

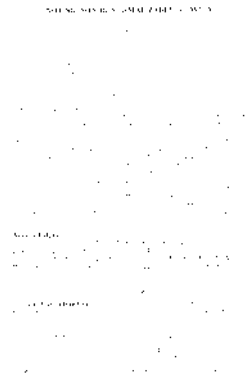
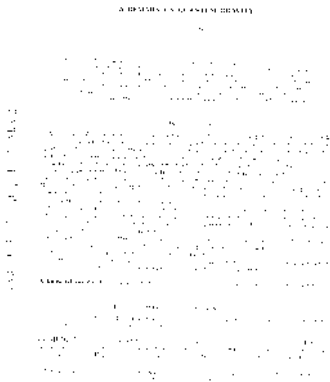
Classical Gravity at High Precision

Johannes Blümlein



Dirk Kreimer Fest, Bures sur Yvette, France, November 20, 2020

J. Blümlein, A. Maier, P. Marquard, Phys.Lett.B 800 (2020) 135100,
J. Blümlein, A. Maier, P. Marquard, A. Schäfer, Phys.Lett.B 807 (2020) 135496, Nucl.Phys.B 955 (2020) 115041,
Phys.Lett.B 801 (2020) 135157, 2010.13672 [gr-qc].



Dirk has published very interesting papers on gravity. I remember excellent lectures by him on this topic at Clay Institute during Summer 2008.



Dirk regularly contributed to DESY conferences over decades: Loops & Legs, QCD09. KMPB initiated by him is an important platform for scientific exchange. We met at Boston, Linz, Les Houches, Bures, and many other places. Often, topical review books collected the achievements made.

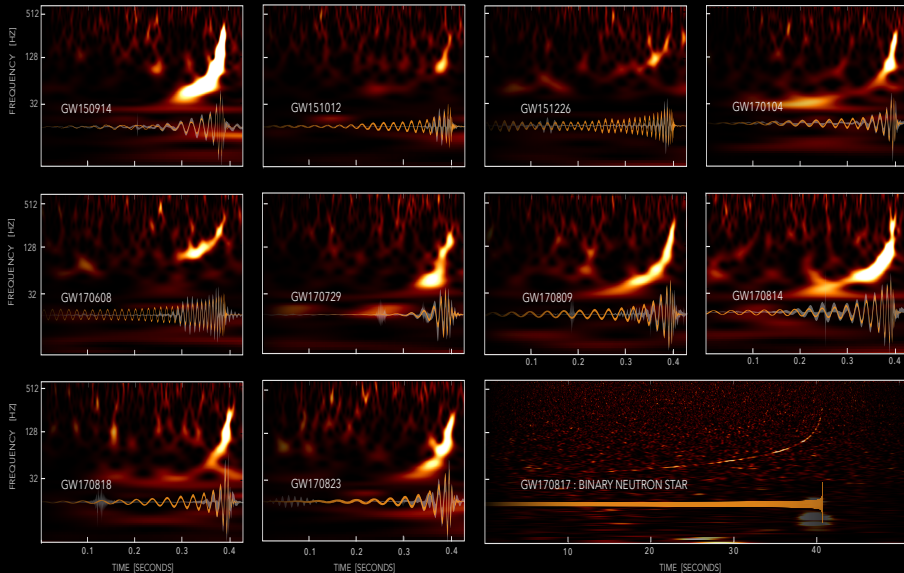
We are very happy to have Dirk at Berlin!

Happy Birthday, Dirk!
And many happy returns!

The motion of two gravitating masses

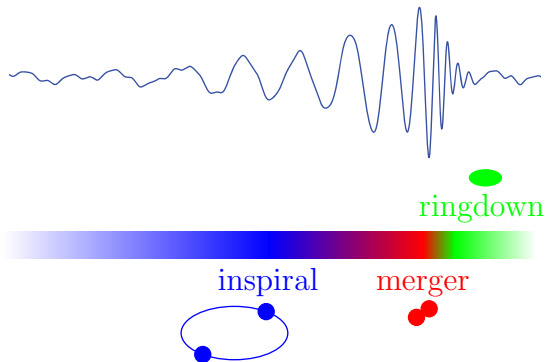
- Conservative part
- Radiation part
- **Consider:** the case of low velocities $v_i \ll c$
- The form of the orbit becomes more and more complicated, invoking higher and higher order corrections.
 - **Newtonian level** : elliptic motion
 - **1PN** : movement of the perihelion (first observed for Mercury)
 - $\vdots$
 - **current level** : $\implies$ **5PN (6PN)**
 - test mass limit: all orders; $O(\nu)$ terms via self force: to 21.5 PN.
- We will consider the **radiation free case** of the inspiral phase in the following.
- The whole dynamics is important for the description of gravitational wave signals.
- $\implies$ **Derive the Hamiltonian Dynamics to 5PN.**

GRAVITATIONAL-WAVE TRANSIENT CATALOG-1



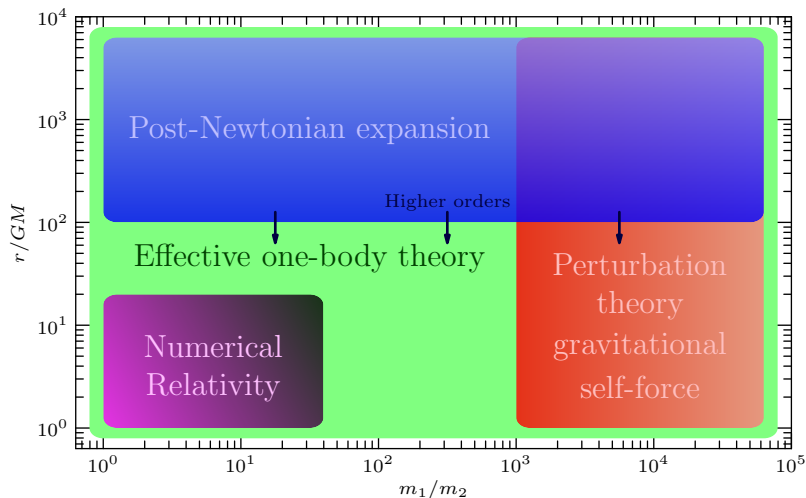
Gravitational waves

[LIGO Scientific Collaboration and Virgo Collaboration 2016]



