming a barotropic relation** $\nabla p = \rho \nabla h$ with an enthalpy h we obtain**:**

$$\boldsymbol{P}_2 \times \boldsymbol{\Omega}_2 = \frac{1}{2}\nabla P_2^2 + \rho_2^2 \nabla(\phi + h) + \frac{\rho_2}{\rho_1}\left[(\nabla \cdot \boldsymbol{P}_1)\boldsymbol{P}_2 + \nabla \cdot \boldsymbol{\Pi}\right], \tag{3}$$

Where

$$\boldsymbol{\Omega}_2 := \nabla \times \boldsymbol{P}_2 \tag{4}$$

We shall show that a generalized Beltrami Condition, demanding that Ω_2 parallels P_2 ***is a unique recourse to avoid singular energy-density in disk-jet geometry.***

In the disk: a radial flow (<< V_θ) is caused by viscosity – a finite dissipation breaks the conservation of angular momentum and enables flow to cause accretion;

flow $\boldsymbol{V}$ is primarily azimuthal **(** θ **),** the viscosity force can be approximated as (*under assumptions of* ***azimuthal symmetry*** **&** $\nabla \cdot \boldsymbol{V} \approx 0$ **,** η *being a shear viscosity coefficient* **) :**

$$-\nabla \cdot \boldsymbol{\Pi} \approx -\nabla \times (\rho\eta\nabla \times \boldsymbol{V}), \tag{5}$$

Beltrami model of a Disk-Jet System

In a Keplerian thin disk $V \approx V_0\, r^{-1/2}\, \boldsymbol{e}_\theta$ **and** $\nabla(\rho\eta)$ **is approximately vertical,** ***we estimate:***

$$-\nabla\cdot\mathbf{\Pi} \approx -\rho\eta\nabla\times(\nabla\times\boldsymbol{V}) = -\rho\eta V_0\frac{3}{4}r^{-5/2}\boldsymbol{e}_\theta. \qquad (6)$$

hence, we may write $-\nabla\cdot\mathbf{\Pi} = -\nu\boldsymbol{P}$

with $\nu > 0$ **===> viscosity force is primarily in azimuthal (toroidal) direction , it can be**

balanced by term $(\rho_2/\rho_1)(\nabla\cdot\boldsymbol{P}_1)\boldsymbol{P}_2$ **extracted from the inertia term!**

Using steady state Mass Conservation Law $\nabla\cdot\boldsymbol{P} = 0,$ we observe

$$\rho_1^{-1}\nabla\cdot\boldsymbol{P}_1 = \rho_2\boldsymbol{V}\cdot\nabla\rho_2^{-1} = -\boldsymbol{V}\cdot\nabla\log\rho_2.$$

Hence, the balance of the viscosity and the partial inertia term demands

$$\boldsymbol{V}\cdot\nabla\log\rho_2 = \nu, \qquad (7)$$

which determines $\boldsymbol{\rho_2}$ **;** the straightforward algebra gives the relation for $\boldsymbol{\rho_1}$

$$\boldsymbol{V}\cdot\nabla\log\rho_1 = -\nabla\cdot\boldsymbol{V} - \nu. \qquad (8)$$

Near the axis poloidal component of flow acquires appreciable vertical (z) component.
Let's unearth the mechanism that collimates the flow.

After above balance the remaining terms in (3) do not have azimuthal components – hence, extracting the partial inertia term from total one we have separated the toroidal component .

The vorticity $\boldsymbol{\Omega_2}$ includes a singular factor $\nabla \times \boldsymbol{V} \propto r^{-3/2}\boldsymbol{e}_z$

To eliminate divergence of $\boldsymbol{P}_2 \times \boldsymbol{\Omega}_2$ **near the axis** $\boldsymbol{P_2}$ ***must align*** $\boldsymbol{\Omega_2}$ **i.e.**

Beltrami Condition **must be satisfied** ($\boldsymbol{\lambda}$ is a certain scalar function)

$$\boldsymbol{\Omega}_2 = \lambda \boldsymbol{P}_2 \tag{9}$$

The flow $\boldsymbol{V} = \boldsymbol{P}_2/\rho_2$ **is, therefore, collimated by the generalized vorticity** $\boldsymbol{\Omega_2}$ **creating a jet!**

The remaining potential forces in (3) must balance ===> ***Bernoulli Condition***

$$\frac{1}{2\rho_2^2}\nabla P_2^2 + \nabla(\phi + h) = \nabla\left(\frac{1}{2}V^2 + \phi + h\right) + V^2\nabla \log \rho_2 = 0. \tag{10}$$

The determining equations are: *Eq.* **(7)** determines "artificial ingredient" $\boldsymbol{\rho_2}$ for given $\boldsymbol{v}$;
$\boldsymbol{V}(= \boldsymbol{P}_2/\rho_2)$ is governed by (9); after determining $\boldsymbol{V}$ and $\boldsymbol{\rho_2}$ we solve (10) to determine $\boldsymbol{h}$.

Remark on *Beltrami Condition*

Here the essential part of the Beltrami condition is its poloidal component, which dictates the poloidal flow so as to eliminate the toroidal component of the inertia term $\boldsymbol{P_2} \times \boldsymbol{\Omega_2}$.

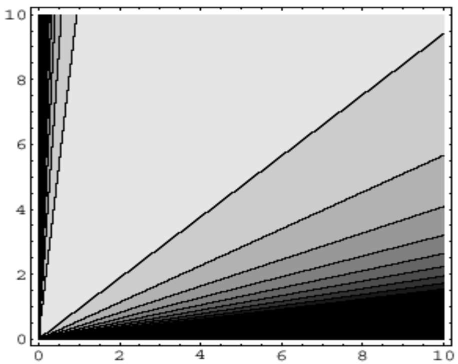
As for the toroidal flow $V_\theta\, e_\theta$, which yields primarily a radial (centrifugal) inertia force, the "Beltrami condition" brings about an extra constraint.

In fact, if we were to use the conventional vorticity $\boldsymbol{\Omega} = \boldsymbol{\nabla} \times \boldsymbol{V}$ (i.e., if $\rho_2 = 1$) and estimate the centrifugal force of the Keplerian flow $V = V_0 r^{-1/2} e_\theta$, the term $\boldsymbol{P} \times \boldsymbol{\Omega}$ contributes a half of the total centrifugal force $(\boldsymbol{V} \cdot \boldsymbol{\nabla})P = \rho V_0^2 / r$, while the term $\boldsymbol{\nabla} V^2/2$ on the right-hand side of (3) contributes the remaining half; combining these two terms, we obtain the right balance with the gravity $-\rho MG/r^2$.

