The Euler characteristic of $\mathrm{Out}(F_n)$ and the Hopf algebra of graphs

Michael Borinsky, Nikhef

November 20, IHES

Algebraic Structures in Perturbative Quantum Field Theory

joint work with Karen Vogtmann

arXiv:1907.03543

Happy birthday Dirk!

Introduction I: Groups

Automorphisms of groups

- Take a group G

Automorphisms of groups

- Take a group G
- An automorphism of G, $\rho \in \mathrm{Aut}(G)$ is a bijection

$$\rho : G \to G$$

$$\text{such that } \rho(x \cdot y) = \rho(x) \cdot \rho(y) \text{ for all } x, y \in G$$

Automorphisms of groups

- Take a group G
- An automorphism of G, $\rho \in \mathrm{Aut}(G)$ is a bijection

$$\rho : G \to G$$

$$\text{such that } \rho(x \cdot y) = \rho(x) \cdot \rho(y) \text{ for all } x, y \in G$$

- Normal subgroup: $\mathrm{Inn}(G) \triangleleft \mathrm{Aut}(G)$, the inner automorphisms.

Automorphisms of groups

- Take a group G
- An automorphism of G, $\rho \in \mathrm{Aut}(G)$ is a bijection

$$\rho : G \to G$$

$$\text{such that } \rho(x \cdot y) = \rho(x) \cdot \rho(y) \text{ for all } x, y \in G$$

- Normal subgroup: $\mathrm{Inn}(G) \triangleleft \mathrm{Aut}(G)$, the inner automorphisms.
- We have, $\rho_h \in \mathrm{Inn}(G)$

$$\begin{aligned} \rho_h : & G \to G, \\ & g \mapsto h^{-1} g h \end{aligned}$$

for each $h \in G$.

Automorphisms of groups

- Take a group G
- An automorphism of G, $\rho \in \text{Aut}(G)$ is a bijection

$$\rho : G \to G$$

$$\text{such that } \rho(x \cdot y) = \rho(x) \cdot \rho(y) \text{ for all } x, y \in G$$

- Normal subgroup: $\text{Inn}(G) \triangleleft \text{Aut}(G)$, the inner automorphisms.
- We have, $\rho_h \in \text{Inn}(G)$

$$\begin{aligned} \rho_h : & G \to G, \\ & g \mapsto h^{-1}gh \end{aligned}$$

  for each $h \in G$.
- Outer automorphisms: $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$

Automorphisms of the free group

- Consider the free group with n generators

$$F_n = \langle a_1, \ldots, a_n \rangle$$

E.g. $a_1 a_3^{-5} a_2 \in F_3$

Automorphisms of the free group

- Consider the free group with n generators

$$F_n = \langle a_1, \ldots, a_n \rangle$$

  E.g. $a_1 a_3^{-5} a_2 \in F_3$
- The group $\mathrm{Out}(F_n)$ is our main object of interest.

Automorphisms of the free group

- Consider the free group with n generators

$$F_n = \langle a_1, \ldots, a_n \rangle$$

  E.g. $a_1 a_3^{-5} a_2 \in F_3$
- The group $\mathrm{Out}(F_n)$ is our main object of interest.
- Generated by

$$\begin{array}{lllll} & a_1 \mapsto a_1 a_2 & a_2 \mapsto a_2 & a_3 \mapsto a_3 & \ldots \\ \text{and} & a_1 \mapsto a_1^{-1} & a_2 \mapsto a_2 & a_3 \mapsto a_3 & \ldots \end{array}$$

  and permutations of the letters.