Post-Newtonian expansion

Other approaches



adapted from [Buonanno 2018]

Post-Newtonian expansion

Theory status

- 1PN (v^2): [Lorentz, Droste 1917; Einstein, Infeld, Hoffmann 1938]
- 2PN (v^4): [Chandrasekhar, Esposito, Nutku 1969–1970; Okamura, Ohta, Kimura, Hiida 1973–1974, final form: Damour 1982]
- 2.5PN (v^5): [Damour, Deruelle 1981–1983]
- 3PN (v^6): [Damour, Jaranowski, Schäfer, 1997–2001; Andrade, Blanchet, Iyer, Faye 2000–2002]
- 3.5PN (v^7): [Iyer, Will 1993–1995]
- 4PN (v^8):
 - ADM Hamiltonian formalism [Damour, Jaranowski, Schäfer 2014–2016]
 - Fokker Lagrangian in harmonic coordinates [Bernard, Blanchet, Bohé, Faye, Marchant, Marsat 2017]
 - Non-relativistic effective field theory [Foffa, Mastrolia, Sturani, Sturm 2017–2019]
- Many more contributions, e.g. spin effects, radiation

Non-relativistic effective field theory [Goldberger, Rothstein 2004, Kol, Smolkin, 2010]

From **path integral** to the **post-Newtonian** Hamiltonians and
Observables by **Methods from Quantum Field Theory**

General Relativity

- Here: point-like objects
 - No spin
 - No finite-size effects
- Harmonic gauge fixing: $\partial_\mu(\sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}) = 0$
 $g = \det(g^{\mu\nu})$
- Dimensional regularisation: $d = 3 - 2\epsilon$

$$S_{\text{GR}} = S_{\text{EH}} + S_{\text{GF}} + S_{\text{pp}}$$

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g} R$$

$$S_{\text{GF}} = - \frac{1}{32\pi G} \int d^{d+1}x \sqrt{-g} \Gamma_\mu \Gamma^\mu$$

$$S_{\text{pp}} = - \sum_i m_i \int d\tau_i = - \sum_i m_i \int dt \sqrt{-g_{\mu\nu} \frac{\partial x_i^\mu}{\partial t} \frac{\partial x_i^\nu}{\partial t}}$$

$$R = g^{\mu\nu} R_{\mu\nu}$$

$$\Gamma^\mu = g^{\alpha\beta} \Gamma^\mu_{\alpha\beta}$$

Why can QFT methods be of help here ?

- Any dynamical physical theory is based on the Lagrange formalism and can be derived from the principle of the least action.
- Near the non-relativistic limit, Einstein gravity possesses an expansion in Newton's constant and the velocities of the involved macroscopic bodies $v_i \ll c$.
- One can consider the associated **path integral** representation [Feynman & Hibbs 1965, Zinn-Justin 2002] and derive the perturbative expansions from it.
- This will lead to the associated effective field theory representation, not necessarily built on the **(flat space) gravitons**.
- From order to order (potentially) **new interaction vertices** will appear.
- This systematic approach will, however, perturbatively represent the **full theory**.
- The key-issue is to reduce to the associated master integrals and to calculate them.
- Current level: **5-loop integrals** in $d = 3 - 2\varepsilon$ (ranging to **6PN**)

Non-relativistic effective theory

[Goldberger, Rothstein 2004]

Similar to non-relativistic QCD

[Caswell, Lepage 1985; Pineda, Soto 1997; Luke, Manohar, Rothstein 2000; ...]

Full theory:

General relativity

$$S_{\text{GR}} = S_{\text{EH}} + S_{\text{GF}} + S_{pp} \quad \longrightarrow$$

potential gravitons:

$$k_0 \sim \frac{v}{r}, \vec{k} \sim \frac{1}{r}$$

radiation gravitons:

$$k_0 \sim \frac{v}{r}, \vec{k} \sim \frac{v}{r}$$

Effective theory:

NRGR

$$S_{\text{NRGR}} = \int dt \frac{1}{2} m_i v_i^2 + \frac{G m_1 m_2}{r} + \dots$$

classical potentials: potential terms

radiation gravitons: tail terms

Potential matching

Expansion of action

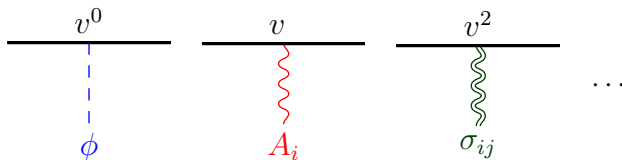
Expand S_{GR} in $v \sim \sqrt{Gm/r} \ll 1$, e.g.

$$S_{\text{pp}} = - \sum_i m_i \int dt \sqrt{-g_{\mu\nu} \frac{\partial x_i^\mu}{\partial t} \frac{\partial x_i^\nu}{\partial t}} = - \sum_i m_i \int dt \sqrt{-g_{00}} + \mathcal{O}(v_i)$$

Coupling to **spatial components** of metric **suppressed**

Temporal Kaluza-Klein decomposition [Kol, Smolkin 2010]: **10 fields**.

$$g^{\mu\nu} = e^{2\phi} \begin{pmatrix} -1 & A_j \\ A_i & e^{-2\frac{d-1}{d-2}\phi} (\delta_{ij} + \sigma_{ij}) - A_i A_j \end{pmatrix}$$