Hence, the conventional Beltrami condition $P \times \Omega = 0$ would lead to an inadequate estimate of the toroidal flow. However, our ***Generalized Beltrami* Condition,** based on the generalized vorticity $\boldsymbol{\Omega}_2$ (including $\rho_2 \neq 1$), **can be made consistent with the Keplerian velocity.**

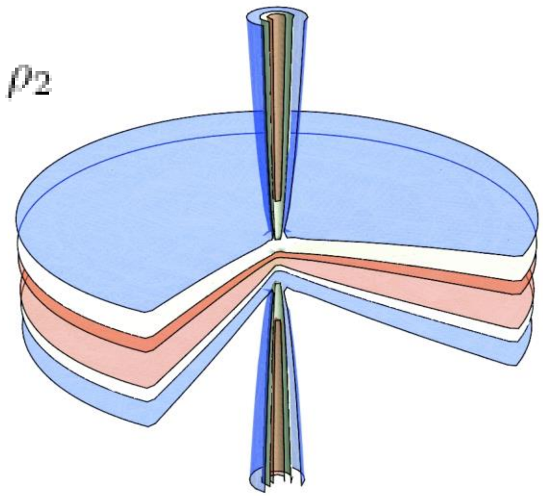
A similarity solution modeling fundamental disk-jet structure

Invoking the Clebsch parameterization, in an axisymmetric geometry, the divergence-free vector $\boldsymbol{P}$ may be parameterized as $\boldsymbol{P} = \nabla\psi \times \nabla\theta + I\nabla\theta$, $I = \rho r V_\theta$, Both ψ and I do not depend on θ. Since $(\boldsymbol{P} \cdot \nabla)\,\psi = 0$, the level sets (contours) of ψ are the streamlines of $\boldsymbol{P}$ (or those of $\boldsymbol{V} = \boldsymbol{P}/\rho$). In a disk region, $rV \propto r^{1/2}$, while ρ is a strongly localized function with respect to z .



The momentum field (streamlines of poloidal component of P) of the similarity solution

$$\rho = \rho_\perp \rho_\parallel \rho_2$$



$$\tau := \frac{z}{r} \quad (r > 0),$$

$$\sigma := \sqrt{r^2 + z^2} \, .$$

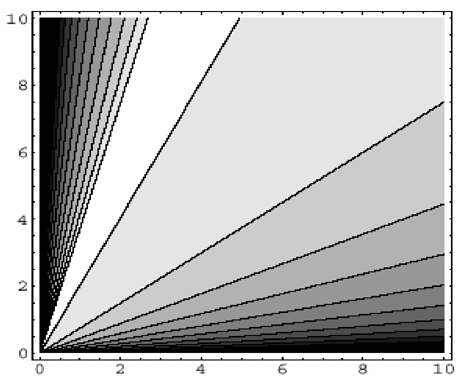
$$(\nabla\tau \cdot \nabla\sigma = 0)$$

The density ρ in the similarity solution (*log* scale)

We assume $\rho_{||}(\sigma) \approx z^{-2/3}$ and $\rho_2 \approx r^{-1/2}$.

A levelset surface of ρ is shown in the domain $r < 5$ $|z| < 5.$

Dividing $\boldsymbol{P}$ by the density ρ , we obtain the velocity field $\boldsymbol{V}$



The distribution of $\rho_\perp$ *of the similarity solution*

Conclusion for Universal Disk-Jet Structure Problem

Invoking the simplest (minimum) model of MHD we have shown that:

- **the combination of a thin disk and narrowly-collimated jet is the unique structure that is amenable to the singularity of the Keplerian vorticity - *the Beltrami condition* - the alignment of flow and generalized vorticity - characterizes the geometry.**

- **The conventional vorticity is generalized to subtract the viscosity force causing the accretion and the centrifugal force of the Keplerian velocity.**

- **We have found an analytic solution in which the *generalized vorticity* is purely kinematic ($\Omega = \nabla \times P_2$ with the momentum P_2 modified by the viscosity effect).**

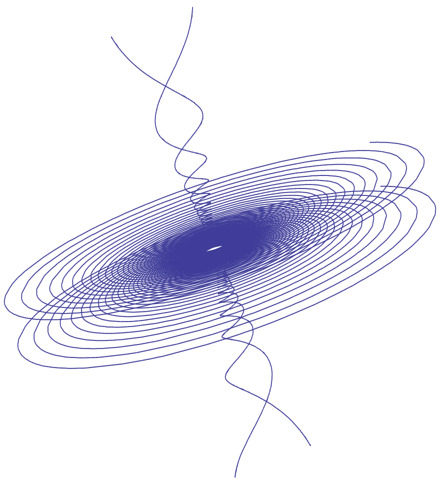
- **The principal force that ejects the jet is the hydrodynamic pressure dominated by $V_\theta^2/2$**

- Present study describes a pure fluid-mechanical model of jet collimation, **magnetic field**, thrusting the center of the disk, **to "guide"** (*and twist, as often observed*) **the flow of charged gas** (plasma), **can be easily invoked** (*Sahatashvili & Yoshida 2011*). **Here the *fluid generalized vorticity* plays the same role of *a magnetic field*.**

- **In charged disk additional magnetic force may contribute to jet acceleration if the self-consistently generated large-scale magnetic field is sufficiently large – such structures can be described by the generalized model.**

The Generalized Beltrami Vortex

Identifying the disk-jet structure as a Generalized Beltrami Vortex, we will be able to understand the self-organization process in terms of the *Generalized Helicity* ==>



The streamlines of *Generalized Beltrami Flow* – in a disk-jet system the accreting flow and jet align parallel to *a generalized vorticity*

***Important lesson* - the *helicity* of the *generalized vorticity* is the key parameter that characterizes the self-organizing of a disk-jet system.**

Comment on the Singularity of Keplerian Velocity

The Keplerian velocity / similarity solution has a singularity at the origin (where gravity potential $\Phi = -MGr^{-1}$ diverges) **which disconnects the disk part & the jet part.**

To "connect" both subsystems we need a singular perturbation that dominates the small-scale hierarchy on which the disk and jet regions are connected smoothly
Shiraishi et al. (2009)

The connection point must switch the topology of the flow.

The topological difference of 2 sub-systems' vector fields demands "decoupling" of them.

Instead of disconnecting them by a singularity, *we will have to consider a small-scale structure in which the topological switch can occur*.

