Anomalies: New Structures and Applications

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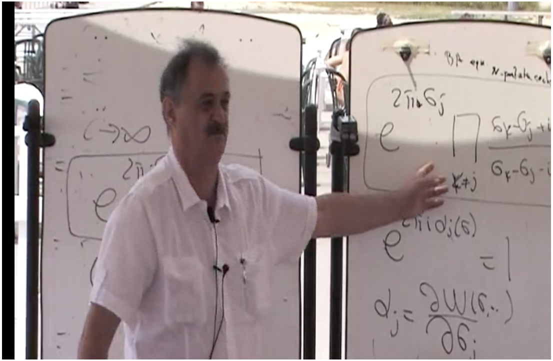


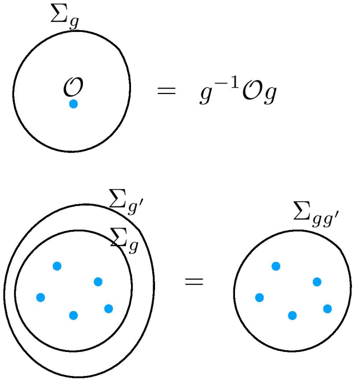
It's been a pleasure to get to know Samson over the last couple of years. I learned some very nice physics from him. Looking forward to many more productive years! Happy Birthday.

The presentation is based mostly on

- arXiv:1703.00501 (with Gaiotto, Kapustin, Seiberg)
- arXiv:1710.03258 (with Gomis and Seiberg)
- arXiv:1802.10130 (with Bashmakov, Gomis, Sharon)
- In Preparation (with Ohmori, Roumpedakis, Seifnashri)

An abstract way to think about Noether's theorem is as follows: Suppose we have a theory with symmetry group G then for every group element $g \in G$ we have a co-dimension 1 topological surface

$$\Sigma_g$$



Σ_g
$\mathcal{O}$
$= \; g^{-1}\mathcal{O}g$
$\Sigma_{g'}$
Σ_g
$=$
$\Sigma_{gg'}$

We also have to think about more complicated objects such as junctions and intersections of such surfaces. For continuous groups Noether's theorem allows us to write

$$\Sigma_g = e^{i\alpha^a \int_\Sigma \star j^a}$$

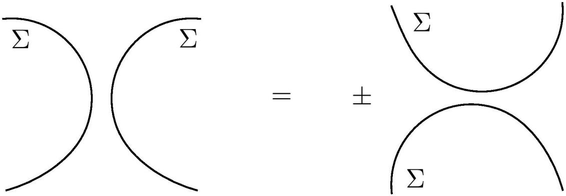
We then study the junctions and intersections by correlation functions of the currents

$$\langle jjj \ldots j \rangle \ .$$

An important phenomenon is that of a 't Hooft anomaly. In this abstract language, for instance in $d = 2$ for a $\mathbb{Z}_2$ symmetry, this is the statement that the splitting-reconnection move is nontrivial.

In the continuous case the anomaly can be also stated in terms of properties of correlation functions $\langle jjj...j \rangle$, which is how we usually learn the subject.

$$\mathbb{Z}_2 = \{1, \Sigma\}$$



$'+'$ corresponds to an ordinary non-anomalous $\mathbb{Z}_2$ symmetry in 2d and $'-'$ corresponds to an anomalous $\mathbb{Z}_2$ symmetry. [Lin,Shao]

Indeed in the case ' $-$ ' we cannot gauge the symmetry since gauging the symmetry in particular means we have to make the cross configuration transparent, but it's ambiguous.

Anomalies in higher dimensions are similarly classified by various phases relating possible configurations of the symmetry operators. This is completely general and applies equally to continuous and discrete groups.

- In the continuous case the anomalies appear only in theories with massless particles in even space-time dimensions.
- In the discrete case it can appear in all space-time dimensions (including in QM) and fermions are not needed!
- In the continuous case it was realized early on that the anomalies are related to group co-cycles (Wess-Zumino, Faddeev-Shatashvili...) but that actually extends to the discrete case. For instance, in 2d, the anomalies are classified by $H^3(G; U(1))$. More generally, cobordism groups appear [Kapustin et al.].

