Tree-like equations from the Connes-Kreimer Hopf algebra and the combinatorics of chord diagrams

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The Connes-Kreimer Hopf algebra $\mathcal{H}_{CK}$

A combinatorial Hopf algebra introduced by Kreimer in the context of renormalization in quantum field theory.

The free commutative algebra freely generated over F by the set of rooted trees.

Product: concatenation of forests, coproduct:

$$\Delta(t) = \sum_{\substack{C \subseteq V(t) \\ C \text{ antichain}}} \left(\prod_{v \in C} t_v \right) \otimes \left(t \setminus \prod_{v \in C} t_v \right),$$

where t_v is the subtree of t rooted at v.

The universal property of $\mathcal{H}_{CK}$

One reason Connes-Kreimer is important: it possesses a universal property unique up to isomorphism among Hopf algebras.

For a coalgebra A, a linear map $L : A \to A$ is a Hochschild 1-cocycle if

$$\Delta \circ L = (\mathrm{id} \otimes L) \circ \Delta + L \otimes 1.$$

For $\mathcal{H}_{CK}$, the add-a-root operator B_+ is a 1-cocycle.

Theorem (Connes-Kreimer)

Let A be a commutative algebra over F and $L : A \to A$ be a linear map. Then there exists a unique algebra homomorphism $\rho_L : \mathcal{H}_{CK} \to A$ such that $\rho_L \circ B_+ = L \circ \rho_L$. Furthermore, if A is a bialgebra and L is a 1-cocycle then ρ_L is a bialgebra homomorphism, and if A is also a Hopf algebra then ρ_L is a Hopf algebra homomorphism.

Hopf subalgebras of $\mathcal{H}_{CK}$

In a series of papers, Loïc Foissy examined subalgebras of $\mathcal{H}_{CK}$ generated by a family recursive equations of the form

$$T(x) = xB_+(\phi(T(x)))$$

with $\phi(z) \in F[[z]]$ with $\phi(0) = 1$.

Writing $t_n = [x^n]T(x)$, Foissy characterized when the subalgebra $A = F[t_1, t_2, \ldots]$ of $\mathcal{H}_{CK}$ is Hopf.

Theorem (Foissy)

A is a Hopf subalgebra if and only if $\phi(z) = (1 + abz)^{-1/b}$ for some $a, b \in F$.

Tree-like equations

We apply the universal property to the polynomial algebra $F[y]$ and a linear map $L : F[y] \to F[y]$.

This gives the algebra homomorphism $\rho_L : \mathcal{H}_{CK} \to F[y]$, which when applied to a tree equation gives

$$G(x, y) = xL(\phi(G(x, y))).$$

ρ_L corresponds to the Feynman rules, mapping each Feynman graph to its associated Feynman integral via the tree of subdivergences.

1-cocycle operators

Suppose we want meaningful combinatorial solutions; then need to restrict L to some specific class of linear maps.

The universal property points the way: consider 1-cocycles arising from coalgebra structures on $F[y]$.

Two classic graded coaglebras on polynomials: binomial coalgebra and divided power coalgebra.

Coproduct of the binomial coalgebra:

$$\Delta(y^n) = \sum_{k=0}^{n} \binom{n}{k} y^k \otimes y^{n-k}.$$

1-cocycle operators

Lemma

If L_{bin} is a 1-cocycle for the binomial coalgebra on $R[y]$, then $\exists F(z) \in R[[z]]$ such that

$$L_{bin}(y^n) = \int_0^y F\left(\frac{d}{dt}\right) t^n dt.$$

Coproduct of the divided power coalgebra: $\Delta(y^n) = \sum_{k=0}^{n} y^k \otimes y^{n-k}$.

Lemma

If L_{div} is a 1-cocycle for the divided power coalgebra on $R[y]$, then $\exists F(z) \in R[[z]]$ such that

$$L_{div}(y^n) = yF\left(\frac{\delta}{\delta y}\right) y^n.$$

Solving tree-like equations

So, we are interested in solving

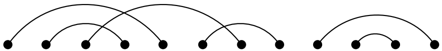
$$G(x,y) = xL_{bin}(\phi(G(x,y))) \quad \text{and} \quad G(x,y) = xL_{div}(\phi(G(x,y))).$$

Keeping the Feynman integral unspecified, the former corresponds to the Dyson-Schwinger equation for a class of Feynman graphs generated by recursively inserting at one place in a single primitive.

We want to solve these with weighted generating functions indexed by some nice combinatorial objects.

Chord diagrams

A **chord diagram** C of size n is a perfect matching of $\{1, 2, \ldots, 2n\}$.

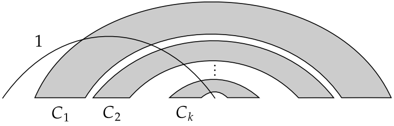


The **directed intersection graph** $G(C)$ has the chords as vertices and two chords are adjacent if and only if they cross.

A chord $c \in C$ is **terminal** if it has no outgoing edges in $G(C)$.

A diagram is **1-terminal** if it has exactly one terminal chord.

Intersection order:



The solution

There are exactly two induced cycle diagrams, the **top cycle** and **bottom cycle**:

Theorem

The functional equation $G(x,y) = xL_{div}(\phi(G(x,y)))$ *is uniquely solved by*

$$G(x,y) = \sum_{C \in \mathcal{C}_{top}} f_C \phi_C x^{|C|} L_{div}(y^{t_1(C)-1}),$$

where $\mathcal{C}_{top}$ *is the set of top-cycle-free chord diagrams and*

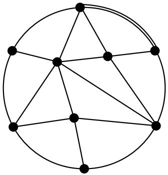
$$f_C = f_0^{|C|-k} f_{t_2(C)-t_1(C)} f_{t_3(C)-t_2(C)} \cdots f_{t_k(C)-t_{k-1}(C)}.$$

Counting top-cycle-free chord diagrams

This motivates determining the number of top-cycle-free diagrams of size n and $t_1 = k$.

To do that, it suffices to find a bijection to some other combinatorial objects that have already been counted.

A **triangulation** T is a plane graph in which every bounded face is a triangle. We root T at a distinguished boundary edge to get **rooted triangulations**.



Counting top-cycle-free chord diagrams

Theorem (Brown)

The number of rooted triangulations with n interior vertices and $m+3$ exterior vertices is

$$\frac{2(2m+3)!(4n+2m+1)!}{m!(m+2)!n!(3n+2m+3)!}.$$

Theorem

There exists a bijection between

- *connected top-cycle-free diagrams with n chords and $t_1 = k$, and*
- *rooted triangulations with $n-k$ interior vertices and $k+1$ exterior vertices.*

Thank you!