Potential matching

Diagrammatic expansion

Equate amplitude in effective and full theory:

$$\begin{aligned}
 & \text{Diagram 1: Two horizontal lines with a vertical orange bar between them. To the left of the bar is a downward arrow labeled } q, \text{ and to the right is } -iV. \\
 & + \frac{1}{2!} \text{Diagram 2: Two horizontal lines with two vertical orange bars between them.} \\
 & + \frac{1}{3!} \text{Diagram 3: Two horizontal lines with three vertical orange bars between them.} \\
 & + \dots \\
 = & \text{Diagram 4: Two horizontal lines with a vertical dashed blue line between them.} \\
 & + \text{Diagram 5: Two horizontal lines with a vertical red wavy line between them.} \\
 & + \text{Diagram 6: Two horizontal lines with a triangular dashed blue loop between them.} \\
 & + \text{Diagram 7: Two horizontal lines with two vertical dashed blue lines between them.} \\
 & + \text{Diagram 8: Two horizontal lines with an X-shaped dashed blue loop between them.} \\
 & + \dots
 \end{aligned}$$

All momenta potential, $p_0 \sim \frac{v}{r} \ll p_i \sim \frac{1}{r}$

$\hookrightarrow$ expand propagators:

$$\frac{1}{\vec{p}^2 - p_0^2} = \frac{1}{\vec{p}^2} + \frac{p_0^2}{\vec{p}^4} + \mathcal{O}(v^4)$$

Potential matching

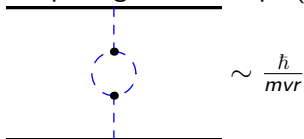
Diagrammatic expansion

$$\begin{aligned} V &= i \log \left(1 + \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{wavy} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \triangle \\ \text{---} \end{array} \right. \\ &\quad \left. + \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{X} \\ \text{---} \end{array} + \dots \right) \\ &= i \left(\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \underbrace{\begin{array}{c} \text{---} \\ \text{wavy} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \triangle \\ \text{---} \end{array}}_{1\text{PN}} + \dots \right) \end{aligned}$$

Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)

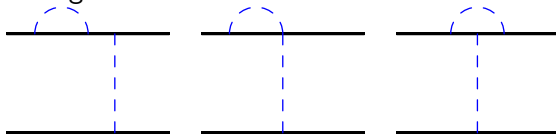


- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections



Absorbed into renormalisation of sources

- No source-reducible diagrams [Fischler 1977]

Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

Initially *time-ordered* diagrams:

$$\frac{\text{Diagram 1}}{\Theta(y^0 - x^0)} = \frac{1}{2} \left(\frac{\text{Diagram 2}}{\Theta(y^0 - x^0)} + \frac{\text{Diagram 3}}{\Theta(x^0 - y^0)} \right) = \frac{1}{2} \text{Diagram 4}$$

The diagrammatic equation illustrates the time-ordering of a graviton exchange between two sources. On the left, a diagram shows two horizontal black lines representing sources. A dashed blue line (graviton) connects them, with a blue arrow pointing from the bottom source to the top source. Below this diagram is the label $\Theta(y^0 - x^0)$. This is set equal to $\frac{1}{2}$ times the sum of two diagrams in parentheses. The first diagram in the sum is identical to the left one, with a blue arrow pointing from bottom to top and label $\Theta(y^0 - x^0)$. The second diagram has a blue arrow pointing from top to bottom and label $\Theta(x^0 - y^0)$. The sum is then set equal to $\frac{1}{2}$ times a final diagram on the right, which shows the same two sources connected by a dashed blue line but without a blue arrow.

Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

Initially *time-ordered* diagrams:

The diagrammatic equation illustrates the relationship between time-ordered diagrams and a squared diagram. It consists of the following parts from left to right:

- A diagram with two horizontal black lines and two vertical dashed blue lines. The left vertical line has an upward arrow, and the right vertical line has a downward arrow.
- A plus sign (+).
- A diagram with two horizontal black lines and two vertical dashed blue lines. The left vertical line has a downward arrow, and the right vertical line has an upward arrow.
- An equals sign (=).
- A diagram with two horizontal black lines and two vertical dashed blue lines. The left vertical line has an upward arrow, and the right vertical line has a downward arrow.
- An equals sign (=).
- A fraction $\frac{1}{2}$.
- A diagram with two horizontal black lines and two vertical dashed blue lines. Both vertical lines have upward arrows.
- An equals sign (=).
- A fraction $\frac{1}{2}$.
- An open parenthesis (().
- A diagram with two horizontal black lines and two vertical dashed blue lines. Both vertical lines have upward arrows.
- A closed parenthesis ()).
- A superscript 2 (²).

Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

Initially *time-ordered* diagrams:

$$\begin{array}{c} \text{Diagram 1} + \text{Diagram 2} = \text{Diagram 3} = \frac{1}{2} \text{Diagram 4} = \frac{1}{2} \left(\text{Diagram 5} \right)^2 \\ \text{Diagram 1: Two horizontal lines with two vertical dashed lines. The left dashed line has an upward arrow, the right has a downward arrow.} \\ \text{Diagram 2: Two horizontal lines with two vertical dashed lines. The left dashed line has a downward arrow, the right has an upward arrow.} \\ \text{Diagram 3: Two horizontal lines with two vertical dashed lines. Both dashed lines have upward arrows.} \\ \text{Diagram 4: Two horizontal lines with two vertical dashed lines. Both dashed lines have upward arrows.} \\ \text{Diagram 5: Two horizontal lines with one vertical dashed line. The dashed line has an upward arrow.} \end{array}$$
$$-iV = \log \left(1 + \text{Diagram 3} + \frac{1}{2} \left(\text{Diagram 5} \right)^2 + \dots \right) = \text{Diagram 3} + \dots$$

Effective Field Theory Calculations

Known results

Confirmation of previous results:

- 1PN: [1917 (1938)] [Goldberger, Rothstein 2004, Kol, Smolkin, 2010]
- 2PN: [1969/70] [Gilmore, Ross 2008]
- 3PN: [1997/2001] [Foffa, Sturani 2011]
- 4PN:
 - “static” contribution $v = 0$:
[Foffa, Mastrolia, Sturani, Sturm 2016; Damour, Jaranowski 2017]
 - $v \neq 0$: [2014] [Foffa, Sturani 2019; Foffa, Porto, Rothstein, Sturani 2019, JB, Maier, Marquard, Schäfer, 2020]

New:

- **5PN static contribution:**

[Foffa, Mastrolia, Sturani, Sturm, Torres Bobadilla 27 Feb 2019; Blümlein, Maier, Marquard 28 Feb 2019]

Effective Field Theory Calculations

Number of diagrams at 5PN: potential terms

#loops	QGRAF	source irred.	no source loops	no tadpoles	masters
0	3	3	3	3	0
1	72	72	72	72	1
2	3286	3286	3286	2702	1
3	81526	62246	60998	41676	1
4	545812	264354	234934	116498	7
5	332020	128080	101570	27582	4

Generation of 962719 Feynman diagrams [QGRAF](#) [Nogueira 1993];
performing the Lorentz algebra and further steps [FORM 3.0](#) [Vermaseren
2001-]; Reduction to master intergrals [Crusher](#) [Marquard, Seidel].

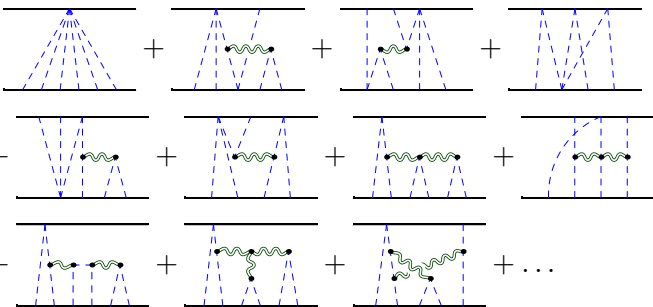
[188533](#) diagrams are finally contributing.

There are factorizing subsets [Foffa, Sturani, Torres Bobadilla 2020_v2], to which we agree.

For all the contributions up to 4PN see [JB, Maier, Marquard, Schäfer, 2020].

Effective Field Theory Calculations

Static 5PN calculation

$$-iV_{5\text{PN}}^S =$$


The diagram illustrates the static 5PN calculation as a sum of Feynman diagrams. The diagrams are arranged in three rows, separated by plus signs. Each diagram is enclosed in a rectangular box with two horizontal black lines representing the worldlines of the masses. Blue dashed lines represent graviton exchanges, and green wavy lines represent matter field exchanges. The diagrams show various topologies of graviton and matter field exchanges between the two worldlines, contributing to the 5PN order. The first row contains four diagrams, the second row contains four diagrams, and the third row contains three diagrams followed by an ellipsis (...).

Effective Field Theory Calculations

Feynman rules

$$\text{---} \overset{p}{\text{---}} = -\frac{i}{2c_d \vec{p}^2}$$

$$\text{---} \overset{i_1 i_2}{\text{---}} \overset{j_1 j_2}{\text{---}} = -\frac{i}{2\vec{p}^2} (\delta_{i_1 j_1} \delta_{i_2 j_2} + \delta_{i_1 j_2} \delta_{i_2 j_1} + (2 - c_d) \delta_{i_1 i_2} \delta_{j_1 j_2})$$

$$\text{---} \overset{m_i}{\text{---}} = -i \frac{m_i}{m_{\text{Pl}}^n}$$

$$\text{---} \overset{p_1}{\text{---}} \text{---} \overset{i_1 i_2}{\text{---}} \text{---} \overset{p_2}{\text{---}} = i \frac{c_d}{2m_{\text{Pl}}} (V_{\phi\phi\sigma}^{i_1 i_2} + V_{\phi\phi\sigma}^{i_2 i_1})$$

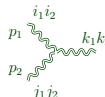
$$V_{\phi\phi\sigma}^{i_1 i_2} = \vec{p}_1 \cdot \vec{p}_2 \delta^{i_1 i_2} - 2p_1^{i_1} p_2^{i_2}$$

$$\text{---} \overset{p_1}{\text{---}} \text{---} \overset{j_1 j_2}{\text{---}} \text{---} \overset{p_2}{\text{---}} \text{---} \overset{i_1 i_2}{\text{---}} = i \frac{c_d}{16m_{\text{Pl}}^2} (V_{\phi\phi\sigma\sigma}^{i_1 i_2, j_1 j_2} + V_{\phi\phi\sigma\sigma}^{i_2 i_1, j_1 j_2} + V_{\phi\phi\sigma\sigma}^{i_1 i_2, j_2 j_1} + V_{\phi\phi\sigma\sigma}^{i_2 i_1, j_2 j_1})$$

$$V_{\phi\phi\sigma\sigma}^{i_1 i_2, j_1 j_2} = \vec{p}_1 \cdot \vec{p}_2 (\delta^{i_1 i_2} \delta^{j_1 j_2} - 2\delta^{i_1 j_1} \delta^{i_2 j_2}) - 2(p_1^{i_1} p_2^{j_2} \delta^{i_1 j_2} + p_1^{j_1} p_2^{i_2} \delta^{i_1 i_2}) + 8\delta^{i_1 j_1} p_1^{i_2} p_2^{j_2}$$

