Fokker-Planck operators and the center of the enveloping algebra

Jean-Michel Bismut

Institut de Mathématique d'Orsay

To Samson

Integrability, Anomalies and Quantum Field Theory

The Euler characteristic

The Euler characteristic

- X compact Riemannian manifold.

The Euler characteristic

- X compact Riemannian manifold.
- Euler characteristic $\chi(X) = \sum (-1)^i \dim H^i(X, \mathbf{R})$.

The Euler characteristic

- X compact Riemannian manifold.
- Euler characteristic $\chi(X) = \sum (-1)^i \dim H^i(X, \mathbf{R})$.
- $g \in \mathrm{Diff}(X)$ acts on $H^\cdot(X, \mathbf{R})$.

The Euler characteristic

- X compact Riemannian manifold.
- Euler characteristic $\chi(X) = \sum (-1)^i \dim H^i(X, \mathbf{R})$.
- $g \in \mathrm{Diff}(X)$ acts on $H^\cdot(X, \mathbf{R})$.
- Lefschetz number $L(g) = \mathrm{Tr_s}^{H^\cdot(X,\mathbf{R})}[g]$.

The heat equation method

The heat equation method

- $\left(\Omega^{\cdot}\left(X, \mathbf{R}\right), d^X\right)$ de Rham complex, whose cohomology is $H^{\cdot}\left(X, \mathbf{R}\right)$.

The heat equation method

- $\left(\Omega^{\cdot}\left(X, \mathbf{R}\right), d^X\right)$ de Rham complex, whose cohomology is $H^{\cdot}\left(X, \mathbf{R}\right)$.
- g^{TX} Riemannian metric, d^{X*} formal adjoint of d^X, $D^X = d^X + d^{X*}$ Dirac operator.

The heat equation method

- $\left(\Omega^{\cdot}(X,\mathbf{R}), d^X\right)$ de Rham complex, whose cohomology is $H^{\cdot}(X,\mathbf{R})$.
- g^{TX} Riemannian metric, d^{X*} formal adjoint of d^X, $D^X = d^X + d^{X*}$ Dirac operator.
- $D^{X,2} = \left[d^X, d^{X*}\right]$ Hodge Laplacian.

The heat equation method

- $\left(\Omega^{\cdot}\left(X,\mathbf{R}\right),d^{X}\right)$ de Rham complex, whose cohomology is $H^{\cdot}\left(X,\mathbf{R}\right)$.
- g^{TX} Riemannian metric, d^{X*} formal adjoint of d^{X}, $D^{X}=d^{X}+d^{X*}$ Dirac operator.
- $D^{X,2}=\left[d^{X},d^{X*}\right]$ Hodge Laplacian.
- McKean-Singer: For $s>0$, $L\left(g\right)=\mathrm{Tr_s}\left[g\exp\left(-sD^{X,2}\right)\right]$ (does not depend on $s>0$).

The heat equation method

- $\left(\Omega^{\cdot}\left(X,\mathbf{R}\right),d^{X}\right)$ de Rham complex, whose cohomology is $H^{\cdot}\left(X,\mathbf{R}\right)$.
- g^{TX} Riemannian metric, d^{X*} formal adjoint of d^{X}, $D^{X}=d^{X}+d^{X*}$ Dirac operator.
- $D^{X,2}=\left[d^{X},d^{X*}\right]$ Hodge Laplacian.
- McKean-Singer: For $s>0$, $L\left(g\right)=\operatorname{Tr_s}\left[g\exp\left(-sD^{X,2}\right)\right]$ (does not depend on $s>0$).
- As $s\to0$, Lefschetz fixed point formula

$$L\left(g\right)=\int_{X_g}\pm e\left(TX_g\right).$$

The heat operator of X

The heat operator of X

- X compact Riemannian manifold.

The heat operator of X

- X compact Riemannian manifold.
- Δ^X Laplacian on X.

The heat operator of X

- X compact Riemannian manifold.
- Δ^X Laplacian on X.
- For $t > 0$, $g = \exp\left(t\Delta^X/2\right)$ heat operator acting on $C^\infty(X, \mathbf{R})$.

Four questions

Four questions

1. Is $\mathrm{Tr}^{C^\infty(X,\mathbf{R})}[g]$ an Euler characteristic?

Four questions

1. Is $\mathrm{Tr}^{C^\infty(X,\mathbf{R})}[g]$ an Euler characteristic?
2. Can I write a formula of the type

Four questions

1. Is $\mathrm{Tr}^{C^\infty(X,\mathbf{R})}[g]$ an Euler characteristic?
2. Can I write a formula of the type

$$\mathrm{Tr}^{C^\infty(X,\mathbf{R})}[g] = \mathrm{Tr_s}^R\left[g\exp\left(-D_{R,b}^2/2\right)\right].$$

Four questions

1. Is $\mathrm{Tr}^{C^\infty(X,\mathbf{R})}[g]$ an Euler characteristic?
2. Can I write a formula of the type

$$\mathrm{Tr}^{C^\infty(X,\mathbf{R})}[g] = \mathrm{Tr_s}^R\left[g\exp\left(-D^2_{R,b}/2\right)\right].$$

3. By making $b\to+\infty$, do we obtain Selberg's trace formula ? (extension of Poisson formula).

Four questions

1. Is $\mathrm{Tr}^{C^\infty(X,\mathbf{R})}[g]$ an Euler characteristic?
2. Can I write a formula of the type

$$\mathrm{Tr}^{C^\infty(X,\mathbf{R})}[g] = \mathrm{Tr_s}^R\left[g\exp\left(-D^2_{R,b}/2\right)\right].$$

3. By making $b \to +\infty$, do we obtain Selberg's trace formula ? (extension of Poisson formula).
4. Is Selberg trace formula a Lefschetz formula?

The analogy

The analogy

$$\underbrace{L(g)\,|_{s=+\infty}}_{\text{global}} \xrightarrow{\mathrm{Tr_s}\left[g\exp\left(-sD^{X,2}\right)\right]|_{s>0}} \underbrace{\text{Fixed point formula}}_{\text{local}}|_{s=0}.$$

The analogy

$$\underbrace{L(g)\,|_{s=+\infty}}_{\text{global}} \xrightarrow{\operatorname{Tr_s}\left[g\exp\left(-sD^{X,2}\right)\right]|_{s>0}} \underbrace{\text{Fixed point formula}}_{\text{local}}|_{s=0}.$$

$$\underbrace{\operatorname{Tr}[g]_{b=0}}_{\text{global}} \xrightarrow{\operatorname{Tr_s}\left[g\exp\left(-D_b^{R,2}\right)\right]|_{b>0}} \underbrace{\text{Selberg t.f.}|_{b=+\infty}}_{\text{local via closed geodesics}}.$$

Selberg's trace formula

Selberg's trace formula

- X Riemann surface of constant scalar curvature -2, l_γ length of closed geodesics γ.