Hall effect (scaled by ε) **or viscosity/resistivity** (scaled by reciprocal Reynolds number) **yields a singular perturbation ===> the generalized magneto-Bernoulli mechanism may effectively accelerate the jet-flow.**

The mechanism of singular perturbation & the local structure of the disk-jet connection point may differ depending on the fluid condition near the central object.

Hydrodynamic Jet formation from Protostellar Accretion Disks

Jets streaming from young stellar objects (*YSO*s) are spectacular manifestations of the star formation process. Believed to be powered by protostellar ADs, they carry away the matter, energy and angular momentum of accreting matter promoting the development of a protostar. Properties of these disk-jet systems are inferred from the observations of Herbig-Haro (HH) objects, T-Tauri and Herbig Ae/Be stars, protostellar and protoplanetary disks.

Collimated bipolar outflows from YSO are known to be parsec-scale, non-relativistic and supersonic by nature. The radial velocity of the jet-flow varies from 20 km/s up to 450 km/s for HH jets (Hartigan et al. 2011; Plunkett et al. 2015; Podio et al. 2016; Jhan and Lee 2016; Bjerkeli et al. 2016; Reiter et al. 2017). **These jets are launched in the vicinity of a protostar, as close as 0.03au** (Lee et al. 2017).

A vast number of collimated jets are detected in molecular clouds pointing to the embedded protostars. Surveys of these jets (Ioannidis and Froebrich 2012; Smith et al. 2014; Zhang et al. 2014) provide a large unbiased observational data that can be used to test the theoretical models of the disk-jet structures

Today we know that morphology, sizes and velocities of the jet-outflows can be used to estimate the mass, luminosity and/or age of the YSOs (see Bally 2016and references therein).

Hydrodynamic Jet formation from Protostellar Accretion Disks 2

Another class of wide-angle hydrodynamic outflows from HH objects decelerate either with increasing the angle away from the central axis of the flow, or increasing distance along axis (Lizano et al. 1988). **It is known that these hydrodynamic outflows of neutral atoms are intrinsically linked with the process of star formation** (Ruden et al. 1990, Shu et al. 1991).

A class of the hydrodynamic jet solution from viscous accretion flows has been studied by Scott and Lovelace (1987). **Self-similar solutions of the model give velocity field being inverse proportional to the radial distance from the center. Matching the profiles of circular jets** (Squire 1951) **this solution deviates from the Keplerian profile and can be effectively used outside the accretion disk area.**

Another solution to the formation of hydrodynamic jets has been studied by Hernandez et al. (2014) showing that under certain conditions hydrodynamic accretion disk can develop instabilities. Using perturbation analysis authors have shown that developed instability may lead to the formation of pair of bipolar jet-outflows.

Axisymmetric Outflows, winds from Accretion Disks

***Instability mechanism is unlikely to support a steady disk-jet structure*, and can possibly be used in the description of short time outbursts from the hydrodynamic accretion flows in the vicinity of central object.**

Recently Clarke and Alexander (2016) have used **a self-similar approach for the description of the axisymmetric hydrodynamic outflows from hydrodynamic accretion disks**. Authors have shown **isothermal disk-wind structure** that can be developed for the power low density profile that well matches the results of numerical simulations in the wind area.

Since the viscous accretion disk model was proposed (Shakura and Sunyaev 1973), with an observed high ejection efficiency, it became natural to assume that jets are driven magnetically from an accretion disk — *when magnetic field is advected inwards by accreting material or/and generated locally by some mechanism, the centrifugal force due to rotation may boost the jet along the magnetic field lines up to a super-Alfvénic speed* (Begelman et al. 1984; Begelman 1993, 1998; Celotti and Blandford 2001; Kudoh and Shibata 1987; Kuwabara et al. 2005; Anderson et al. 2005).

Axisymmetric Outflows, winds from Accretion Disks 2

Blandford and Payne (1982) studied the magneto-centrifugal acceleration along the magnetic field lines and demonstrated that magnetic field results into instability of particles at Kepler orbit leading to the Jet formation in the disk center (*with the opening angle of the jet ≤30◦*).

They were the first to show the braking of matter in the azimuthal direction inside disk and outflow acceleration above disk surface guided by the poloidal magnetic field components. Toroidal components of magnetic field then collimate flow.

In the case of **the fully turbulent Keplerian disk the poloidal magnetic field tends to drift outward** (Lovelace et al. 1994; Lubow et al. 1994; Bisnovatyi-Kogan & Lovelace 2007) so that its value cannot significantly exceed the strength of the large-scale seed magnetic field.

Then, **according to the hydro-magnetic models, the magnetic fields provide a natural mechanical link between disks and jets and can account for the launching, confinement and collimation of jets** (see e.g. Blandford and Rees 1974; Blandford and Znajek 1977; Blandford and Payne 1982; Blandford 1994; Krasnopolsky et al. 1999; Zanni et al. 2007) — ***the angular momentum, energy and mass can be removed from the accreting flow***.

The collimation is provided by the stratified thermal pressure from an external medium while the acceleration efficiency then depends on the pressure gradient of the medium.

Simplest HD Model with Turbulent Viscosity

In **Arshilava et al. (2019) we present the results of theoretical study of the disk-jet structure formation for YSOs based on the Beltrami Flow model (Shatashvili & Yoshida 2011) using the Turbulent Viscosity approach (Shakura & Sunyaev 1973) as the main reason of accretion**; **disk was assumed un-magnetized, hence, there is no pre-existed global magnetic field.**

YSO disk-jet structure, according to observations, is quite a long-lived object. Hence, the steady state solutions could well describe its behavior. **Equations governing the dynamics of the stationary viscous compressible fluid rotating around a central gravitating object can be written as follows:**

($\boldsymbol{P} = \rho \boldsymbol{V}$ **momentum density);**

$$(\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla H - \nabla\Phi + \frac{1}{\rho}\nabla\cdot\mathrm{T}\,, \qquad \nabla\cdot(\rho\mathbf{V}) = 0\,, \tag{11}$$

$\boldsymbol{\rho}$ - the mass density, $\boldsymbol{V}$ - flow velocity, $\boldsymbol{T}$ - the effective viscosity tensor, $\boldsymbol{\phi}$ - the gravity potential.