Mapping class group

- Another example of an outer automorphism group: the mapping class group

Mapping class group

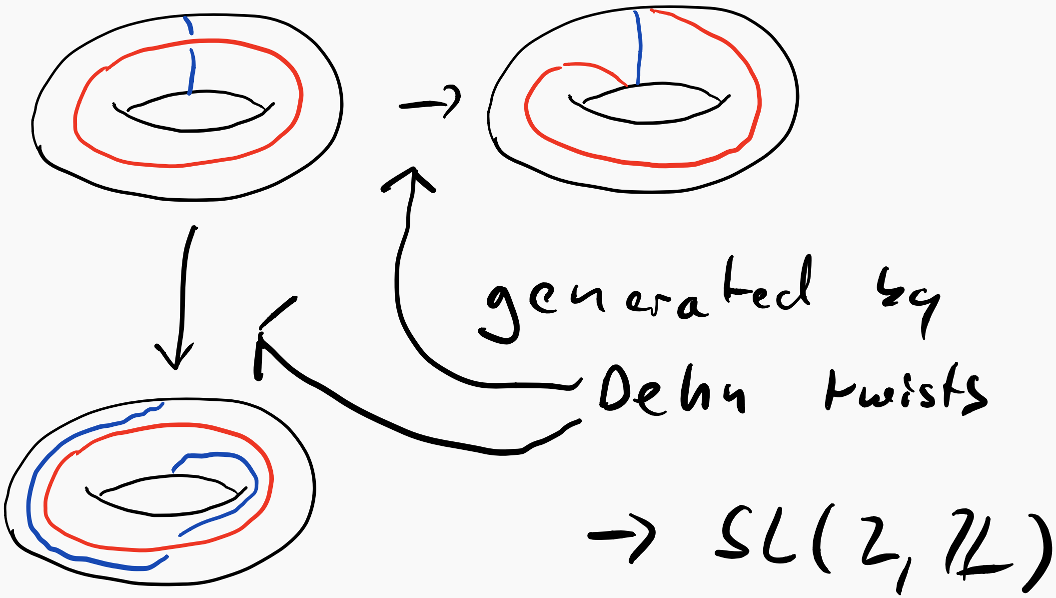
- Another example of an outer automorphism group: the mapping class group
- The group of homeomorphisms of a closed, connected and orientable surface S_g of genus g up to isotopies

$$\mathrm{MCG}(S_g) := \mathrm{Out}(\pi_1(S_g))$$

Example: Mapping class group of the torus

$$\mathsf{MCG}(\mathbb{T}^2) = \mathsf{Out}(\pi_1(\mathbb{T}^2))$$

The group of homeomorphisms $\mathbb{T}^2 \to \mathbb{T}^2$ up to an isotopy:



Introduction II: Spaces

How to study such groups?

How to study groups such as MCG(S) or Out(F_n)?

How to study such groups?

How to study groups such as MCG(S) or Out(F_n)?

Main idea

Realize G as symmetries of some geometric object.

`Due to Stallings, Thurston, Gromov, ... (1970-)`

For the mapping class group: Teichmüller space

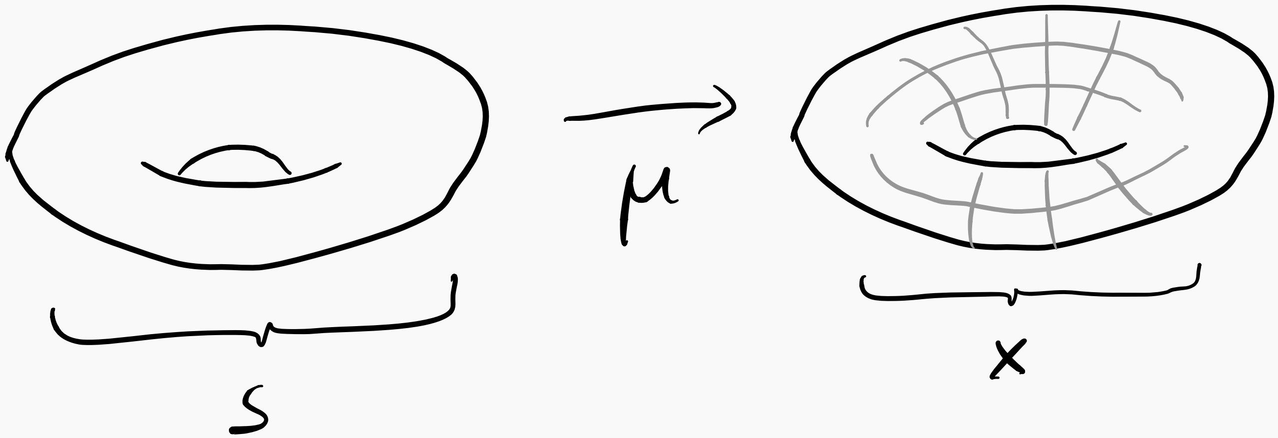
Let S be a closed, connected and orientable surface.