Effective Field Theory Calculations

Feynman rules



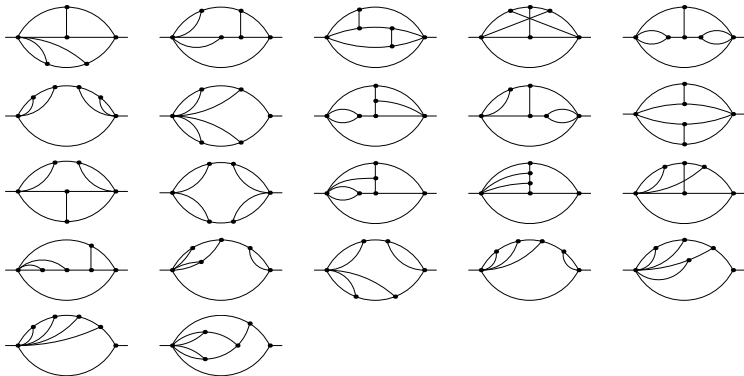
$$\begin{aligned}
 & \text{Diagram} = \frac{i}{32m_{\text{Pl}}} (\tilde{V}_{\sigma\sigma\sigma\sigma}^{i_1 i_2, j_1 j_2, k_1 k_2} + \tilde{V}_{\sigma\sigma\sigma\sigma}^{i_2 i_1, j_1 j_2, k_1 k_2}) \\
 & \tilde{V}_{\sigma\sigma\sigma\sigma}^{i_1 i_2, j_1 j_2, k_1 k_2} = V_{\sigma\sigma\sigma\sigma}^{i_1 i_2, j_1 j_2, k_1 k_2} + V_{\sigma\sigma\sigma\sigma}^{i_1 i_2, j_2 j_1, k_1 k_2} + V_{\sigma\sigma\sigma\sigma}^{i_1 i_2, j_1 j_2, k_2 k_1} + V_{\sigma\sigma\sigma\sigma}^{i_1 i_2, j_2 j_1, k_2 k_1} \\
 & V_{\sigma\sigma\sigma\sigma}^{i_1 i_2, j_1 j_2, k_1 k_2} = (\vec{p}_1^2 + \vec{p}_1 \cdot \vec{p}_2 + \vec{p}_2^2) \left(-\delta^{j_1 j_2} \left(2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2} \right) \right. \\
 & \quad \left. + 2 \left[\delta^{i_1 j_1} \left(4\delta^{i_2 k_1} \delta^{j_2 k_2} - \delta^{i_2 j_2} \delta^{k_1 k_2} \right) - \delta^{i_1 i_2} \delta^{j_1 k_1} \delta^{j_2 k_2} \right] \right) \\
 & \quad + 2 \left\{ 4 \left(p_1^{k_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2} \right) \delta^{i_1 j_1} \delta^{j_2 k_1} \right. \\
 & \quad \left. + 2 \left[\left(p_1^{i_1} + p_2^{i_1} \right) p_2^{i_2} \delta^{j_1 k_1} \delta^{j_2 k_2} - p_1^{k_1} p_2^{k_2} \delta^{i_1 j_1} \delta^{i_2 j_2} \right] \right. \\
 & \quad \left. + \delta^{j_1 j_2} \left[p_1^{k_1} p_2^{k_2} \delta^{i_1 i_2} + 2 \left(p_1^{k_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2} \right) \delta^{i_1 k_1} - \left(p_1^{i_1} + p_2^{i_1} \right) p_2^{i_2} \delta^{k_1 k_2} \right] \right. \\
 & \quad \left. + p_2^{j_2} \left(4p_1^{i_2} \delta^{i_1 k_1} \delta^{j_1 k_2} + p_1^{j_1} \left(2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2} \right) \right. \right. \\
 & \quad \left. \left. + 2 \left[\delta^{i_1 j_1} \left(p_1^{i_2} \delta^{k_1 k_2} - 2p_1^{k_2} \delta^{i_2 k_1} \right) - p_1^{k_2} \delta^{i_1 i_2} \delta^{j_1 k_1} \right] \right] \right. \\
 & \quad \left. + p_1^{j_2} \left(p_1^{i_1} \left(2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2} \right) - 4p_2^{i_2} \delta^{i_1 k_1} \delta^{j_1 k_2} \right. \right. \\
 & \quad \left. \left. + 2 \left[p_2^{k_2} \delta^{i_1 i_2} \delta^{j_1 k_1} + \delta^{i_1 j_1} \left(2p_2^{k_2} \delta^{i_2 k_1} - p_2^{i_2} \delta^{k_1 k_2} \right) \right] \right] \right\}, \quad c_d = 2 \frac{d-1}{d-2}, \quad m_{\text{Pl}} = \sqrt{32\pi G}
 \end{aligned}$$

Effective Field Theory Calculations

Diagram families

Massless propagators:

$$P_f(q) = \int \frac{d^d l_1}{\pi^{d/2}} \cdots \frac{d^d l_5}{\pi^{d/2}} \frac{\mathcal{N}(q, l_1, \dots, l_5)}{\vec{p}_1^{2a_1} \cdots \vec{p}_{10}^{2a_{10}}}$$



Effective Field Theory Calculations

Reduction to master integrals

Apply integration-by-parts relations (crusher): [Marquard & Seidel, unpubl.],
[Chetyrkin, Tkachov 1981; Laporta 2000]

$$\begin{aligned} V_{5\text{PN}}^S &= \tilde{c}_0 \text{ (5 internal lines) } + \tilde{c}_1 \text{ (4 internal lines, 2 vertices) } + \tilde{c}_2 \text{ (3 internal lines, 3 vertices) } + \dots \\ &\stackrel{\text{IBP}}{=} c_0 \text{ (5 internal lines) } + c_1 \text{ (4 internal lines, 1 vertex) } + c_2 \text{ (3 internal lines, 2 vertices) } \\ &\quad + c_3 \text{ (3 internal lines, 3 vertices) } + \mathcal{O}(\epsilon) \end{aligned}$$

$\tilde{c}_i, c_j$: Laurent series in $\epsilon = \frac{3-d}{2}$, polynomials in m_1, m_2, r^{-1}, G^{-1}

Effective Field Theory Calculations

Calculation of master integrals

Master integrals factorise, e.g.

$$\begin{aligned}
 & \text{Diagram 1} = \left[\text{Diagram 2} \right]_{q^2=1}^2 \times \text{Diagram 3} \\
 & \text{Diagram 4} = \left[\text{Diagram 5} \right]_{q^2=1} \times \text{Diagram 6}
 \end{aligned}$$

Diagram 1: A circle with four internal lines connecting two vertices on the left to two vertices on the right.