Selberg's trace formula

- X Riemann surface of constant scalar curvature -2, l_γ length of closed geodesics γ.
-

$$\underbrace{\operatorname{Tr}\left[\exp\left(t\Delta^X/2\right)\right]}_{\text{Laplacian}} = \underbrace{\frac{\exp\left(-t/8\right)}{2\pi t}\operatorname{Vol}\left(\Sigma\right)}_{\text{geodesic flow}}$$

Selberg's trace formula

- X Riemann surface of constant scalar curvature -2, l_γ length of closed geodesics γ.
-

$$\underbrace{\operatorname{Tr}\left[\exp\left(t\Delta^X/2\right)\right]}_{\text{Laplacian}} = \underbrace{\frac{\exp\left(-t/8\right)}{2\pi t}\operatorname{Vol}\left(\Sigma\right)}_{\text{geodesic flow}}$$

$$\int_{\mathbf{R}} \exp\left(-y^2/2t\right)\frac{y/2}{\sinh\left(y/2\right)}\frac{dy}{\sqrt{2\pi t}}$$

Selberg's trace formula

- X Riemann surface of constant scalar curvature -2, l_γ length of closed geodesics γ.
- $$\underbrace{\mathrm{Tr}\left[\exp\left(t\Delta^X/2\right)\right]}_{\text{Laplacian}} = \underbrace{\frac{\exp\left(-t/8\right)}{2\pi t}\mathrm{Vol}\left(\Sigma\right)}_{\text{geodesic flow}}$$
$$\int_{\mathbf{R}} \exp\left(-y^2/2t\right)\frac{y/2}{\sinh\left(y/2\right)}\frac{dy}{\sqrt{2\pi t}}$$
$$+\sum_{\gamma\neq 0}\frac{\mathrm{Vol}_\gamma}{\sqrt{2\pi t}}\frac{\exp\left(-\ell_\gamma^2/2t - t/8\right)}{2\sinh\left(\ell_\gamma/2\right)}.$$

Selberg's trace formula

- X Riemann surface of constant scalar curvature -2, l_γ length of closed geodesics γ.
- $$\underbrace{\operatorname{Tr}\left[\exp\left(t\Delta^X/2\right)\right]}_{\text{Laplacian}} = \underbrace{\frac{\exp\left(-t/8\right)}{2\pi t}\operatorname{Vol}\left(\Sigma\right)}_{\text{geodesic flow}}$$
$$\int_{\mathbf{R}} \exp\left(-y^2/2t\right)\frac{y/2}{\sinh\left(y/2\right)}\frac{dy}{\sqrt{2\pi t}}$$
$$+\sum_{\gamma\neq 0}\frac{\operatorname{Vol}_\gamma}{\sqrt{2\pi t}}\frac{\exp\left(-\ell_\gamma^2/2t - t/8\right)}{2\sinh\left(\ell_\gamma/2\right)}.$$
- Right-hand side orbital integrals.

The symmetric space X

The symmetric space X

- G real reductive group, K maximal compact subgroup, $X = G/K$ symmetric space.

The symmetric space X

- G real reductive group, K maximal compact subgroup, $X = G/K$ symmetric space.
- $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ Cartan splitting equipped with bilinear form $B > 0$ on $\mathfrak{p}$, < 0 on $\mathfrak{k}$ ...

The symmetric space X

- G real reductive group, K maximal compact subgroup, $X = G/K$ symmetric space.
- $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ Cartan splitting equipped with bilinear form $B > 0$ on $\mathfrak{p}$, < 0 on $\mathfrak{k}$ ...
- ... descends to bundle of Lie algebras $TX \oplus N$ on X.

The symmetric space X

- G real reductive group, K maximal compact subgroup, $X = G/K$ symmetric space.
- $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ Cartan splitting equipped with bilinear form $B > 0$ on $\mathfrak{p}$, < 0 on $\mathfrak{k}$ ...
- ... descends to bundle of Lie algebras $TX \oplus N$ on X.

Example

$G = \mathrm{SL}_2(\mathbf{R})$, $K = S^1$, X upper half-plane, $TX \oplus N$ of dimension 3.

The analysis on $G \times_K \mathfrak{g}$

The analysis on $G \times_K \mathfrak{g}$

- The analysis will be done on $G \times_K \mathfrak{g}$...

The analysis on $G \times_K \mathfrak{g}$

- The analysis will be done on $G \times_K \mathfrak{g}$...
- ...which is the total space $\widehat{\mathcal{X}}$ of $TX \oplus N$ over $X = G/K$.

The analysis on $G \times_K \mathfrak{g}$

- The analysis will be done on $G \times_K \mathfrak{g}$...
- ...which is the total space $\widehat{\mathcal{X}}$ of $TX \oplus N$ over $X = G/K$.
- Two separate constructions on G and on $\mathfrak{g}$.

Casimir and Kostant

Casimir and Kostant

- $C^{\mathfrak{g}} = -\sum e_i^* e_i$ Casimir (differential operator on G), positive on $\mathfrak{p}$, negative on $\mathfrak{k}$.

Casimir and Kostant

- $C^{\mathfrak{g}} = -\sum e_i^* e_i$ Casimir (differential operator on G), positive on $\mathfrak{p}$, negative on $\mathfrak{k}$.
- $\widehat{c}(\mathfrak{g})$ Clifford algebra of $(\mathfrak{g}, -B)$ acts on $\Lambda^\cdot(\mathfrak{g}^*)$.

Casimir and Kostant

- $C^{\mathfrak{g}} = -\sum e_i^* e_i$ Casimir (differential operator on G), positive on $\mathfrak{p}$, negative on $\mathfrak{k}$.
- $\widehat{c}(\mathfrak{g})$ Clifford algebra of $(\mathfrak{g}, -B)$ acts on $\Lambda^{\cdot}(\mathfrak{g}^*)$.
- $U(\mathfrak{g})$ enveloping algebra (left-invariant differential operators on G).