For the mapping class group: Teichmüller space

Let S be a closed, connected and orientable surface.

$\Rightarrow$ A point in Teichmüller space $T(S)$ is a pair, (X, μ)

- A Riemann surface X.
- A *marking*: a homeomorphism $\mu : S \to X$.

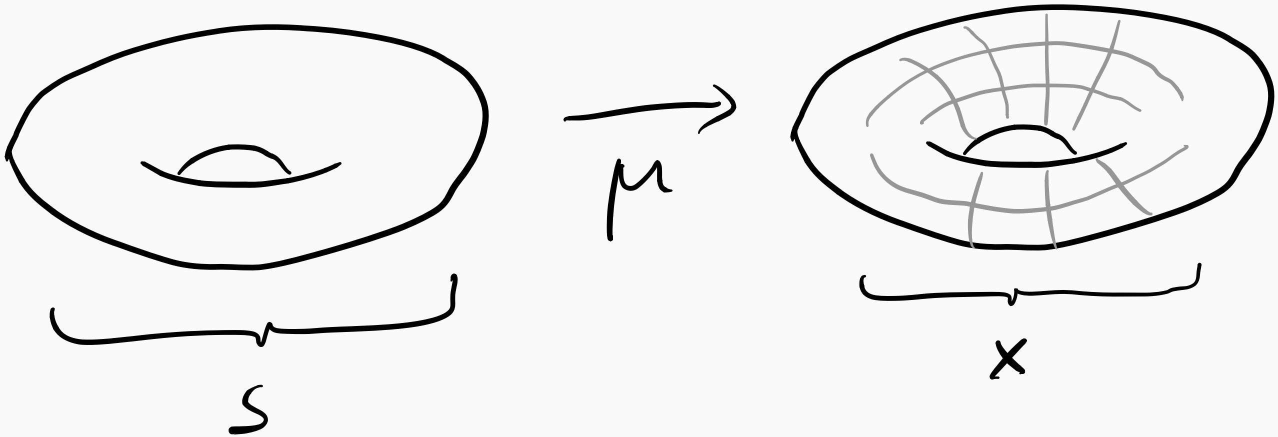


For the mapping class group: Teichmüller space

Let S be a closed, connected and orientable surface.

$\Rightarrow$ A point in Teichmüller space $T(S)$ is a pair, (X, μ)

- A Riemann surface X.
- A *marking*: a homeomorphism $\mu : S \to X$.



MCG(S) acts on $T(S)$ by composing to the marking:

$$(X, \mu) \mapsto (X, \mu \circ g^{-1}) \text{ for some } g \in \text{MCG}(S).$$

For Out(F_n): Outer space

Idea: Mimic previous construction for Out(F_n).

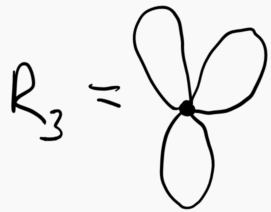
`Culler, Vogtmann (1986)`

For Out(F_n): Outer space

Idea: Mimic previous construction for Out(F_n).

Culler, Vogtmann (1986)

Let R_n be the rose with n petals.

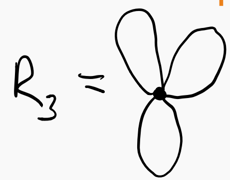


For Out(F_n): Outer space

Idea: Mimic previous construction for Out(F_n).

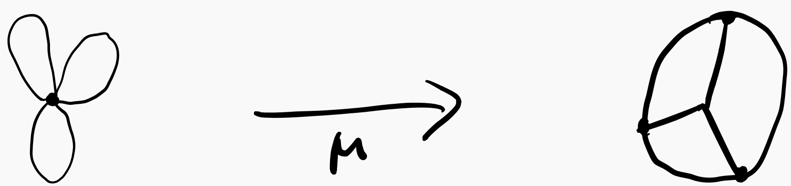
Culler, Vogtmann (1986)

Let R_n be the rose with n petals.



$\Rightarrow$ A point in Outer space $\mathcal{O}_n$ is a pair, (G, μ)

- A connected graph G with a length assigned to each edge.
- A marking: a *homotopy* $\mu : R_n \to G$.