Diagram 2: A circle with a horizontal line connecting two vertices, with a momentum arrow q pointing right.

Diagram 3: A circle with two vertices, each having a line extending outwards. The top arc is labeled $d-3$ and the bottom arc is labeled $d-3$.

Diagram 4: A circle with four internal lines connecting two vertices on the left to two vertices on the right, with an additional vertical line connecting the two internal vertices.

Diagram 5: A circle with a horizontal line connecting two vertices, with a momentum arrow q pointing right, and two vertical lines connecting the two vertices to the horizontal line.

Diagram 6: A circle with two vertices, each having a line extending outwards. The top arc is labeled $2d-8$ and the bottom arc is labeled 1 .

$$\begin{aligned}
 & \text{Diagram 7} = \int \frac{d^d l}{\pi^{d/2}} \frac{1}{((q-l)^2)^a} \frac{1}{(l^2)^b} \\
 & = \frac{1}{(q^2)^{a+b-d/2}} \frac{\Gamma(\frac{d}{2}-a) \Gamma(\frac{d}{2}-b) \Gamma(a+b-\frac{d}{2})}{\Gamma(a)\Gamma(b)\Gamma(d-a-b)}
 \end{aligned}$$

Diagram 7: A circle with two vertices, each having a line extending outwards. The top arc is labeled a and the bottom arc is labeled b . A momentum arrow q points right from the left vertex.

 known from 4PN [Foffa, Mastrolia, Sturani, Sturm 2016; Damour, Jaranowski 2017] up to the power in ε presently needed.

Effective Field Theory Calculations

Results for master integrals

$$\text{Diagram 1} = e^{5\epsilon\gamma_E} \frac{\Gamma(6 - \frac{5d}{2}) \Gamma^6(-1 + \frac{d}{2})}{\Gamma(-6 + 3d)}$$

$$\text{Diagram 2} = e^{5\epsilon\gamma_E} \frac{\Gamma(7 - \frac{5d}{2}) \Gamma(3 - d) \Gamma(2 - \frac{d}{2}) \Gamma^7(-1 + \frac{d}{2}) \Gamma(5 - 2d)}{\Gamma(5 - \frac{3}{2}d) \Gamma(-2 + d) \Gamma(-3 + \frac{3}{2}d) \Gamma(-7 + 3d)}$$

$$\text{Diagram 3} = e^{5\epsilon\gamma_E} \frac{\Gamma(7 - \frac{5d}{2}) \Gamma^2(3 - d) \Gamma^7(-1 + \frac{d}{2}) \Gamma(-6 + \frac{5d}{2})}{\Gamma(6 - 2d) \Gamma^2(-3 + \frac{3d}{2}) \Gamma(-7 + 3d)}$$

$$\text{Diagram 4} = 6\pi^{7/2} \left[\frac{2}{\epsilon} - 4 - 4\ln(2) - (48 + 8\ln(2) - 4\ln^2(2) - 105\zeta_2) \epsilon + \mathcal{O}(\epsilon^2) \right]$$

$$\begin{aligned} V_{5\text{PN}}^S \stackrel{\epsilon \rightarrow 0}{=} & \frac{G^6}{r^6 \pi^{7/2}} (m_1 m_2) \left\{ \frac{15}{32} (m_1^5 + m_2^5) \left[\text{Diagram 1} \right]_{\epsilon^0} + \frac{91}{4} m_1 m_2 (m_1^3 + m_2^3) \left[\text{Diagram 1} \right]_{\epsilon^0} \right. \\ & + m_1^2 m_2^2 (m_1 + m_2) \left(\left[\frac{293}{4} \text{Diagram 1} - \frac{45}{16} \text{Diagram 2} + \frac{45}{32} \text{Diagram 3} \right]_{\epsilon^0} \right. \\ & \left. \left. + \left[\frac{519}{16} \text{Diagram 2} - \frac{627}{32} \text{Diagram 3} + 2 \text{Diagram 4} \right]_{\epsilon^{-1}} \right) \right\} \end{aligned}$$

The Static Potential

Result

$$V_N^S = -\frac{G}{r} m_1 m_2$$

$$V_{1PN}^S = \frac{G^2}{2r^2} m_1 m_2 (m_1 + m_2)$$

$$V_{2PN}^S = -\frac{G^3}{r^3} m_1 m_2 \left[\frac{1}{2} (m_1^2 + m_2^2) + 3m_1 m_2 \right]$$

$$V_{3PN}^S = \frac{G^4}{r^4} m_1 m_2 \left[\frac{3}{8} (m_1^3 + m_2^3) + 6m_1 m_2 (m_1 + m_2) \right]$$

$$V_{4PN}^S = -\frac{G^5}{r^5} m_1 m_2 \left[\frac{3}{8} (m_1^4 + m_2^4) + \frac{31}{3} m_1 m_2 (m_1^2 + m_2^2) + \frac{141}{4} m_1^2 m_2^2 \right]$$

$$V_{5PN}^S = \frac{G^6}{r^6} m_1 m_2 \left[\frac{5}{16} (m_1^5 + m_2^5) + \frac{91}{6} m_1 m_2 (m_1^3 + m_2^3) + \frac{653}{6} m_1^2 m_2^2 (m_1 + m_2) \right]$$

All ζ -values cancel. This is not the case for velocity corrections, to which we turn now.