Barotropic Equation of State was used to derive the Enthalpy $\nabla H = \frac{1}{\rho}\nabla\mathcal{P}\,,$ (12)

Energy densities are normalized by unit kinetic energy $\mathcal{E}_0 = \frac{\rho_0 V_0^2}{2}.$

Scale parameter is $\epsilon = \frac{\delta_i}{L_0}$, $\boldsymbol{L_0}$ - system size, $\delta_i = mc/\sqrt{4\pi e^2\rho_0}$

$\boldsymbol{V_0}$ and $\boldsymbol{\rho_0}$ are the representative flow velocity and mass density in the disk.

Model Equations for Disk-Jet Structure of YSOs

To seek for the steady state solutions of the disc-jet structures persisting around a central accreting object we introduce a so-called " ideal" and "reduced" factors of the "local" density following Shatashvili & Yoshida (2011); Yoshida & Shatashvili (2012): $\rho = \rho_I \rho_R$.

We introduce "Ideal" and "Reduced" momenta: $\mathbf{P}_I = \rho_I \mathbf{V}$, $\mathbf{P}_R = \rho_R \mathbf{V}$ **===>** (13)

and we write:

$$\mathbf{P}_R \times (\nabla \times \mathbf{P}_R) = \\ = \frac{1}{2}\nabla P_R^2 + \rho_R^2 \nabla (H + \Phi) + \frac{\rho_R}{\rho_I}\left[\mathbf{P}_R(\nabla \cdot \mathbf{P}_I) + \nabla \cdot \mathrm{T}\right] . \tag{14}$$

In** Conventional formulation of Fluid Mechanics **we choose - $\rho_R = 1$, **&** $\rho_I = \rho$

The geometry of the problem allows the assumption for the reduced momentum to obey the Beltrami Condition implying that the Reduced Momentum is aligning along its corresponding Generalized Vorticity (see e.g. Shatashvili & Yoshida 2011; Yoshida & Shatashvili 2012 and references therein):

$$\mathbf{P}_R = \lambda \left(\nabla \times \mathbf{P}_R\right) , \tag{15}$$

makimg *l.h.s.* of (14) strictly zero. λ is a Beltrami Parameter related to reduced momentum.

We seek for the solution of the $\boldsymbol{P_R}$ that will make the last term zero on the r.h.s. of the Eq. (4), thus, fully determined by the viscosity effect:

$$\mathbf{P}_R(\nabla \cdot \mathbf{P}_I) + \nabla \cdot \mathrm{T} = 0 . \tag{16}$$

Turblent Viscosity model

Using (15,16) the Eq. (14) reduces to so called **Generalized Bernoulli Condition:**

$$\frac{1}{2}\nabla P_R^2 + \rho_R^2 \nabla \left(H + \Phi \right) = 0 \,. \tag{17}$$

We employ the turbulent viscosity model, when the small scale turbulence creates the anomalous dissipation that can be described by using the *α*–viscosity model introduced by Shakura & Sunyaev (1973).

In contrast to the standard model, **we use the effective viscosity model both in the disk and jet as well as in the disk-jet transition areas.** We employ cylindrical coordinates (*r; φ; z*) to describe the disk-jet system. Hence**, assuming the strong azimuthal rotation**, we have:

$$T_{r\varphi} = T_{\varphi r} = \nu_{\rm t} \rho \left[r \frac{\partial}{\partial r} \left(\frac{V_\varphi}{r} \right) + \frac{1}{r} \frac{\partial V_r}{\partial \varphi} \right] \,, \tag{18}$$

$$T_{z\varphi} = T_{\varphi z} = \nu_{\rm t} \rho \left[\frac{\partial V_\varphi}{\partial z} + \frac{1}{r} \frac{\partial V_z}{\partial \varphi} \right] \,, \tag{19}$$

where $\nu_{\rm t}$ is a turbulent viscosity tensor. **We introduce background constant pressure** $\mathcal{P}_0$ and

$$\mathcal{P} = \mathcal{P}_0 + p \,.$$

In this limit turbulent stress tensor is split too $T_{ik} = \bar{T}_{ik} + t_{ik} \,.$

The classical *α*–viscosity model links turbulent viscosity stress tensor to the pressure using the constant parameter α_0 **:** $\bar{T}_{r\varphi} = \alpha_0 \mathcal{P}_0 \,.$

Turbulent Viscosity Model - 2

Assuming axisymmetric flow $V_\varphi = r\Omega_K(r,z)$, **rotating locally with Keplerian angular velocity**:

$$\Omega_K^2(r,z) = \frac{GM_\star}{(r^2+z^2)^{3/2}} \,, \tag{20}$$

We can extend the standard turbulent viscosity model to derive:

$$t_{ik} \equiv \left\langle \frac{3}{2}\nu_t \rho \Omega_K(r,z) \right\rangle \frac{r^2}{r^2+z^2} - \alpha_0 \mathcal{P}_0 \,. \tag{21}$$

Hence, in the axisymmetric case, where the azimuthal gradients in the Eqs. (18,19) can be neglected, the viscous stress tensor elements can be calculated as follows:

$$t_{r\varphi} = \frac{r^2}{r^2+z^2}\,\beta\, p \,, \qquad t_{z\varphi} = \frac{rz}{r^2+z^2}\,\beta\, p \,. \tag{22}$$

Here p can be positive/negative, corresponding to the stronger/weaker turbulence compared to the background turbulent steady state.

Assumption of the strictly Keplerian local angular velocity of the rotation (see Eq. (20)) can be justified in the rotationally supported flow, where radial pressure gradients can be negligible compared to the centrifugal force.

Such situation is realized in slowly accreting flows, where background pressure is known to vary slowly, i.e., $\mathcal{P}_0 = const.$ **.**

Model equations for Disk-Jet Structure

We expand flow velocity using the axisymmetric stream function and Keplerian azimuthal circulation, **dictated by the geometry of observed YSOs disk-jet structure and the continuity Eq**. as:

$$\mathbf{V} = \frac{1}{\rho}\left(\nabla\psi \times \nabla\varphi\right) + rV_\varphi \nabla\varphi\,. \tag{23}$$

The ψ introduced in such a way matches the stream function of the actual momentum $\mathbf{P} = \rho\mathbf{V}$ of the flow. The Eq. (16) takes the form:

$$\frac{V_\varphi}{r}\left(\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\ln\rho_R - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\ln\rho_R\right) = \beta\,\frac{r^2}{r^2+z^2}\left[\frac{\partial p}{\partial r} + \frac{z}{r}\frac{\partial p}{\partial z} + \frac{2\beta}{r}p\right], \tag{24}$$

While the Generaliozed *Bernoulli Condition* *is reduced to*

$$\nabla\mathcal{E}_m + \left(V_\varphi^2 + \frac{(\nabla\psi)^2}{r^2\rho^2}\right)\nabla\ln\rho_R = 0\,,$$ **with Total mechanical Energy** $$\mathcal{E}_m \equiv \Phi + \frac{V_\varphi^2}{2} + \frac{(\nabla\psi)^2}{2r^2\rho^2}\,. \tag{25}$$

The Beltrami Conditions (15) is written as:

$$\nabla\times\nabla\psi\times\nabla\varphi + \nabla\times\left[(\rho r V_\varphi)\nabla\varphi\right] + \nabla\psi\times\nabla\varphi\times\nabla\ln\rho_I \tag{26}$$

$$+ (\rho r V_\varphi)\nabla\varphi\times\nabla\ln\rho_I = \lambda\left(\nabla\psi\times\nabla\varphi + (\rho r V_\varphi)\nabla\varphi\right)\,.$$

Hence, the final describing system of equations is reduced to the Eqs. (24, 25), and (26).