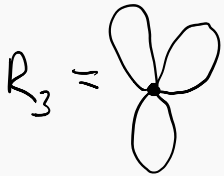


For $\mathrm{Out}(F_n)$: Outer space

Idea: Mimic previous construction for $\mathrm{Out}(F_n)$.

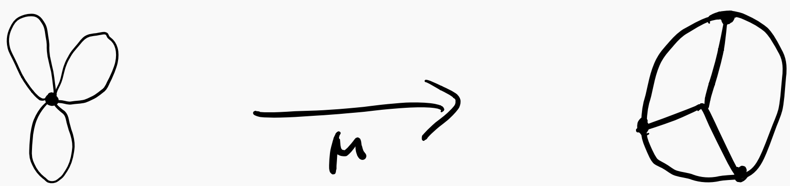
Culler, Vogtmann (1986)

Let R_n be the rose with n petals.



$\Rightarrow$ A point in Outer space $\mathcal{O}_n$ is a pair, (G, μ)

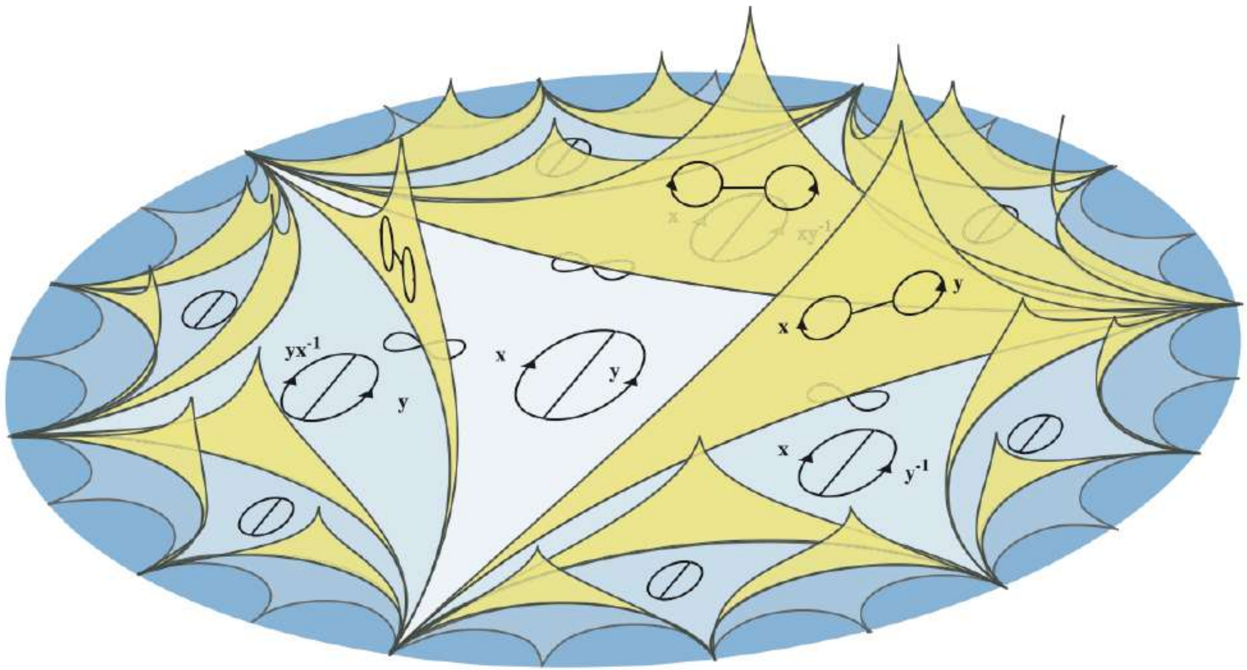
- A connected graph G with a length assigned to each edge.
- A marking: a *homotopy* $\mu : R_n \to G$.