What are the implications of the discrete anomalies? In the continuous case in even space-time dimensions we know that there must be some massless particles (Banks et al., Coleman-Grossman). In the case of discrete anomalies, the anomalies must be still matched but they can matched by massless particles, by discrete symmetry breaking or, sometimes, by a symmetric vacuum with a TFT (but see [Cordova-Ohmori]).

For example our 2d $\mathbb{Z}_2$ anomaly implies either massless particles or symmetry breaking.

This story is very clean and rather well understood. It admits generalizations in several directions.

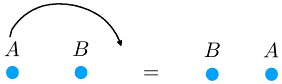
- co-dimension p topological operators.
- non-invertible topological operators.

For the usual co-dimension 1 symmetry defects we have $\Sigma_g \Sigma_{g^{-1}} = 1$. But in general there could be topological operators which have no inverse. In addition, there could be topological operators of higher co-dimension which may or may not be invertible.

Both of these generalizations are very useful in analyzing gauge theory dynamics. The notion of anomaly is extendable to such defects and one can prove anomaly matching theorems. There is also a generalization of notions such as gauging, symmetry breaking etc. Also, the formalism of anomaly inflow is very useful.

When we speak of invertible topological co-dimension 2 operators this is called "one-form symmetry."

[Gaiotto-Kapustin-Seiberg-Willett] The corresponding topological operators form an Abelian group.



$$AB = BA$$

Many familiar gauge theories have such a one-form symmetry.

- A co-dimension 2 topological operator can wrap a line and in this way give a new line:

$$\Sigma^{d-2} L = g^{-1} L g \ .$$

- Therefore, if the gauge group has a center which does not act on the matter fields the center will be a subgroup of the one-form symmetry group of the theory.
- The one-form symmetry group may be larger than the center of the gauge group.
- We can couple the co-dimension 2 surfaces to two-form gauge fields B_{mn} in the same way that we couple ordinary currents to one-form gauge fields.

Consider pure Yang-Mills theory in 4 dimensions

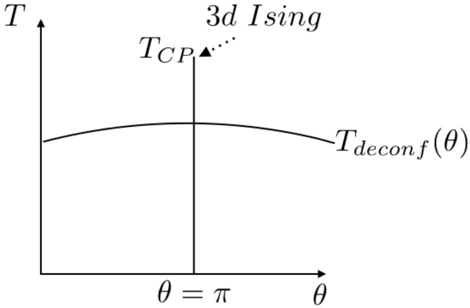
$$\mathcal{L} = \frac{1}{4g^2} F \wedge \star F + i\frac{\theta}{8\pi^2} F \wedge F$$

Since $\int_M F \wedge F \in 8\pi^2\mathbb{Z}$ for any closed, spin M, we have that θ and $\theta + 2\pi\mathbb{Z}$ describe the very same theory.

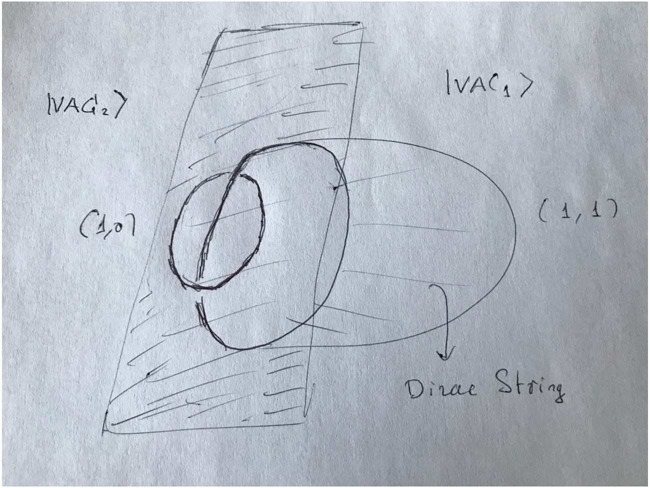
The one-form symmetry is $\mathbb{Z}_N$. In addition at $\theta = 0, \pi$ there is time-reversal symmetry since θ and $-\theta$ are the same theory. Turns out there is an anomaly at $\theta = \pi$ involving these two symmetries.