Casimir and Kostant

- $C^{\mathfrak{g}} = -\sum e_i^* e_i$ Casimir (differential operator on G), positive on $\mathfrak{p}$, negative on $\mathfrak{k}$.
- $\widehat{c}(\mathfrak{g})$ Clifford algebra of $(\mathfrak{g}, -B)$ acts on $\Lambda^{\cdot}(\mathfrak{g}^*)$.
- $U(\mathfrak{g})$ enveloping algebra (left-invariant differential operators on G).
- $\widehat{D}^{\mathrm{Ko}} \in \widehat{c}(\mathfrak{g}) \otimes U(\mathfrak{g})$ Dirac operator of Kostant.

Casimir and Kostant

- $C^{\mathfrak{g}} = -\sum e_i^* e_i$ Casimir (differential operator on G), positive on $\mathfrak{p}$, negative on $\mathfrak{k}$.
- $\widehat{c}(\mathfrak{g})$ Clifford algebra of $(\mathfrak{g}, -B)$ acts on $\Lambda^{\cdot}(\mathfrak{g}^*)$.
- $U(\mathfrak{g})$ enveloping algebra (left-invariant differential operators on G).
- $\widehat{D}^{\mathrm{Ko}} \in \widehat{c}(\mathfrak{g}) \otimes U(\mathfrak{g})$ Dirac operator of Kostant.
- $\kappa^{\mathfrak{g}}(U, V, W) = B([U, V], W)$ closed 3-form.

Casimir and Kostant

- $C^{\mathfrak{g}} = -\sum e_i^* e_i$ Casimir (differential operator on G), positive on $\mathfrak{p}$, negative on $\mathfrak{k}$.
- $\widehat{c}(\mathfrak{g})$ Clifford algebra of $(\mathfrak{g}, -B)$ acts on $\Lambda^{\cdot}(\mathfrak{g}^*)$.
- $U(\mathfrak{g})$ enveloping algebra (left-invariant differential operators on G).
- $\widehat{D}^{\mathrm{Ko}} \in \widehat{c}(\mathfrak{g}) \otimes U(\mathfrak{g})$ Dirac operator of Kostant.
- $\kappa^{\mathfrak{g}}(U, V, W) = B([U, V], W)$ closed 3-form.
- $\widehat{D}^{\mathrm{Ko}} = \widehat{c}(e_i^*) e_i + \frac{1}{2}\widehat{c}(-\kappa^{\mathfrak{g}})$.

A formula of Kostant

A formula of Kostant

Theorem (Kostant)

$$\widehat{D}^{\mathrm{Ko},2} = -C^{\mathfrak{g}} + B^{*}\left(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}\right).$$

A formula of Kostant

Theorem (Kostant)

$$\widehat{D}^{\mathrm{Ko},2} = -C^{\mathfrak{g}} + B^{*}\left(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}\right).$$

Remark

$\widehat{D}^{\mathrm{Ko}}$ acts on $C^{\infty}\left(G, \Lambda^{\cdot}\left(\mathfrak{g}^{*}\right)\right)$, while $C^{\mathfrak{g}}$ acts on $C^{\infty}\left(G, \mathbf{R}\right)$.

Wick rotation and harmonic oscillator on $\mathfrak{g}_i$

Wick rotation and harmonic oscillator on $\mathfrak{g}_i$

- On $\mathfrak{g}_i = \mathfrak{p} \oplus i\mathfrak{k}$, $H^{\mathfrak{g}_i}$ harmonic oscillator on $\mathfrak{g}_i$.

Wick rotation and harmonic oscillator on $\mathfrak{g}_i$

- On $\mathfrak{g}_i = \mathfrak{p} \oplus i\mathfrak{k}$, $H^{\mathfrak{g}_i}$ harmonic oscillator on $\mathfrak{g}_i$.
- $H^{\mathfrak{g}_i} = \frac{1}{2}\left(-\Delta^{\mathfrak{g}_i} + |Y|^2 - n\right)$.

Wick rotation and harmonic oscillator on $\mathfrak{g}_i$

- On $\mathfrak{g}_i = \mathfrak{p} \oplus i\mathfrak{k}$, $H^{\mathfrak{g}_i}$ harmonic oscillator on $\mathfrak{g}_i$.
- $H^{\mathfrak{g}_i} = \frac{1}{2}\left(-\Delta^{\mathfrak{g}_i} + |Y|^2 - n\right)$.
- Witten operators $\underline{d}^{\mathfrak{g}_i} = d^{\mathfrak{g}_i} + Y\wedge, \underline{d}^{\mathfrak{g}_i *} = d^{\mathfrak{g}_i *} + i_Y$ acts on $C^\infty\left(\mathfrak{g}, \Lambda^{\cdot}\left(\mathfrak{g}_i^*\right)\right)$

Wick rotation and harmonic oscillator on $\mathfrak{g}_i$

- On $\mathfrak{g}_i = \mathfrak{p} \oplus i\mathfrak{k}$, $H^{\mathfrak{g}_i}$ harmonic oscillator on $\mathfrak{g}_i$.
- $H^{\mathfrak{g}_i} = \frac{1}{2}\left(-\Delta^{\mathfrak{g}_i} + |Y|^2 - n\right)$.
- Witten operators $\underline{d}^{\mathfrak{g}_i} = d^{\mathfrak{g}_i} + Y\wedge, \underline{d}^{\mathfrak{g}_i *} = d^{\mathfrak{g}_i *} + i_Y$ acts on $C^\infty\left(\mathfrak{g}, \Lambda^\cdot\left(\mathfrak{g}_i^*\right)\right)$
- $D^{\mathfrak{g}_i} = \underline{d}^{\mathfrak{g}_i} + \underline{d}^{\mathfrak{g}_i *}$ Dirac like operator on $\mathfrak{g}_i$.