$\mathrm{Out}(F_n)$ acts on $\mathcal{O}_n$ by composing to the marking:

$$(G, \mu) \mapsto (G, \mu \circ g^{-1}) \text{ for some } g \in \mathrm{Out}(F_n) = \mathrm{Out}(\pi_1(R_n)).$$

$\mathcal{O}_2$



Vogtmann 2008

Examples of applications of Outer space

- The group $\mathrm{Out}(F_n)$
- Moduli spaces of punctured surfaces
- Tropical curves
- Invariants of symplectic manifolds
- Classical modular forms
- (Mathematical) physics → Marko's talk
- Graph complexes → Francis' talk

Invariants

Algebraic invariants

- $H_\bullet(\mathrm{Out}(F_n);\mathbb{Q}) \simeq H_\bullet(\mathcal{O}_n \,/\, \mathrm{Out}(F_n);\mathbb{Q}) = H_\bullet(\mathcal{G}_n;\mathbb{Q})$, as $\mathcal{O}_n$ is contractible Culler, Vogtmann (1986).

Algebraic invariants

- $H_\bullet(\mathrm{Out}(F_n);\mathbb{Q}) \simeq H_\bullet(\mathcal{O}_n \,/\, \mathrm{Out}(F_n);\mathbb{Q}) = H_\bullet(\mathcal{G}_n;\mathbb{Q})$, as $\mathcal{O}_n$ is contractible Culler, Vogtmann (1986).

$\Rightarrow$ Study $\mathrm{Out}(F_n)$ using $\mathcal{G}_n$!

Algebraic invariants

- $H_\bullet(\mathrm{Out}(F_n);\mathbb{Q}) \simeq H_\bullet(\mathcal{O}_n \,/\, \mathrm{Out}(F_n);\mathbb{Q}) = H_\bullet(\mathcal{G}_n;\mathbb{Q})$, as $\mathcal{O}_n$ is contractible Culler, Vogtmann (1986).

$\Rightarrow$ Study $\mathrm{Out}(F_n)$ using $\mathcal{G}_n$!

- One simple invariant: Euler characteristic

Further motivation to look at Euler characteristic of $\mathrm{Out}(F_n)$

Consider the abelization map $F_n \to \mathbb{Z}^n$.

Further motivation to look at Euler characteristic of $\mathrm{Out}(F_n)$

Consider the abelization map $F_n \to \mathbb{Z}^n$.

$\Rightarrow$ Induces a group homomorphism

$$\mathrm{Out}(F_n) \to \ \mathrm{Out}(\mathbb{Z}^n)$$

Further motivation to look at Euler characteristic of $\mathrm{Out}(F_n)$

Consider the abelization map $F_n \to \mathbb{Z}^n$.

$\Rightarrow$ Induces a group homomorphism

$$\mathrm{Out}(F_n) \to \underbrace{\mathrm{Out}(\mathbb{Z}^n)}_{=\mathrm{GL}(n,\mathbb{Z})}$$

Further motivation to look at Euler characteristic of $\mathrm{Out}(F_n)$

Consider the abelization map $F_n \to \mathbb{Z}^n$.

$\Rightarrow$ Induces a group homomorphism

$$1 \to \mathcal{T}_n \to \mathrm{Out}(F_n) \to \underbrace{\mathrm{Out}(\mathbb{Z}^n)}_{=\mathrm{GL}(n,\mathbb{Z})} \to 1$$

Further motivation to look at Euler characteristic of $\mathrm{Out}(F_n)$

Consider the abelization map $F_n \to \mathbb{Z}^n$.

$\Rightarrow$ Induces a group homomorphism

$$1 \to \mathcal{T}_n \to \mathrm{Out}(F_n) \to \underbrace{\mathrm{Out}(\mathbb{Z}^n)}_{=\mathrm{GL}(n,\mathbb{Z})} \to 1$$

- $\mathcal{T}_n$ the 'non-abelian' part of $\mathrm{Out}(F_n)$ is interesting.

Further motivation to look at Euler characteristic of $\mathrm{Out}(F_n)$

Consider the abelization map $F_n \to \mathbb{Z}^n$.

$\Rightarrow$ Induces a group homomorphism

$$1 \to \mathcal{T}_n \to \mathrm{Out}(F_n) \to \underbrace{\mathrm{Out}(\mathbb{Z}^n)}_{=\mathrm{GL}(n,\mathbb{Z})} \to 1$$

- $\mathcal{T}_n$ the 'non-abelian' part of $\mathrm{Out}(F_n)$ is interesting.
- By the short exact sequence above

$$\chi(\mathrm{Out}(F_n)) = \chi(\mathrm{GL}(n,\mathbb{Z}))\,\chi(\mathcal{T}_n)$$

Further motivation to look at Euler characteristic of $\mathrm{Out}(F_n)$

Consider the abelization map $F_n \to \mathbb{Z}^n$.