At $\theta = 0$ we expect confinement (unbroken one-form symmetry) and a trivially gapped spectrum at low temperatures and deconfinement (broken one-form symmetry) at high temperatures. At $\theta = \pi$ the confined phase cannot be trivial! and it has to remain nontrivial all the way at least until we de-confine.

$$T_{CP} \geq T_{deconf}(\theta = \pi)$$



For ordinary symmetries, in the spontaneously broken phase, we are familiar with the restrictions on pion couplings that stem from the anomaly ($KK \to \pi\pi\pi$). Here we have spontaneous breaking of time reversal at $\theta = \pi$, and one similarly finds an interesting constraint on the domain wall physics!


|VAC₂⟩
|VAC₁⟩
(1,0)
(1,1)
Dirac String

A way to match the one-form symmetry anomaly inflow to the wall is to declare that the wall supports

$$SU(N)_1$$

TFT. This means that on the wall, the quarks are deconfined and furthermore the fundamental quark becomes an anyon with

$$spin = \frac{1}{2N} \ .$$

Let us now consider another example with interesting dynamics that is constrained by one-form symmetry. We are now in 3 dimensions and we consider QCD with an adjoint Majorana fermion

$$\mathcal{L} = \frac{1}{4g^2} F \wedge \star F + i\Psi^T \gamma^\mu D_\mu \Psi + M\Psi^T \Psi \ .$$

Particularly interesting is the massless theory, $M = 0$, which has $\mathcal{N} = 1$ supersymmetry. It is the minimally supersymmetric Yang Mills theory in 3d.

We take the gauge group to be $SU(N)$ with even N. (When we write ^{T} we include multiplication by γ^0.)

The Witten index is $I_{Witten} = 0$. One would therefore think that there is a massless uncharged Goldstino particle in the infrared λ_α but otherwise the theory confines.

This naive expectation about 3d SYM is met with some difficulties.

- The $M = 0$ theory has time reversal symmetry such that $T^2 = (-1)^F$. There is an anomaly for the time reversal symmetry that is measured in $\mathbb{Z}_{16}$ [Review by Witten]. The theory at hand has

$$N^2 - 1 \mod 16$$

while the Goldstino only contributes $1 \mod 16$.

- There is a $\mathbb{Z}_N$ one-form symmetry with anomaly

$$\frac{N}{2} \mod N \; .$$

A single Majorana fermion and a confined vacuum cannot match it. In fact, the theory cannot be confined!!

There's a simple conjecture about the infrared dynamics that fixes both problems simultaneously. We postulate that

$$SU(N) + \Psi \longrightarrow \lambda \otimes U(N/2)_{N/2,N} \ .$$

Above

$$U(N/2)_{N/2,N} = \frac{SU(N/2)_{N/2} \times U(1)_{N^2/2}}{\mathbb{Z}_{N/2}}$$

The proposal looks weird at first since the gauge theory has T and the infrared has a Chern-Simons theory, which violates T. But in fact this special TFT has an isomorphism:

$$U(N/2)_{N/2,N} \simeq U(N/2)_{-N/2,-N} \ .$$

This proposal implies deconfinement and simultaneously saves all the anomalies and symmetries. It has connections to domain wall physics in four dimensions, it has generalizations to other gauge groups and other matter content, and it leads to some interesting fermion-fermion dualities which we will not discuss here.

We have so far seen applications of higher symmetries and anomalies involving two-form gauge fields. Now let us discuss the topic of non-invertible topological operators.

A very educational example is to study the adjoint 2d model:

$$\mathcal{L} = \frac{1}{4g^2} F \wedge \star F + i\Psi^T \gamma^\mu D_\mu \Psi + M\Psi^T \Psi \ .$$

This model again has a one-form symmetry $\mathbb{Z}_N$. This corresponds to some topological local operators $\mathcal{O}_i(x)$, $i = 1, ..., N$.
A very interesting case is the massless case $M = 0$, where an additional chiral $\mathbb{Z}_2$ symmetry is present. It acts by

$$\mathbb{Z}_2 : \Psi_+ \to -\Psi_+$$

$$\mathbb{Z}_2 : \Psi_- \to \Psi_-$$

The $\mathbb{Z}_2$ symmetry has interesting anomalies involving also the one-form symmetry.