Wick rotation and harmonic oscillator on $\mathfrak{g}_i$

- On $\mathfrak{g}_i = \mathfrak{p} \oplus i\mathfrak{k}$, $H^{\mathfrak{g}_i}$ harmonic oscillator on $\mathfrak{g}_i$.
- $H^{\mathfrak{g}_i} = \frac{1}{2}\left(-\Delta^{\mathfrak{g}_i} + |Y|^2 - n\right)$.
- Witten operators $\underline{d}^{\mathfrak{g}_i} = d^{\mathfrak{g}_i} + Y\wedge, \underline{d}^{\mathfrak{g}_i *} = d^{\mathfrak{g}_i *} + i_Y$ acts on $C^\infty\left(\mathfrak{g}, \Lambda^{\cdot}\left(\mathfrak{g}_i^*\right)\right)$
- $D^{\mathfrak{g}_i} = \underline{d}^{\mathfrak{g}_i} + \underline{d}^{\mathfrak{g}_i *}$ Dirac like operator on $\mathfrak{g}_i$.
- $\frac{1}{2}\left[\underline{d}^{\mathfrak{g}_i}, \underline{d}^{\mathfrak{g}_i *}\right] = H^{\mathfrak{g}_i} + N^{\Lambda^{\cdot}\left(\mathfrak{g}_i^*\right)}$.

The operator $\mathfrak{D}_b$

The operator $\mathfrak{D}_b$

- $\mathfrak{D}_b$ combination of Dirac operators on G and $\mathfrak{g}$.

The operator $\mathfrak{D}_b$

- $\mathfrak{D}_b$ combination of Dirac operators on G and $\mathfrak{g}$.
- $\mathfrak{D}_b$ acts on $C^\infty\left(G \times \mathfrak{g}, \Lambda^\cdot\left(\mathfrak{g}^*_{\mathbf{C}}\right)\right)$.

The operator $\mathfrak{D}_b$

- $\mathfrak{D}_b$ combination of Dirac operators on G and $\mathfrak{g}$.
- $\mathfrak{D}_b$ acts on $C^\infty\left(G \times \mathfrak{g}, \Lambda^{\cdot}\left(\mathfrak{g}_{\mathbf{C}}^*\right)\right)$.
- $\mathfrak{D}_b = \widehat{D}^{\mathrm{Ko}} + ic\left(\left[Y^{\mathfrak{k}}, Y^{\mathfrak{p}}\right]\right) + \frac{1}{b}\left(\underline{d}^{\mathfrak{g}_i} + \underline{d}^{\mathfrak{g}_i *}\right)$.

The operator $\mathfrak{D}_b$

- $\mathfrak{D}_b$ combination of Dirac operators on G and $\mathfrak{g}$.
- $\mathfrak{D}_b$ acts on $C^\infty\left(G \times \mathfrak{g}, \Lambda^\cdot\left(\mathfrak{g}^*_{\mathbf{C}}\right)\right)$.
- $\mathfrak{D}_b = \widehat{D}^{\mathrm{Ko}} + ic\left(\left[Y^{\mathfrak{k}}, Y^{\mathfrak{p}}\right]\right) + \frac{1}{b}\left(\underline{d}^{\mathfrak{g}_i} + \underline{d}^{\mathfrak{g}_i *}\right)$.
- $\mathfrak{D}_b$ K-invariant.

The operator $\mathfrak{D}_b$

- $\mathfrak{D}_b$ combination of Dirac operators on G and $\mathfrak{g}$.
- $\mathfrak{D}_b$ acts on $C^\infty\left(G \times \mathfrak{g}, \Lambda^\cdot\left(\mathfrak{g}^*_{\mathbf{C}}\right)\right)$.
- $\mathfrak{D}_b = \widehat{D}^{\mathrm{Ko}} + ic\left(\left[Y^{\mathfrak{k}}, Y^{\mathfrak{p}}\right]\right) + \frac{1}{b}\left(\underline{d}^{\mathfrak{g}_i} + \underline{d}^{\mathfrak{g}_i *}\right)$.
- $\mathfrak{D}_b$ K-invariant.
- The quadratic term is related to the quotienting by K.

The hypoelliptic Laplacian

The hypoelliptic Laplacian

- Set $\mathcal{L}_b = \frac{1}{2}\left(-\widehat{D}^{\mathrm{Ko},2} + \mathfrak{D}_b^2\right)$.

The hypoelliptic Laplacian

- Set $\mathcal{L}_b = \frac{1}{2}\left(-\widehat{D}^{\mathrm{Ko},2} + \mathfrak{D}_b^2\right)$.
- Quotient the construction by K.

The hypoelliptic Laplacian

- Set $\mathcal{L}_b = \frac{1}{2}\left(-\widehat{D}^{\mathrm{Ko},2} + \mathfrak{D}_b^2\right)$.
- Quotient the construction by K.
- $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ descends on X to $TX \oplus N$.

The hypoelliptic Laplacian

- Set $\mathcal{L}_b = \frac{1}{2}\left(-\widehat{D}^{\mathrm{Ko},2} + \mathfrak{D}_b^2\right)$.
- Quotient the construction by K.
- $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ descends on X to $TX \oplus N$.
- $\widehat{\pi} : \widehat{\mathcal{X}} \to X$ total space of $TX \oplus N$.

The hypoelliptic Laplacian

- Set $\mathcal{L}_b = \frac{1}{2}\left(-\widehat{D}^{\mathrm{Ko},2} + \mathfrak{D}_b^2\right)$.
- Quotient the construction by K.
- $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ descends on X to $TX \oplus N$.
- $\widehat{\pi} : \widehat{\mathcal{X}} \to X$ total space of $TX \oplus N$.
- $\mathcal{L}_b^X = \frac{1}{2}\left(-\widehat{D}^{\mathrm{Ko},2} + \mathfrak{D}_b^{X,2}\right)$ acts on

$$C^\infty\left(\widehat{\mathcal{X}}, \widehat{\pi}^* \Lambda^\cdot \left(T^*X \oplus N^*\right)\right).$$

The hypoelliptic Laplacian

- Set $\mathcal{L}_b = \frac{1}{2}\left(-\widehat{D}^{\mathrm{Ko},2} + \mathfrak{D}_b^2\right)$.
- Quotient the construction by K.
- $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ descends on X to $TX \oplus N$.
- $\widehat{\pi} : \widehat{\mathcal{X}} \to X$ total space of $TX \oplus N$.
- $\mathcal{L}_b^X = \frac{1}{2}\left(-\widehat{D}^{\mathrm{Ko},2} + \mathfrak{D}_b^{X,2}\right)$ acts on

$$C^\infty\left(\widehat{\mathcal{X}}, \widehat{\pi}^* \Lambda^\cdot\left(T^*X \oplus N^*\right)\right).$$

Remark

Using the fiberwise Bargmann isomorphism, $\mathcal{L}_b^X$ acts on

$$C^\infty\left(X, S^\cdot\left(T^*X \oplus N^*\right) \otimes \Lambda^\cdot\left(T^*X \oplus N^*\right)\right).$$