$\Rightarrow$ Induces a group homomorphism

$$1 \to \mathcal{T}_n \to \mathrm{Out}(F_n) \to \underbrace{\mathrm{Out}(\mathbb{Z}^n)}_{=\mathrm{GL}(n,\mathbb{Z})} \to 1$$

- $\mathcal{T}_n$ the 'non-abelian' part of $\mathrm{Out}(F_n)$ is interesting.
- By the short exact sequence above

$$\chi(\mathrm{Out}(F_n)) = \underbrace{\chi(\mathrm{GL}(n,\mathbb{Z}))}_{=0}\chi(\mathcal{T}_n) \quad n \geq 3$$

Further motivation to look at Euler characteristic of $\mathrm{Out}(F_n)$

Consider the abelization map $F_n \to \mathbb{Z}^n$.

$\Rightarrow$ Induces a group homomorphism

$$1 \to \mathcal{T}_n \to \mathrm{Out}(F_n) \to \underbrace{\mathrm{Out}(\mathbb{Z}^n)}_{=\mathrm{GL}(n,\mathbb{Z})} \to 1$$

- $\mathcal{T}_n$ the 'non-abelian' part of $\mathrm{Out}(F_n)$ is interesting.
- By the short exact sequence above

$$\chi(\mathrm{Out}(F_n)) = \underbrace{\chi(\mathrm{GL}(n,\mathbb{Z}))}_{=0}\chi(\mathcal{T}_n) \quad n \geq 3$$

$\Rightarrow$ $\mathcal{T}_n$ does not have finitely-generated homology for $n \geq 3$ if $\chi(\mathrm{Out}(F_n)) \neq 0$.

Conjectures

Conjecture Smillie-Vogtmann (1987)

$$\chi(\mathrm{Out}(F_n)) \neq 0 \text{ for all } n \geq 2$$

$$\text{and } |\chi(\mathrm{Out}(F_n))| \text{ grows exponentially for } n \to \infty.$$

based on initial computations by Smillie-Vogtmann (1987) up to $n \leq 11$. Later strengthened by Zagier (1989) up to $n \leq 100$.

Conjectures

Conjecture Smillie-Vogtmann (1987)

$$\chi(\mathrm{Out}(F_n)) \neq 0 \text{ for all } n \geq 2$$

$$\text{and } |\chi(\mathrm{Out}(F_n))| \text{ grows exponentially for } n \to \infty.$$

based on initial computations by Smillie-Vogtmann (1987) up to $n \leq 11$. Later strengthened by Zagier (1989) up to $n \leq 100$.

Conjecture Magnus (1934)

$\mathcal{T}_n$ is not finitely presentable.

Conjectures

Conjecture Smillie-Vogtmann (1987)

$$\chi(\mathrm{Out}(F_n)) \neq 0 \text{ for all } n \geq 2$$

$$\text{and } |\chi(\mathrm{Out}(F_n))| \text{ grows exponentially for } n \to \infty.$$

based on initial computations by Smillie-Vogtmann (1987) up to $n \leq 11$. Later strengthened by Zagier (1989) up to $n \leq 100$.

Conjecture Magnus (1934)

$\mathcal{T}_n$ is not finitely presentable.

In topological terms, i.e. $\dim(H_2(\mathcal{T}_n)) = \infty$,

which implies that $\mathcal{T}_n$ does not have finitely-generated homology.

Conjectures

Conjecture Smillie-Vogtmann (1987)

$$\chi(\mathrm{Out}(F_n)) \neq 0 \text{ for all } n \geq 2$$

$$\text{and } |\chi(\mathrm{Out}(F_n))| \text{ grows exponentially for } n \to \infty.$$

based on initial computations by Smillie-Vogtmann (1987) up to $n \leq 11$. Later strengthened by Zagier (1989) up to $n \leq 100$.