Stationary Solutions in Similarity Variables

We introduce the orthogonal variables

$$\sigma = \sqrt{r^2+z^2}\,, \qquad \tau = z/r\,, \qquad \nabla\tau\cdot\nabla\sigma = 0. \tag{27}$$

It is obvious that $\Phi = \Phi(\sigma) = -\Omega_0^2/\sigma.$

$\psi = \psi(\tau)\,,$:====> **it is possible to separate the variables in the solution.** Then (24) gives

$$V_\varphi \frac{1+\tau^2}{\sigma^2}\frac{\partial\psi}{\partial\tau}\frac{\partial}{\partial\sigma}\ln\rho_R = \left(\frac{\partial}{\partial\sigma}+\frac{2}{\sigma}\right)\frac{\beta\, p}{1+\tau^2}\,, \tag{28}$$

The three components (r, φ, z) of Beltrami Condition (26) , after straightforward algebra, **give**:

$$\frac{\partial}{\partial\sigma}\left(\frac{\sigma}{(1+\tau^2)^{1/2}}\rho V_\varphi\right) = 0\,, \qquad \frac{\partial}{\partial\tau}\ln\left(\frac{\sigma}{(1+\tau^2)^{1/2}}\rho_R V_\varphi\right) = \lambda\frac{(1+\tau^2)^{1/2}}{\sigma\rho V_\varphi}\frac{\partial\psi}{\partial\tau}\,, \tag{29}$$

while r and z components of Bernoulli condition (25) give:

$$\frac{1}{\rho}\frac{\partial p}{\partial\sigma} + \frac{\partial\mathcal{E}_m}{\partial\sigma} + 2\left(\mathcal{E}_m - \Phi\right)\frac{\partial}{\partial\sigma}\ln\rho_R = 0\,, \tag{30}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial\tau} + \frac{\partial\mathcal{E}_m}{\partial\tau} + 2\left(\mathcal{E}_m - \Phi\right)\frac{\partial}{\partial\tau}\ln\rho_R = 0\,, \tag{31}$$

Hence, in similarity variables, the system is reduced to the Eqs. (28-31).

Stationary Solutions in Similarity Variables 2

We seek the solution of the system assuming that **describing physical variables can be factorized using the similarity variables** introduced in previous subsection:

$$\begin{aligned} p(\sigma,\tau) &= p_1(\sigma)p_2(\tau)\,, \\ \rho_{\rm I}(\sigma,\tau) &= \rho_{I1}(\sigma)\rho_{I2}(\tau)\,, \\ \rho_{\rm R}(\sigma,\tau) &= \rho_{R1}(\sigma)\rho_{R2}(\tau)\,, \\ \lambda(\sigma,\tau) &= \lambda_1(\sigma)\lambda_2(\tau)\,. \end{aligned} \tag{32}$$

The azimuthal velocity can be calculated using the Keplerian rotation velocity $V_\varphi(\sigma,\tau) = V_{\rm Kep}$

Where

$$V_{\rm Kep} = \frac{\sigma_0\Omega_0}{(1+\tau^2)^{1/2}}\left(\frac{\sigma}{\sigma_0}\right)^{-1/2} \,. \tag{33}$$

***Ω_0* is the Keplerian angular velocity of the rotation at some characteristic radius *σ_0* in the disk.** Applying this ansatz into Eqs.(28-31), solving them in σ, we find that balance of all terms give:

$$\begin{aligned} p_1(\sigma) &= \sigma^{-5/2}\,, \\ \rho_{\rm I1}(\sigma) &= \sigma^{-1}\,, \\ \rho_{\rm R1}(\sigma) &= \sigma^{-1/2}\,, \\ \lambda_1(\sigma) &= \sigma^{-1}\,. \end{aligned} \tag{34}$$

Introducing notation $W \equiv \dfrac{1}{\rho_2}\dfrac{{\rm d}\psi}{{\rm d}\tau}\,,$

where $\rho_2 \equiv \rho_{I2}\rho_{R2}.$ (35)

We get

$$\beta\left[\beta W(\tau)^2 + \frac{5}{2}\frac{\sigma_0^{3/2}\Omega_0}{(1+\tau^2)^{3/2}}W(\tau) - \beta\frac{\sigma_0^3\Omega_0^2\tau^2}{(1+\tau^2)^4}\right] = 0\,. \tag{36}$$

Disk-Jet Structure – Generalized Beltrami Flow

There are three apparent solutions to this equation:

(i) The solution with $\beta = 0$, corresponding to the background dissipation model $(T_{ik} = \alpha_0 \mathcal{P}_0)$;

(ii) Two separate solutions for the dissipative flow $(\beta \neq 0)$ with

$$W_{\pm}(\tau) = -\frac{5}{4}\frac{\sigma_0^{3/2}}{(1+\tau^2)^{3/2}}\frac{\Omega_0}{\beta}\left[1 \pm \left(1 + \frac{16}{25}\frac{\beta^2\tau^2}{(1+\tau^2)}\right)^{1/2}\right] . \qquad (37)$$

For simplicity we constrain on the small β limit ($\beta << 1$). Then the solutions can be approximated:

$$W_+(\tau) \approx \frac{2}{5}\frac{\tau^2}{(1+\tau^2)^{5/2}}\beta\sigma_0^{3/2}\Omega_0 . \qquad W_-(\tau) \approx -\frac{5}{2}\frac{1}{(1+\tau^2)^{3/2}}\frac{\sigma_0^{3/2}\Omega_0}{\beta} , \qquad (38)$$

Hence, the solutions of our disk-jet model can be calculated as follows:

$$\begin{aligned} \rho(\sigma,\tau) &= \sigma^{-3/2}\rho_2(\tau) \\ p(\sigma,\tau) &= \frac{(1+\tau^2)^{3/2}}{\sigma^{5/2}}\frac{\sigma_0^{3/2}\Omega_0}{\beta}\rho_2(\tau)W(\tau) , \\ V_r(\sigma,\tau) &= -\frac{1+\tau^2}{\sigma^{1/2}}W(\tau) , \\ V_z(\sigma,\tau) &= -\tau\frac{1+\tau^2}{\sigma^{1/2}}W(\tau) . \end{aligned} \qquad (39)$$

(39) is a full set of solutions together with (32) and appropriate choice of W from (37).