The hypoelliptic Laplacian as a deformation

The hypoelliptic Laplacian as a deformation

$$\mathcal{L}_b^X = \frac{1}{2}\left|\left[Y^N, Y^{TX}\right]\right|^2 + \underbrace{\frac{1}{2b^2}\left(-\Delta^{TX\oplus N} + |Y|^2 - n\right)}_{\text{Harmonic oscillator of } TX\oplus N} + \frac{N^{\Lambda^\cdot(T^*X\oplus N^*)}}{b^2}$$

$$+\frac{1}{b}\Bigg(\underbrace{\nabla_{Y^{TX}}}_{\text{geodesic flow}} + \widehat{c}\left(\mathrm{ad}\left(Y^{TX}\right)\right) - c\left(\mathrm{ad}\left(Y^{TX}\right) + i\theta\mathrm{ad}\left(Y^N\right)\right)\Bigg).$$

The hypoelliptic Laplacian as a deformation

$$\mathcal{L}_b^X = \frac{1}{2}\left|\left[Y^N, Y^{TX}\right]\right|^2 + \underbrace{\frac{1}{2b^2}\left(-\Delta^{TX\oplus N} + |Y|^2 - n\right)}_{\text{Harmonic oscillator of } TX\oplus N} + \frac{N^{\Lambda^\cdot(T^*X\oplus N^*)}}{b^2}$$

$$+\frac{1}{b}\left(\underbrace{\nabla_{Y^{TX}}}_{\text{geodesic flow}} + \widehat{c}\left(\mathrm{ad}\left(Y^{TX}\right)\right) - c\left(\mathrm{ad}\left(Y^{TX}\right) + i\theta \mathrm{ad}\left(Y^N\right)\right)\right).$$

Remark

- $\mathcal{L}_b^X$ looks like a Fokker-Planck operator.

The hypoelliptic Laplacian as a deformation

$$\mathcal{L}_b^X = \frac{1}{2}\left|\left[Y^N, Y^{TX}\right]\right|^2 + \underbrace{\frac{1}{2b^2}\left(-\Delta^{TX\oplus N} + |Y|^2 - n\right)}_{\text{Harmonic oscillator of } TX\oplus N} + \frac{N^{\Lambda^\cdot(T^*X\oplus N^*)}}{b^2}$$

$$+\frac{1}{b}\Bigg(\underbrace{\nabla_{Y^{TX}}}_{\text{geodesic flow}} + \widehat{c}\left(\mathrm{ad}\left(Y^{TX}\right)\right) - c\left(\mathrm{ad}\left(Y^{TX}\right) + i\theta\mathrm{ad}\left(Y^N\right)\right)\Bigg).$$

Remark

- $\mathcal{L}_b^X$ looks like a Fokker-Planck operator.
- $b \to 0$, $\mathcal{L}_b^X \to \frac{1}{2}\left(C^{\mathfrak{g},X} - c\right)$: $\widehat{\mathcal{X}}$ collapses to X (B. 2011)

The hypoelliptic Laplacian as a deformation

$$\mathcal{L}_b^X = \frac{1}{2}\left|\left[Y^N, Y^{TX}\right]\right|^2 + \underbrace{\frac{1}{2b^2}\left(-\Delta^{TX\oplus N} + |Y|^2 - n\right)}_{\text{Harmonic oscillator of } TX\oplus N} + \frac{N^{\Lambda^\cdot(T^*X\oplus N^*)}}{b^2}$$

$$+\frac{1}{b}\Bigg(\underbrace{\nabla_{Y^{TX}}}_{\text{geodesic flow}} + \widehat{c}\left(\mathrm{ad}\left(Y^{TX}\right)\right) - c\left(\mathrm{ad}\left(Y^{TX}\right) + i\theta\mathrm{ad}\left(Y^N\right)\right)\Bigg).$$

Remark

- $\mathcal{L}_b^X$ looks like a Fokker-Planck operator.
- $b \to 0$, $\mathcal{L}_b^X \to \frac{1}{2}\left(C^{\mathfrak{g},X} - c\right)$: $\widehat{\mathcal{X}}$ collapses to X (B. 2011)
- $b \to +\infty$, geodesic f. $\nabla_{Y^{TX}}$ dominates $\Rightarrow$ closed geodesics.

The case of locally symmetric spaces

The case of locally symmetric spaces

- $\Gamma \subset G$ cocompact torsion free.

The case of locally symmetric spaces

- $\Gamma \subset G$ cocompact torsion free.
- $Z = \Gamma \setminus X$ compact locally symmetric.

A fundamental identity

A fundamental identity

Theorem

For $t > 0, b > 0$,

$$\mathrm{Tr}^{C^\infty(Z,E)}\left[\exp\left(-t\left(C^{\mathfrak{g},Z} - c\right)/2\right)\right] = \mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b^Z\right)\right].$$

A fundamental identity

Theorem

For $t > 0, b > 0$,

$$\mathrm{Tr}^{C^{\infty}(Z,E)}\left[\exp\left(-t\left(C^{\mathfrak{g},Z} - c\right)/2\right)\right] = \mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b^Z\right)\right].$$

Remark

•This is exactly what we wanted!

A fundamental identity

Theorem

For $t > 0, b > 0$,

$$\mathrm{Tr}^{C^\infty(Z,E)}\left[\exp\left(-t\left(C^{\mathfrak{g},Z} - c\right)/2\right)\right] = \mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b^Z\right)\right].$$

Remark

•This is exactly what we wanted!
• $C^\infty(Z, \mathbf{R})$ has been replaced by a Witten complex over Z, whose cohomology is just $C^\infty(Z, \mathbf{R})$.

A fundamental identity

Theorem

For $t > 0, b > 0$,

$$\mathrm{Tr}^{C^\infty(Z,E)}\left[\exp\left(-t\left(C^{\mathfrak{g},Z} - c\right)/2\right)\right] = \mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b^Z\right)\right].$$

Remark

•This is exactly what we wanted!
• $C^\infty(Z, \mathbf{R})$ has been replaced by a Witten complex over Z, whose cohomology is just $C^\infty(Z, \mathbf{R})$.
• There is a supersymmetric interpretation involving $G_\mathbf{C}$.