Conjecture Magnus (1934)

$\mathcal{T}_n$ is not finitely presentable.

In topological terms, i.e. $\dim(H_2(\mathcal{T}_n)) = \infty$,

which implies that $\mathcal{T}_n$ does not have finitely-generated homology.

Theorem Bestvina, Bux, Margalit (2007)

$\mathcal{T}_n$ does not have finitely-generated homology.

Results: $\chi(\mathrm{Out}(F_n)) \neq 0$

Theorem A MB-Vogtmann (2019)

$\chi(\mathrm{Out}(F_n)) < 0$ for all $n \geq 2$

Theorem A MB-Vogtmann (2019)

$$\chi(\mathrm{Out}(F_n)) < 0 \text{ for all } n \geq 2$$

$$\chi(\mathrm{Out}(F_n)) \sim -\frac{1}{\sqrt{2\pi}} \frac{\Gamma(n-3/2)}{\log^2 n} \text{ as } n \to \infty.$$

Theorem A MB-Vogtmann (2019)

$$\chi(\mathrm{Out}(F_n)) < 0 \text{ for all } n \geq 2$$

$$\chi(\mathrm{Out}(F_n)) \sim -\frac{1}{\sqrt{2\pi}}\frac{\Gamma(n-3/2)}{\log^2 n} \text{ as } n \to \infty.$$

which settles the initial conjecture by Smillie-Vogtmann (1987). Immediate questions:

Theorem A MB-Vogtmann (2019)

$$\chi(\mathrm{Out}(F_n)) < 0 \text{ for all } n \geq 2$$

$$\chi(\mathrm{Out}(F_n)) \sim -\frac{1}{\sqrt{2\pi}} \frac{\Gamma(n - 3/2)}{\log^2 n} \text{ as } n \to \infty.$$

which settles the initial conjecture by Smillie-Vogtmann (1987). Immediate questions:

$\Rightarrow$ Huge amount of unstable homology in odd dimensions.

Theorem A MB-Vogtmann (2019)

$$\chi(\mathrm{Out}(F_n)) < 0 \text{ for all } n \geq 2$$

$$\chi(\mathrm{Out}(F_n)) \sim -\frac{1}{\sqrt{2\pi}} \frac{\Gamma(n-3/2)}{\log^2 n} \text{ as } n \to \infty.$$

which settles the initial conjecture by Smillie-Vogtmann (1987). Immediate questions:

$\Rightarrow$ Huge amount of unstable homology in odd dimensions.

- Only one odd-dimensional class known Bartholdi (2016).

Theorem A MB-Vogtmann (2019)

$$\chi(\mathrm{Out}(F_n)) < 0 \text{ for all } n \geq 2$$

$$\chi(\mathrm{Out}(F_n)) \sim -\frac{1}{\sqrt{2\pi}} \frac{\Gamma(n-3/2)}{\log^2 n} \text{ as } n \to \infty.$$

which settles the initial conjecture by Smillie-Vogtmann (1987). Immediate questions:

$\Rightarrow$ Huge amount of unstable homology in odd dimensions.

- Only one odd-dimensional class known Bartholdi (2016).
- Where does all this homology come from?

This Theorem A follows from an implicit expression for $\chi(\mathrm{Out}(F_n))$:

This Theorem A follows from an implicit expression for $\chi(\mathrm{Out}(F_n))$:

Theorem B MB-Vogtmann (2019)

$$\sqrt{2\pi}e^{-N}N^N \sim \sum_{k\geq 0} a_k(-1)^k\Gamma(N+1/2-k) \text{ as } N\to\infty$$

$$\text{where } \sum_{k\geq 0} a_k z^k = \exp\left(\sum_{n\geq 0}\chi(\mathrm{Out}(F_{n+1}))z^n\right)$$

This Theorem A follows from an implicit expression for $\chi(\mathrm{Out}(F_n))$:

Theorem B MB-Vogtmann (2019)

$$\sqrt{2\pi}e^{-N}N^N \sim \sum_{k\geq 0} a_k(-1)^k\Gamma(N+1/2-k) \text{ as } N\to\infty$$

$$\text{where } \sum_{k\geq 0} a_k z^k = \exp\left(\sum_{n\geq 0}\chi(\mathrm{Out}(F_{n+1}))z^n\right)$$

$\Rightarrow$ $\chi(\mathrm{Out}(F_n))$ are the coefficients of an asymptotic expansion.