Stationary Solutions in Similarity Variables

Eq. (39) indicates the character of the solutions:

Radial–vertical accretion flow for $W_+ > 0$ corresponding to $V_r < 0$ and $V_z < 0$;

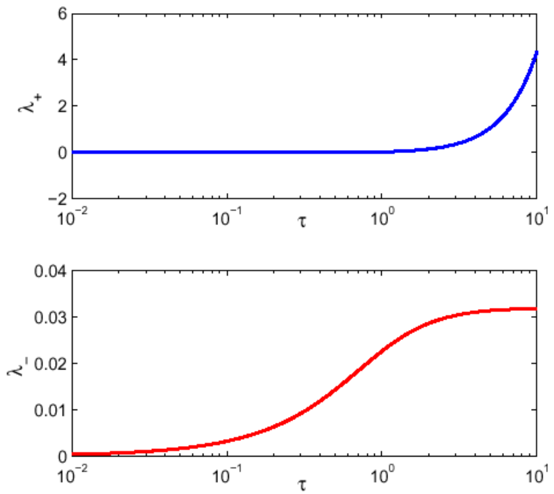
Radial–vertical ejection flow for $W_- < 0$ corresponding to $V_r > 0$ and $V_z > 0$;

We calculated the dependence of the Beltrami parameter on the turbulent viscosity parameter β :

$$\lambda = \left(\frac{\mathrm{d}W}{\mathrm{d}\tau} + \frac{5\tau W}{1+\tau^2}\right)\left((1+\tau^2)W^2 + \frac{\sigma_0^3\Omega_0^2}{(1+\tau^2)^3}\right)^{-1}\sigma_0^{3/2}\Omega_0\,. \qquad (40)$$

interestingly, **both solutions grow with** $\boldsymbol{\beta}$:

$$\lambda_\pm = \lambda(W_\pm) \propto \beta\,.$$



The Beltrami parameters $\lambda(\tau)$ for the disk (top) and the jet (bottom) solutions are shown vs τ for the case when $\beta = 0.01$.

Notice that Beltrami parameter for the disk solution is negligible for low poloidal angles $(\tau << 1)$.

Disk-Jet Structure Solutions – Properties, Realizability

Finally we derive the solutions:

For Disk Flow:

$$V_{rD}(\sigma,\tau) = -\frac{2}{5}\beta\sigma_0\Omega_0\frac{\tau^2}{(1+\tau^2)^{3/2}}\left(\frac{\sigma}{\sigma_0}\right)^{-1/2},$$

$$V_{zD}(\sigma,\tau) = -\frac{2}{5}\beta\sigma_0\Omega_0\frac{\tau^3}{(1+\tau^2)^{3/2}}\left(\frac{\sigma}{\sigma_0}\right)^{-1/2},\qquad (41)$$

For Jet Flow:

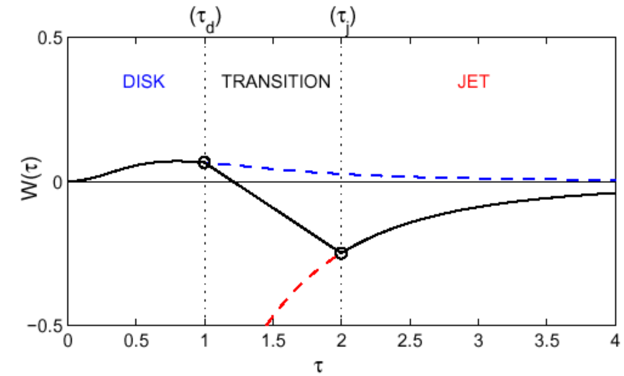
$$V_{rJ}(\sigma,\tau) = \frac{5}{2}\frac{\sigma_0\Omega_0}{\beta}\frac{1}{(1+\tau^2)^{1/2}}\left(\frac{\sigma}{\sigma_0}\right)^{-1/2},\qquad (42)$$

$$V_{zJ}(\sigma,\tau) = \frac{5}{2}\frac{\sigma_0\Omega_0}{\beta}\frac{\tau}{(1+\tau^2)^{1/2}}\left(\frac{\sigma}{\sigma_0}\right)^{-1/2}.$$

To get disk-jet configuration in one solution we need to construct flow that matches disk solution with W_+ at $\tau << 1$ & jet solution for nearly vertical direction $(\tau >> 1)$.

Continuity of the velocity field restricts any jumps in $W(\tau)$.

To describe the disk-jet solution we use three region model:



1.) Disk region for $\tau < \tau_d$ **when** $W(\tau) = W_+(\tau)$,

2.) Transition region for $\tau_d < \tau < \tau_j$ **when** $\beta = 0$,

3.) Jet region for $\tau > \tau_j$ **when** $W(\tau) = W_-(\tau)$

Transition region **- realizability condition allows any values of** $W(\tau)$**.** ***Flow is ballistic.***

Disk-Jet Structure Solutions – Properties, Realizability

Estimated Accretion speed of the flow in the disk region:

$$V_{\rm acc} = \left(V_{rD}^2 + V_{zD}^2\right)^{1/2} = \frac{2}{5}\frac{\tau^2}{(1+\tau^2)^{1/2}}\,\beta V_{\rm Kep}\ , \tag{43}$$

Estimated ejection velocity in the jet region:

$$V_{\rm ej} = \left(V_{rJ}^2 + V_{zJ}^2\right)^{1/2} = \frac{5}{2}\left(1+\tau^2\right)^{1/2}\ \frac{V_{\rm Kep}}{\beta}\ . \tag{44}$$

Thus, **in the low β limit derived solution of disk-jet structure corresponds to slowly accreting flow in the disk ($V_{acc} << V_{Kep}$) with fast outflow in the jet region ($V_{ej} >> V_{Kep}$), matching the properties of astrophysical accretion-ejection flows.**