Splitting the identity

Splitting the identity

1. Each side splits as an infinite sum indexed by conjugacy classes in Γ.

Splitting the identity

1. Each side splits as an infinite sum indexed by conjugacy classes in Γ.
2. The above identity splits as an identity of orbital integrals.

Semisimple orbital integrals

Semisimple orbital integrals

- $\gamma \in G$ semisimple, $[\gamma]$ conjugacy class.

Semisimple orbital integrals

- $\gamma \in G$ semisimple, $[\gamma]$ conjugacy class.
- For $t > 0$, $\mathrm{Tr}^{[\gamma]}\left[\exp\left(-t\left(C^{\mathfrak{g},X} - c\right)/2\right)\right]$ orbital integral of heat kernel on orbit of γ:

$$I\left([\gamma]\right) = \int_{Z(\gamma)\backslash G} \mathrm{Tr}^{E}\left[p_t^X\left(g^{-1}\gamma g\right)\right] dg.$$

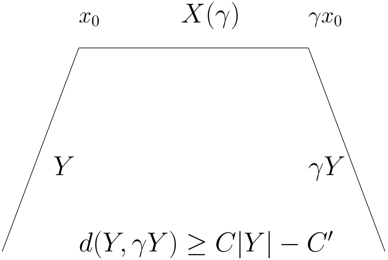
Geometric description of the orbital integral

Geometric description of the orbital integral

$$I(\gamma) = \int_{N_{X(\gamma)/X}} \mathrm{Tr}\left[\gamma p_t^X(Y, \gamma Y)\right] \underbrace{r(Y)}_{\text{Jacobian}} dY.$$

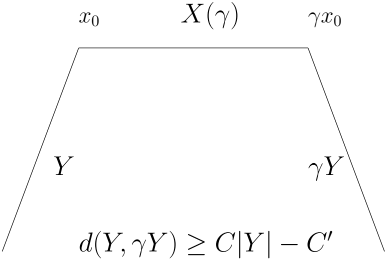
Geometric description of the orbital integral

$$I(\gamma) = \int_{N_{X(\gamma)/X}} \operatorname{Tr}\left[\gamma p_t^X (Y, \gamma Y)\right] \underbrace{r(Y)}_{\text{Jacobian}} dY.$$



Geometric description of the orbital integral

$$I(\gamma) = \int_{N_{X(\gamma)/X}} \operatorname{Tr}\left[\gamma p_t^X(Y, \gamma Y)\right] \underbrace{r(Y)}_{\text{Jacobian}} dY.$$



$$p_t^X(x, x') \leq C \exp(-C' d^2(x, x')).$$

A second fundamental identity

A second fundamental identity

Theorem (B. 2011)

For $b > 0, t > 0$,

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-t\left(C^{\mathfrak{g},X} - c\right)/2\right)\right] = \mathrm{Tr_s}^{[\gamma]}\left[\exp\left(-t\mathcal{L}_b^X\right)\right].$$

A second fundamental identity

Theorem (B. 2011)

For $b > 0, t > 0$,

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-t\left(C^{\mathfrak{g},X} - c\right)/2\right)\right] = \mathrm{Tr_s}^{[\gamma]}\left[\exp\left(-t\mathcal{L}_b^X\right)\right].$$

Remark

The proof uses the fact that $\mathrm{Tr}^{[\gamma]}$ is a trace on the algebra of G-invariants smooth kernels on X with Gaussian decay.

The limit as $b \to +\infty$

The limit as $b \to +\infty$

- After rescaling of Y^{TX}, Y^N, as $b \to +\infty$,
$$\mathcal{L}_b \simeq \frac{b^4}{2} \left| \left[Y^N, Y^{TX} \right] \right|^2 + \frac{1}{2} \left| Y \right|^2 - \underbrace{\nabla_{Y^{TX}}}_{\text{geodesic flow}} .$$

The limit as $b \to +\infty$

- After rescaling of Y^{TX}, Y^N, as $b \to +\infty$,
$\mathcal{L}_b \simeq \frac{b^4}{2} \left| \left[Y^N, Y^{TX} \right] \right|^2 + \frac{1}{2} \left| Y \right|^2 - \underbrace{\nabla_{Y^{TX}}}_{\text{geodesic flow}} .$
- As $b \to +\infty$, the orbital integral localizes near a manifold of geodesics in X associated with γ.

The limit as $b \to +\infty$

- After rescaling of Y^{TX}, Y^N, as $b \to +\infty$,
$$\mathcal{L}_b \simeq \frac{b^4}{2}\left|\left[Y^N, Y^{TX}\right]\right|^2 + \frac{1}{2}\left|Y\right|^2 - \underbrace{\nabla_{Y^{TX}}}_{\text{geodesic flow}}.$$
- As $b \to +\infty$, the orbital integral localizes near a manifold of geodesics in X associated with γ.
- $\gamma = e^a k^{-1}, a \in \mathfrak{p}, k \in K, \mathrm{Ad}\,(k)\, a = a.$

The limit as $b \to +\infty$

- After rescaling of Y^{TX}, Y^N, as $b \to +\infty$,
$\mathcal{L}_b \simeq \frac{b^4}{2} \left| \left[Y^N, Y^{TX} \right] \right|^2 + \frac{1}{2} \left| Y \right|^2 - \underbrace{\nabla_{Y^{TX}}}_{\text{geodesic flow}}$.
- As $b \to +\infty$, the orbital integral localizes near a manifold of geodesics in X associated with γ.
- $\gamma = e^a k^{-1}, a \in \mathfrak{p}, k \in K, \mathrm{Ad}\,(k)\, a = a.$
- $Z\,(\gamma)$ centralizer of γ, $\mathfrak{z}\,(\gamma) = \mathfrak{p}\,(\gamma) \oplus \mathfrak{k}\,(\gamma)$ Lie algebra of $Z\,(\gamma)$.