The 5PN Hamiltonian

The principal calculation steps

■

$$H = H_{\text{pot}} + H_{\text{tail}} = H_{\text{loc}} + H_{\text{nl}}$$

- In the harmonic gauge H_{pot} and H_{tail} both singular.
- The potential terms H_{pot} are calculated as outlined before.
- The singular and logarithmic contributions to H_{tail} are calculated using the method of expansion by regions [Beneke, Smirnov 1997, Jantzen 2011].
- There are no overlap terms between the potential and tail contributions [JB, Maier, Marquard, Schäfer, 2020].
- One adds both terms and obtains
 - a still singular Hamiltonian
 - the **non-local Hamiltonian** H_{nl} .
- Perform a canonical transformation to a pole-free Hamiltonian [then in different coordinates];

The formulae are still rather long. [JB, Maier, Marquard, Schäfer, 2010.13672 [gr-qc].]

The 5PN Hamiltonian

The principal calculation steps

- Perform a canonical transformation of the $O(\nu^0)$ and $O(\nu^{3..5})$ terms from the Hamiltonian starting with harmonic coordinates, H_{harm} , to effective one body (EOB) coordinates, H_{EOB} .
⇒ **Physical equivalence** of these terms. [On an individual diagram basis even the Schwarzschild limit is not easy to obtain: very many diagrams.]
- Prediction of all terms contributing to 3PM and 6PN [JB, Maier, Marquard, Schäfer 2020 3PM, 6PN]; confirmed by [Bini, Damour, Geralico, 2020 6PN loc].
- A **special treatment** is needed for the $O(\nu)$ terms. ✓
- A series of more **local** tail terms contribute at $O(\nu^2)$ and have to dealt with (**work in progress**).
- The **non-local** contributions of the tail terms at 4 & 5 PN haven been calculated. Agreement with [Bini, Damour, Geralico, 2020 5PN].

Observables at 5PN

Periastron advance: local contributions

$$\begin{aligned} K(\hat{E}, j)_{\text{loc}, \text{f}}^{5\text{PN}} \propto & \left\{ \left[\frac{15\nu^2}{16} - \frac{15}{4}\nu^3 + 3\nu^4 \right] \frac{\hat{E}^4}{j^2} + \left[\frac{3465}{16} + \left(-\frac{12160657}{8400} + \frac{15829\pi^2}{256} \right) \nu + \left(-\frac{35569\pi^2}{1024} \right) \nu^2 \right. \right. \\ & + \left(\frac{1107\pi^2}{128} - \frac{7113}{8} \right) \nu^3 + 75\nu^4 \left. \right] \frac{\hat{E}^3}{j^4} + \left[\frac{315315}{32} + \left(-\frac{33023719}{840} + \frac{4899565\pi^2}{4096} \right) \nu \right. \\ & + \left(-\frac{3289285\pi^2}{1024} \right) \nu^2 + \left(\frac{35055\pi^2}{256} - \frac{240585}{32} \right) \nu^3 + \frac{1575}{8}\nu^4 \left. \right] \frac{\hat{E}^2}{j^6} + \left[\frac{765765}{16} \right. \\ & + \left(-\frac{30690127}{240} + \frac{16173395\pi^2}{8192} \right) \nu + \left(-\frac{77646205\pi^2}{8192} \right) \nu^2 + \left(\frac{121975\pi^2}{512} - \frac{271705}{24} \right) \nu^3 \\ & + \frac{2205}{16}\nu^4 \left. \right] \frac{\hat{E}}{j^8} + \left[\frac{2909907}{64} \left(-\frac{61358067}{640} + \frac{1096263\pi^2}{1024} \right) \nu + \left(-\frac{87068961\pi^2}{16384} \right) \nu^2 \right. \\ & \left. \left. + \left(\frac{90405\pi^2}{1024} - \frac{127995}{32} \right) \nu^3 + \frac{3465}{128}\nu^4 \right] \frac{1}{j^{10}} \right\} \eta^{10} + O(\eta^{12}) \end{aligned}$$

- Very recently the origin of the $O(\nu)$ terms has been understood within the present approach.
- Only the rational contributions to the 5PN ν^2 terms are yet open.

Observables at 5PN

Circular orbit: energy and periastron advance

$$\begin{aligned}\hat{E}^{\text{circ}}(j) = & -\frac{1}{2j^2} + \left(-\frac{\nu}{8} - \frac{9}{8}\right) \frac{1}{j^4} \eta^2 + \left(-\frac{\nu^2}{16} + \frac{7\nu}{16} - \frac{81}{16}\right) \frac{1}{j^6} \eta^4 + \left[-\frac{5\nu^3}{128} + \frac{5\nu^2}{64} + \left(\frac{8833}{384}\right.\right. \\ & \left.-\frac{41\pi^2}{64}\right) \nu - \frac{3861}{128} \Big] \frac{1}{j^8} \eta^6 + \left[-\frac{7\nu^4}{256} + \frac{3\nu^3}{128} + \left(\frac{41\pi^2}{128} - \frac{8875}{768}\right) \nu^2 + \left(\frac{989911}{3840}\right.\right. \\ & \left.-\frac{6581\pi^2}{1024}\right) \nu - \frac{53703}{256} \Big] \frac{1}{j^{10}} \eta^8 + \left[\left(r_{\nu^2}^E + \frac{132979\pi^2}{2048}\right) \nu^2 - \frac{21\nu^5}{1024} + \frac{5\nu^4}{1024} + \left(\frac{41\pi^2}{512}\right.\right. \\ & \left.-\frac{3769}{3072}\right) \nu^3 + \left(\frac{3747183493}{1612800} - \frac{31547\pi^2}{1536}\right) \nu - \frac{1648269}{1024} \Big] \frac{1}{j^{12}} \eta^{10} \\ & + \frac{E_{\text{nl}}^{\text{circ}}(j)}{\mu c^2} + O(\eta^{12}),\end{aligned}$$