Notice, **that above expressions for local accretion flows in the disk and local outflows in the jet do not depend on the explicit profile of τ-dependent part of density.**

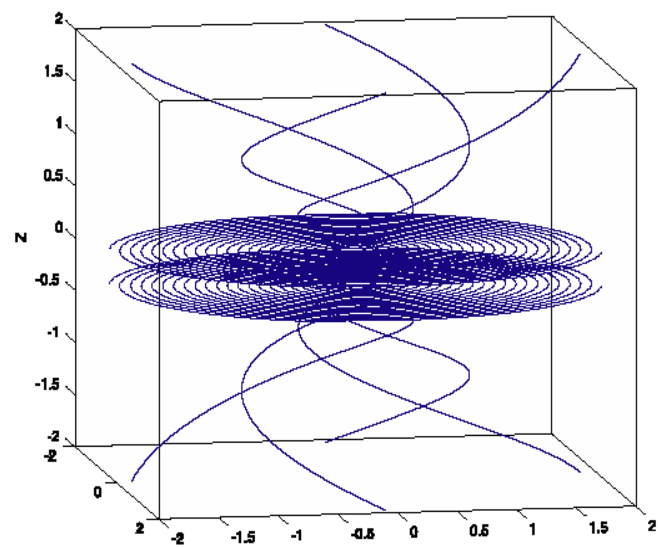
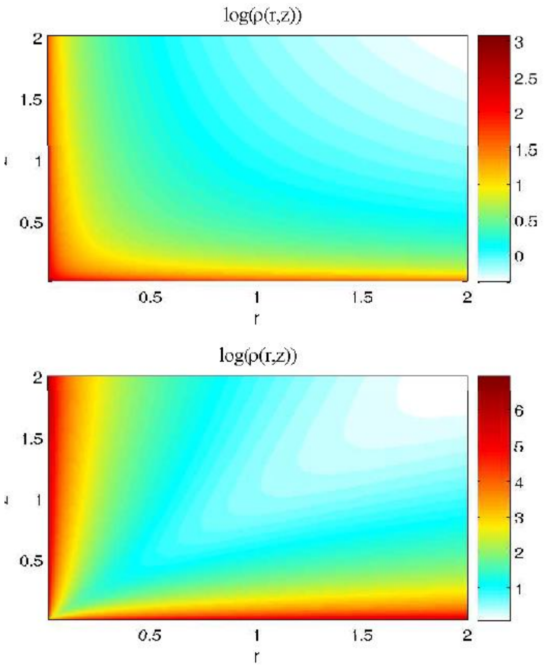


Figure shows the velocity streamlines of the disk-jet structure illustrating accretion-ejection flow at $\tau_d = 1$, $\tau_j = 2$, $\tau_0 = 0.01$; $\beta = 0.02$.

Decrease of β leads to the increase of ratio between the vertical & radial velocities and, consequently, change of the flow geometry.

Disk-Jet Structure Solutions – Properties, Realizability



To close the system of the disk-jet solutions we need to define angular distribution of the density $\rho_2(\tau)$ that will be used to obtain different classes of disk-jet structures. For this purpose we employ the power low distribution (cf. Shatashvili & Yoshida 2011):

$$\rho_2(\tau) = A_{\rm d}(\tau + \tau_0)^{m_{\rm d}} + A_{\rm j}(\tau + \tau_0)^{m_{\rm j}} , \qquad \textbf{(45)}$$

The small parameter $\tau_0 << 1$ is used to avoid the divergence at $\tau = 0$.

Figure shows the total density $\rho(r, z)$ distribution of the disk-jet structure for:

$A_d = 1,\ A_j = 1,\ m_d = -1,\ m_j = 1$ (top) *and* $A_d = 3,\ A_j = 3,\ m_d = -3,\ m_j = 3$ (bottom).

In all cases $\tau_0 = 0.01$. The topology of the density distribution is set by the disk ($m_d < 0$) and the jet ($m_j > 0$) power indices; for $\rho_2(\tau)$ the power-law distribution (45) was used.

Hence, Eqs. (39) and (45) describe classes of disk-jet solutions within our self-similar Beltrami flow model.

Linkage with Observational Properties of YSOs

The purpose of the current study is to find the analytical solutions constructing a reliable modal for disk-jet structure formation that describe basic properties of hydrodynamic jet outflows from YSOs.

Solutions derived in the paper depend on number of parameters. **We evaluated the possibility of linking these parameters with observational properties of YSOs.**

$$\beta^2 = \frac{V_{\rm acc}}{V_{\rm ej}} \,. \qquad (46)$$

Eqs. (43) and (44) reveal the **link of the parameter with kinematic properties of disk-jet structures:**

The value of parameter is constrained by the α_0 **parameter that describes anomalous viscosity due to background stationary turbulence:** $\beta < \alpha_0$. Using a typical value from observational luminosity $\alpha_0 \sim 0.01$ we can get:

$$V_{\rm ej} > 10^4 V_{\rm acc} \,. \qquad (47)$$

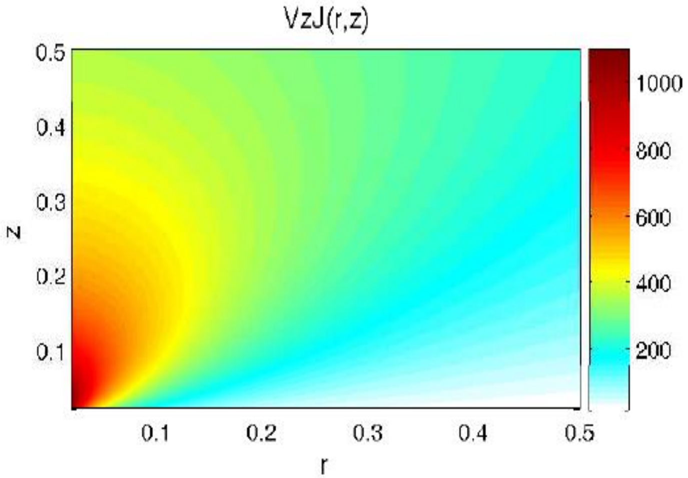
Specific value of the parameter can be inferred from observations, where both the radial accretion and the vertical ejection velocities near the central object can be observed..

Linkage with Observational Properties of YSOs - 2

One of the major properties/features of the astrophysical disk-jet flows is their narrow high velocity vertical jets.