Semisimple orbital integrals

Semisimple orbital integrals

Theorem (B. 2011)

There is an explicit function $\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right), Y_0^{\mathfrak{k}} \in i\mathfrak{k}(\gamma)$, such that

Semisimple orbital integrals

Theorem (B. 2011)

There is an explicit function $\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right), Y_0^{\mathfrak{k}} \in i\mathfrak{k}(\gamma)$, such that

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-t\left(C^{\mathfrak{g},X}-c\right)/2\right)\right] = \frac{\exp\left(-|a|^2/2t\right)}{(2\pi t)^{p/2}} \int_{i\mathfrak{k}(\gamma)} \mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right) \mathrm{Tr}\left[\rho^E\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right] \exp\left(-\left|Y_0^{\mathfrak{k}}\right|^2/2t\right)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.$$

Semisimple orbital integrals

Theorem (B. 2011)

There is an explicit function $\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right), Y_0^{\mathfrak{k}} \in i\mathfrak{k}(\gamma)$, such that

$$\operatorname{Tr}^{[\gamma]}\left[\exp\left(-t\left(C^{\mathfrak{g},X} - c\right)/2\right)\right] = \frac{\exp\left(-\left|a\right|^2/2t\right)}{(2\pi t)^{p/2}} \int_{i\mathfrak{k}(\gamma)} \mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right) \operatorname{Tr}\left[\rho^E\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right] \exp\left(-\left|Y_0^{\mathfrak{k}}\right|^2/2t\right)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.$$

Note the integral on $i\mathfrak{k}(\gamma)$...

The function $\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right), Y_0^{\mathfrak{k}} \in i\mathfrak{k}(\gamma)$

The function $\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right), Y_0^{\mathfrak{k}} \in i\mathfrak{k}(\gamma)$

Definition

$$\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right) = \frac{1}{\left|\det\left(1-\operatorname{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{\widehat{A}\left(\operatorname{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}(\gamma)}\right)}{\widehat{A}\left(\operatorname{ad}\left(Y_0^{\mathfrak{k}}\right)_{\mathfrak{k}(\gamma)}\right)}$$

$$\left[\frac{1}{\det\left(1-\operatorname{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{z}_0^\perp(\gamma)}}\right.$$

$$\left.\frac{\det\left(1-\operatorname{Ad}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\left(1-\operatorname{Ad}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{1/2}.$$

The center of the enveloping algebra

The center of the enveloping algebra

- $U(\mathfrak{g})$ enveloping algebra.

The center of the enveloping algebra

- $U(\mathfrak{g})$ enveloping algebra.
- $Z(\mathfrak{g})$ center of $U(\mathfrak{g})$ commutative algebra.

The center of the enveloping algebra

- $U(\mathfrak{g})$ enveloping algebra.
- $Z(\mathfrak{g})$ center of $U(\mathfrak{g})$ commutative algebra.
- $S^{\cdot}(\mathfrak{g})$ algebra of polynomials on $\mathfrak{g}^*$.

The center of the enveloping algebra

- $U(\mathfrak{g})$ enveloping algebra.
- $Z(\mathfrak{g})$ center of $U(\mathfrak{g})$ commutative algebra.
- $S^{\cdot}(\mathfrak{g})$ algebra of polynomials on $\mathfrak{g}^*$.
- $I^{\cdot}(\mathfrak{g})$ invariant polynomials on $\mathfrak{g}^*$.

The Duflo isomorphism

The Duflo isomorphism

- There is a filtered canonical isomorphism of algebras $\tau_D : I^{\cdot}(\mathfrak{g}) \simeq Z(\mathfrak{g})$.

The Duflo isomorphism

- There is a filtered canonical isomorphism of algebras $\tau_D : I^{\cdot}(\mathfrak{g}) \simeq Z(\mathfrak{g})$.
- $\tau_D^{-1} C^{\mathfrak{g}} = -B^* + B^*(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}})$.

Center and enveloping algebras

Center and enveloping algebras

- $L \in Z(\mathfrak{g})$, $\tau_D^{-1} L \in I^{\cdot}(\mathfrak{g})$.

Center and enveloping algebras

- $L \in Z(\mathfrak{g})$, $\tau_D^{-1} L \in I^{\cdot}(\mathfrak{g})$.
- $\tau_D^{-1} L \in I^{\cdot}(\mathfrak{g})$ restricts to an element of $I^{\cdot}(\mathfrak{z}(\gamma))$.

Center and enveloping algebras

- $L \in Z(\mathfrak{g})$, $\tau_D^{-1} L \in I^{\cdot}(\mathfrak{g})$.
- $\tau_D^{-1} L \in I^{\cdot}(\mathfrak{g})$ restricts to an element of $I^{\cdot}(\mathfrak{z}(\gamma))$.
- $L^{\mathfrak{z}(\gamma)}$ associated differential operator on $\mathfrak{z}(\gamma)$ with constant coefficients.

The formula for general orbital integrals

The formula for general orbital integrals

$\mu \in \mathcal{S}^{\text{even}}(\mathbf{R})$.

The formula for general orbital integrals

$\mu \in \mathcal{S}^{\text{even}}(\mathbf{R})$.

Theorem (B., Shu SHEN 2019)

The formula for general orbital integrals

$\mu \in \mathcal{S}^{\mathrm{even}}(\mathbf{R})$.

Theorem (B., Shu SHEN 2019)

$$\mathrm{Tr}^{[\gamma]}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]$$

The formula for general orbital integrals

$\mu \in \mathcal{S}^{\mathrm{even}}(\mathbf{R})$.

Theorem (B., Shu SHEN 2019)

$$\mathrm{Tr}^{[\gamma]}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right] = L^{\mathfrak{z}(\gamma)}\mu\left(\sqrt{\left(C^{\mathfrak{g}}\right)^{\mathfrak{z}(\gamma)}+A}\right) \left[\mathcal{J}_{\gamma}\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^{E}\left[\rho^{E}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]\delta_a\right](0).$$

Principle of the proof

Principle of the proof

- We start from the formula with $L = 1$.

Principle of the proof

- We start from the formula with $L = 1$.
- When $\gamma \in G$ is regular, the formula of B. 2011 simplifies dramatically, and can be differentiated in γ.

Principle of the proof

- We start from the formula with $L = 1$.
- When $\gamma \in G$ is regular, the formula of B. 2011 simplifies dramatically, and can be differentiated in γ.
- Use fundamental results of Harish-Chandra to evaluate the action of a differential operator on B. 2011, and obtain our formula for γ regular.

Principle of the proof

- We start from the formula with $L = 1$.
- When $\gamma \in G$ is regular, the formula of B. 2011 simplifies dramatically, and can be differentiated in γ.
- Use fundamental results of Harish-Chandra to evaluate the action of a differential operator on B. 2011, and obtain our formula for γ regular.
- If $\gamma \in G$ semisimple arbitrary, use the previous result for γ regular and limit results of Harish-Chandra to obtain the general formula.