In the 90s, this model was studied in great detail by Gross, Klebanov, Matytsin, Smilga, Hashimoto, and many others.

Essentially the main point in the paper of [Gross-Klebanov-Matytsin-Smilga] is that in the corresponding $PSU(N)$ gauge theory the θ angle disappears due to a chiral rotation and one finds that all the Wilson loops are deconfined. This is incorrect as we only have a chiral $\mathbb{Z}_2$ and what we may do at most is $\theta \to \theta + N/2 \mod N$ for even N but we cannot remove the theta angle completely (except for $SU(2)$ gauge theory, where their argument holds).

This mistake was recently pointed out in [Cherman-Jacobson-Tanizaki-Ünsal]. The minimal scenario that saturates the obvious anomalies is therefore confinement of the fundamental Wilson line, but deconfinement of

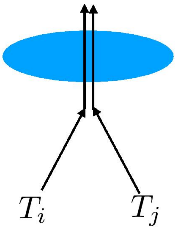
$$W^{N/2} .$$

In addition, the $\mathbb{Z}_2$ symmetry is spontaneously broken.

This might seem plausible at first but additional constraints arise from non-invertible topological lines. The theory of $N^2 - 1$ free fermions has very many topological line operators (in fact they were not classified) and some fraction of them survives gauging $SU(N)$. See also [Bachas-Monnier].

Given a set of topological lines T_i we obtain a *fusion category* which in particular has some fusion rules

$$T_i T_j = \sum F_{ij}^k T_k$$


T_i
T_j

If the lines are topological then the fusion category must be present also in the infrared. This is due to a rigidity theorem: the fusion categories do not have continuous parameters. See [Chang-Lin-Shao-Wang-Yin] for many nice examples.

A simple scenario as the one we mentioned above therefore cannot possibly capture the full story.
This fusion category **conjecturally** forces the theory to have an exponentially large number of vacua $\sim e^N$. The splittings between these vacua on a circle of radius R are of the order e^{-gR}. All the Wilson lines

$$1, W, W^2 ..., W^{N-1}$$

flow to some such topological lines in the infrared and hence they are all deconfined. The topological theory in the infrared can be explicitly identified with $(Spin(N)_1/SU(N)_N)/\sim$. The $\sim$ stands for an additional "quantum" $\mathbb{Z}_2$ quotient.
This TFT is not the same as the $SU(N)_N/SU(N)_N$ theory.

We see that the non-invertible lines lead to a surprising constraint on the infrared physics and to deconfinement of massless 2d adjoint QCD. The exponentially many vacua lead to a vanishing Hagedorn temperature at infinite N. As soon as we turn on $M \neq 0$ the topological lines conjecturally disappear and we remain with one confined vacuum.

Conclusions and Homework for Samson

The notion of symmetry should be generally extended to include co-dimension p topological surfaces which are not necessarily invertible. We saw several examples of recent applications of these ideas. What is the mathematics that governs this most general setup? Can we classify the possibilities? In two dimensions the answer is given by fusion categories but in higher dimensions the most general axioms are not laid out.

Conclusions and Homework for Samson

Anomalies for ordinary symmetries are governed by group co-cycles. For higher symmetries it is a little more complicated. For systems with non-invertible lines it is yet more complicated. In all cases though it boils down to the rigidity of the corresponding structures (i.e. quantization of some coefficients).

What is the anomaly inflow formalism for general non-invertible lines? (interesting ideas in [Thorngren-Wang]) Is there a $d+2$ dimensional analog of an anomaly polynomial?

Conclusions and Homework for Samson

We described some tantalizing facts about 2d Adjoint QCD. How about the (1,1) supersymmetric version of the theory? How about adding quartic fermion interactions? Is the model integrable at $M=0$? Large N? Are there new dualities such as those in 3d?

Thank You!
And Happy Birthday Again!