To illustrate the outflow properties of our solutions we plot the vertical velocity distribution of the jet flow. The outflow velocity is maximal at the basis of the jet-flow (launching), while the vertical flow velocity decreases with radial distance, similar to the Keplerian profile $(\propto (\sigma/\sigma_0)^{-1/2})$. **===>**

Our Solutions can describe the formation of the disk-jet structure **(Shatashvili & Yoshida 2011), while the effects of the formed jet acceleration and collimation occur in the vertical outflows farther out from the central object and requires more general dynamical model including the heating/cooling processes.**



Vertical jet velocity of the jet solution $V_{zJ}(r, z) /(\sigma_0\Omega_0)$. Here $\beta=0.01$ and $\sigma_0 = 1$. *Maximal velocity of the outflow is reached near the vertical axis above the disk plane, at the top edge of transition region.*

Linkage with Observational Properties of YSOs - 3

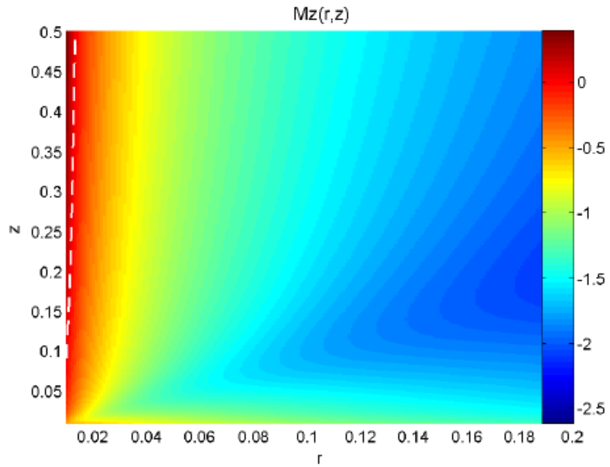
The standard mechanism of the jet acceleration through purely hydrodynamic mechanism is the Laval nozzle, when adiabatic expansion of the supersonic flow leads to its acceleration. To estimate the feasibility of this mechanism for the derived in this model disk-jet structure we calculate the local Mach number of the vertical jet flow defined as follows:

$$M_z = \frac{V_{zJ}}{[(\mathcal{P}_0 + p)/\rho]^{1/2}} \ . \qquad (48)$$

Note that pressure variation induced by disk-jet solutions is negative in the jet region ($p < 0$ when $W\_(\tau) < 0$). Hence, swirling solution in the jet region leads to the decrease of the pressure and corresponding sound speed, thus increasing the local Mach number of the flow. Background pressure can be normalized on the pressure of the self-similar solution at $\sigma = \sigma_0$ and $\tau = \tau_j$:

$$p_0 \approx \frac{5}{2}\frac{\sigma_0^{1/2}\Omega_0^2}{\beta^2} A_j \tau_j^{m_j} \ ,$$

assuming that $\tau_0 << \tau_j$.



Vertical Mach Number of jet flow:
Vertical dashed line shows the area where Mach number reveals supersonic flow $M_z > 1$

$\mathcal{P}_0 / p_0 = 10^5$

Outflow Mach Number

The decrease of the vertical outflow velocity coincides with simultaneous decrease of the sound speed, thus, leading to the possible increase of the local Mach number. In the considered extreme limit the flow can reach supersonic velocities in the narrow jet region at $\tau >> 1$. The increase of the background pressure would decrease the local Mach number to the subsonic values. At wider polar angles $(\tau > 1)$ vertical flow is subsonic and thus should be decelerating away from the central object

$$V_z(\sigma) \propto \sigma^{-1/2} .$$

Indeed, such wide-angle rotating outflows near the central object together with narrow collimated jets are consistent with ALMA observations of the HH objects (Arce et al., 2013).

Moreover, the outer parabolic shape of the wide-angle outflow near the central object seen in color gradients illustrated in Figure for Mach Number is observed recently for the Class 0protostellar system by Lee et al. (2018).

In realistic disk-jet systems jet flow streaming away from the central object is likely to undergo cooling (effect not considered in our model) that will further reduce the local sound speed and may render the outflow velocity to become supersonic. In this case farther adiabatic expansion will lead to the jet acceleration – property inherent to the protostellar disk outflows.

Discussions and Conclusions

Thus, **solution derived in present study using Beltrami flow configuration can successfully mimic slow radial sub-Keplerian accretion flow in the disk region and fast narrow super-Keplerian outflow in the jet region. The decrease of the parameter β leads to the increase of the ratio between the vertical and radial velocities and, consequently, change of the flow geometry**.

It seems that **our disk-jet structure depends on the thermal properties of the disk flow.**

Local Mach number of the outflow depends on the background pressure in the jet area. At low pressure, i.e., low temperature values jet outflow may reach supersonic amplitudes close to the central axis of the flow at high poloidal angles. At higher values of the temperature the outflow is subsonic. Considered disk-jet solution describes the formation of the high velocity outflow from slowly accreting Keplerian flow. The further kinematic acceleration and collimation of the jet flow may be due to the effects not considered in the current minimal model.

In case of YSOs, our solution shows weaker jets at the later stage of the evolution of protostar, when the temperature of the central object and corresponding disk matter increases.

Summary and Future Perspectives

We constructed the analytic configuration of the hydrodynamic jet from Young Stellar Objects (YSOs) using the Beltrami-Bernoulli flow model for disk-jet structure formation.

For this purpose we used the extended turbulent viscosity model and derived several classes of analytic solutions using the flow parametrization in the self-similar variables.

Derived solution describes the disk-jet structure flow with **jet properties linked to the properties of the accretion disk flow.** Ratio of the disk accretion and jet ejection velocities is controlled by the turbulence parameter (9,11,35), while the ejection velocity increases with the decrease of local sound velocity and the jet launching radius.

Derived solutions can be used to analyze the astrophysical jets from YSOs and link the properties of the outflows with the local conditions at the inner edge of the accretion disk flows.

Summary and Future Perspectives

Disk-jet solutions derived in the present study describe the astrophysical disk-jet structures with low ionization, where the main energy source of outflow should come from non-magnetic processes. These should include hydrodynamic jets from protostellar accretion disks and YSOs in general.

Our analytic model links the accretion and ejection rates of the flow, thus allowing to propose the specific predictions for observed structures.

We believe, that additional effects of magnetic fields (self-consistently generated or advected, or their combination) will make the solution of the problem only richer; the consequent problems of jet acceleration and heating could be also discussed then. Due to Hall effect the generalized magneto-Bernoulli mechanism may effectively accelerate the flow.

THANK YOU !