The function $\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right), Y_0^{\mathfrak{k}} \in i\mathfrak{k}(\gamma)$

The function $\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right), Y_0^{\mathfrak{k}} \in i\mathfrak{k}(\gamma)$

Definition

$$\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right) = \frac{1}{\left|\det\left(1-\operatorname{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{\widehat{A}\left(\operatorname{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}(\gamma)}\right)}{\widehat{A}\left(\operatorname{ad}\left(Y_0^{\mathfrak{k}}\right)_{\mathfrak{k}(\gamma)}\right)}$$

$$\left[\frac{1}{\det\left(1-\operatorname{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{z}_0^\perp(\gamma)}}\right.$$

$$\left.\frac{\det\left(1-\operatorname{Ad}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\left(1-\operatorname{Ad}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{1/2}.$$

The function $\mathcal{J}_\gamma$ for γ regular

The function $\mathcal{J}_\gamma$ for γ regular

Theorem (B., Shu SHEN)

The function $\mathcal{J}_\gamma$ for γ regular

Theorem (B., Shu SHEN)

$$\mathcal{J}_\gamma\left(h_{\mathfrak{k}}\right) = \frac{(-1)^{\left|R^{\mathrm{im}}_{\mathfrak{p},+}\right|}\epsilon_D\left(\gamma\right)\prod_{\alpha\in R^{\mathrm{re}}_{+}}\xi^{1/2}_\alpha\left(k^{-1}\right)}{D_H\left(\gamma\right)}$$

$$\frac{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}}\left(\xi^{1/2}_\alpha\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)-\xi^{-1/2}_\alpha\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)}{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}}\left(\xi^{1/2}_\alpha\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)-\xi^{-1/2}_\alpha\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)}.$$

The function $\mathcal{J}_\gamma$ for γ regular

Theorem (B., Shu SHEN)

$$\mathcal{J}_\gamma\left(h_\mathfrak{k}\right) = \frac{(-1)^{\left|R^{\mathrm{im}}_{\mathfrak{p},+}\right|}\epsilon_D\left(\gamma\right)\prod_{\alpha\in R^{\mathrm{re}}_+}\xi_\alpha^{1/2}\left(k^{-1}\right)}{D_H\left(\gamma\right)} \frac{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}}\left(\xi_\alpha^{1/2}\left(k^{-1}e^{-h_\mathfrak{k}}\right)-\xi_\alpha^{-1/2}\left(k^{-1}e^{-h_\mathfrak{k}}\right)\right)}{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}}\left(\xi_\alpha^{1/2}\left(k^{-1}e^{-h_\mathfrak{k}}\right)-\xi_\alpha^{-1/2}\left(k^{-1}e^{-h_\mathfrak{k}}\right)\right)}.$$

The function $\left(\gamma, h_\mathfrak{k}\right) \in H^{\mathrm{reg}} \times i\mathfrak{h}_\mathfrak{k} \to \mathcal{J}_\gamma\left(h_\mathfrak{k}\right) \in \mathbf{C}$ is smooth.

The function $\mathcal{J}_\gamma$ for γ regular

Theorem (B., Shu SHEN)

$$\mathcal{J}_\gamma\left(h_{\mathfrak{k}}\right)=\frac{(-1)^{\left|R^{\mathrm{im}}_{\mathfrak{p},+}\right|}\epsilon_D\left(\gamma\right)\prod_{\alpha\in R^{\mathrm{re}}_{+}}\xi_\alpha^{1/2}\left(k^{-1}\right)}{D_H\left(\gamma\right)}$$
$$\frac{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}}\left(\xi_\alpha^{1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)-\xi_\alpha^{-1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)}{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}}\left(\xi_\alpha^{1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)-\xi_\alpha^{-1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)}.$$

The function $(\gamma, h_{\mathfrak{k}})\in H^{\mathrm{reg}}\times i\mathfrak{h}_{\mathfrak{k}}\to\mathcal{J}_\gamma\left(h_{\mathfrak{k}}\right)\in\mathbf{C}$ is smooth.

Remark

Cancellations between $\mathfrak{p}$ and $\mathfrak{k}$ parts of $\mathcal{J}_\gamma$.

The Langevin equation

The Langevin equation

- In 1908, on $\mathbf{R}^3$, Langevin introduced the Langevin equation $m\ddot{x} = -\dot{x} + \dot{w}\dots$

The Langevin equation

- In 1908, on $\mathbf{R}^3$, Langevin introduced the Langevin equation $m\ddot{x} = -\dot{x} + \dot{w}\ldots$
- …to reconcile Brownian motion $\dot{x} = \dot{w}$ and classical mechanics: $\ddot{x} = 0$.

The Langevin equation

- In 1908, on $\mathbf{R}^3$, Langevin introduced the Langevin equation $m\ddot{x} = -\dot{x} + \dot{w}\dots$
- $\dots$to reconcile Brownian motion $\dot{x} = \dot{w}$ and classical mechanics: $\ddot{x} = 0$.
- In the theory of the hypoelliptic Laplacian, $m = b^2$ is a mass.

Langevin (C.R. de l'Académie des Sciences 1908)

Langevin (C.R. de l'Académie des Sciences 1908)

Une particule comme celle que nous considérons, grande par rapport à la distance moyenne des molécules du liquide, et se mouvant par rapport à celui-ci avec la vitesse ξ subit une résistance visqueuse égale à $-6\pi\mu a\xi$ d'après la formule de Stokes. En réalité, cette valeur n'est qu'une moyenne, et en raison de l'irrégularité des chocs des molécules environnantes, l'action du fluide sur la particule oscille autour de la valeur précédente, de sorte que l'équation du mouvement est, dans la direction x,

$$m\frac{d^2x}{dt^2} = -6\pi\mu a\frac{dx}{dt} + X. \tag{3}$$

Sur la force complémentaire X nous savons qu'elle est indifféremment positive et négative, et sa grandeur est telle qu'elle maintient l'agitation de la particule que, sans elle, la résistance visqueuse finirait par arrêter.

- P. Langevin, *Sur la théorie du mouvement brownien*, C. R. Acad. Sci. Paris **146** (1908), 530–533.
- J.-M. Bismut, *Hypoelliptic Laplacian and orbital integrals*, Annals of Mathematics Studies, vol. 177, Princeton University Press, Princeton, NJ, 2011. MR 2828080
- J.-M. Bismut and S. Shen, *Geometric orbital integrals and the center of the enveloping algebra*, arXiv 1910.11731 (2019).

Happy birthday, Samson!