$$\begin{aligned}K^{\text{circ}}(j) = & 1 + 3\frac{1}{j^2} \eta^2 + \left(\frac{45}{2} - 6\nu\right) \frac{1}{j^4} \eta^4 + \left[\frac{405}{2} + \left(-202 + \frac{123}{32}\pi^2\right) \nu + 3\nu^2\right] \frac{1}{j^6} \eta^6 + \\ & \left[\frac{15795}{8} + \left(\frac{185767}{3072}\pi^2 - \frac{105991}{36}\right) \nu + \left(-\frac{41}{4}\pi^2 + \frac{2479}{6}\right) \nu^2\right] \frac{1}{j^8} \eta^8 + \left[\frac{161109}{8}\right. \\ & \left.+ \left(-\frac{18144676}{525} + \frac{488373}{2048}\pi^2\right) \nu + \left(r_{\nu^2}^K - \frac{1379075}{1024}\pi^2\right) \nu^2 + \left(-\frac{1627}{6} + \frac{205}{32}\pi^2\right) \nu^3\right] \frac{1}{j^{10}} \eta^{10} \\ & + K_{4+5\text{PN}}^{\text{nl}}(j) + O(\eta^{12}). \quad \eta^2 = 1/c^2.\end{aligned}$$

Observables at 5PN

Determining the missing π^2 contributions

In [Bini, Damour, Geralico, 2020] ["tutti frutti method"] two constants $\bar{d}_5$ and a_6 could not be determined. We have computed their π^2 contributions within our calculation **ab initio** [JB, Maier, Marquard, Schäfer, 2020]. These terms are contained in the potential terms

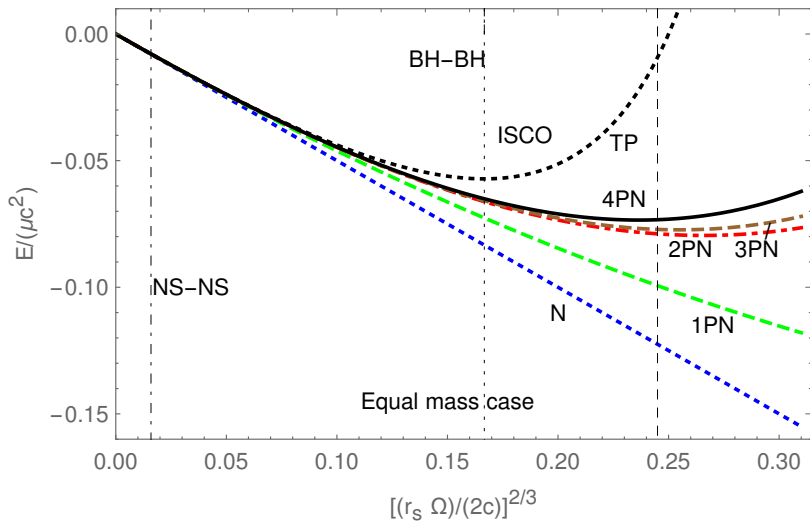
$$\begin{aligned}\bar{d}_5 &= r_{\bar{d}_5} + \frac{306545}{512}\pi^2 \\ a_6 &= r_{a_6} + \frac{25911}{256}\pi^2 .\end{aligned}$$

All terms are in agreement with [Bini, Damour, Geralico, 2020] and the π^2 contributions to $\bar{d}_5$ and a_6 are new.

There are also first 6PN corrections known [Bini, Damour, Geralico, 2020]. Up to the terms of $O(1/r^3)$ they were predicted in [JB, Maier, Marquard, Schäfer 2020, 6PN].

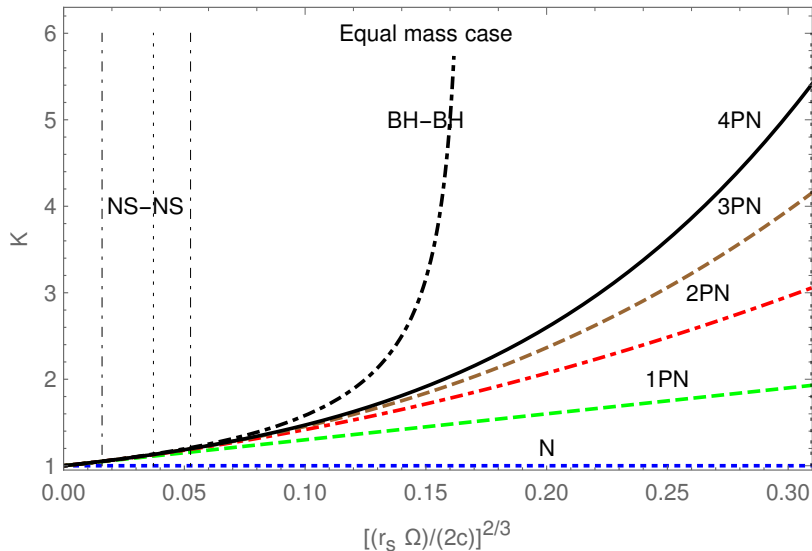
Observables up to 4PN

Numerical results



Observables up to 4PN

Numerical results



Conclusions and Outlook

- The inspiral phase of compact binary systems is well described by the *Post-Newtonian (PN) expansion* $v \sim \sqrt{Gm/r} \ll 1$.
- Effective field theory and calculation methods from **Quantum Field Theory** are very effective for high PN orders.
- The static gravitational potential is known at **five loops**.
- Very recently the **5PN Hamiltonian** has been determined up to a small set of rational terms in $O(v^2)$. These results are obtained performing a **5 loop calculation** of **nearly 200.000** Feynman diagrams with partly very complicated vertices **ab initio**.
- Higher order corrections extend the area perturbatively accessible towards the region where presently only pure numerical methods are applied.
- Fully resummed velocity expressions would be welcome to have: **Post-Minkowskian approach**. They are currently limited to 2-loop order. [Bern et al. 2019; Kälín, Porto, 2020; Damour et al. 2019/20]