This Theorem A follows from an implicit expression for $\chi(\mathrm{Out}(F_n))$:

Theorem B MB-Vogtmann (2019)

$$\sqrt{2\pi} e^{-N} N^N \sim \sum_{k \geq 0} a_k (-1)^k \Gamma(N + 1/2 - k) \text{ as } N \to \infty$$

$$\text{where } \sum_{k \geq 0} a_k z^k = \exp\left(\sum_{n \geq 0} \chi(\mathrm{Out}(F_{n+1})) z^n \right)$$

$\Rightarrow$ $\chi(\mathrm{Out}(F_n))$ are the coefficients of an asymptotic expansion.

- An analytic argument is needed to prove Theorem A from Theorem B.

This Theorem A follows from an implicit expression for $\chi(\mathrm{Out}(F_n))$:

Theorem B MB-Vogtmann (2019)

$$\sqrt{2\pi}e^{-N}N^N \sim \sum_{k\geq 0} a_k(-1)^k\Gamma(N+1/2-k) \text{ as } N\to\infty$$

$$\text{where } \sum_{k\geq 0} a_k z^k = \exp\left(\sum_{n\geq 0}\chi(\mathrm{Out}(F_{n+1}))z^n\right)$$

$\Rightarrow$ $\chi(\mathrm{Out}(F_n))$ are the coefficients of an asymptotic expansion.

- An analytic argument is needed to prove Theorem A from Theorem B.
- In this talk: Focus on proof of Theorem B

Analogy to the mapping class group

Harer-Zagier formula for $\chi(\mathrm{MCG}(S_g))$

Similar result for the mapping class group/moduli space of curves:

Harer-Zagier formula for $\chi(\mathrm{MCG}(S_g))$

Similar result for the mapping class group/moduli space of curves:

Theorem Harer-Zagier (1986)

$$\chi(\mathcal{M}_g) = \chi(\mathrm{MCG}(S_g)) = \frac{B_{2g}}{4g(g-1)} \qquad g \geq 2$$

Harer-Zagier formula for $\chi(\mathrm{MCG}(S_g))$

Similar result for the mapping class group/moduli space of curves:

Theorem Harer-Zagier (1986)

$$\chi(\mathcal{M}_g) = \chi(\mathrm{MCG}(S_g)) = \frac{B_{2g}}{4g(g-1)} \qquad g \geq 2$$

- Original proof by Harer and Zagier in 1986.

Harer-Zagier formula for $\chi(\mathrm{MCG}(S_g))$

Similar result for the mapping class group/moduli space of curves:

Theorem Harer-Zagier (1986)

$$\chi(\mathcal{M}_g) = \chi(\mathrm{MCG}(S_g)) = \frac{B_{2g}}{4g(g-1)} \qquad g \geq 2$$

- Original proof by Harer and Zagier in 1986.
- Alternative proof using topological field theory (TFT) by Penner (1988).

Harer-Zagier formula for $\chi(\mathrm{MCG}(S_g))$

Similar result for the mapping class group/moduli space of curves:

Theorem Harer-Zagier (1986)

$$\chi(\mathcal{M}_g) = \chi(\mathrm{MCG}(S_g)) = \frac{B_{2g}}{4g(g-1)} \qquad g \geq 2$$

- Original proof by Harer and Zagier in 1986.
- Alternative proof using topological field theory (TFT) by Penner (1988).
- Simplified proof by Kontsevich (1992) based on TFT's.

Harer-Zagier formula for $\chi(\mathrm{MCG}(S_g))$

Similar result for the mapping class group/moduli space of curves:

Theorem Harer-Zagier (1986)

$$\chi(\mathcal{M}_g) = \chi(\mathrm{MCG}(S_g)) = \frac{B_{2g}}{4g(g-1)} \qquad g \geq 2$$

- Original proof by Harer and Zagier in 1986.
- Alternative proof using topological field theory (TFT) by Penner (1988).
- Simplified proof by Kontsevich (1992) based on TFT's.

$\Rightarrow$ Kontsevich's proof served as a blueprint for $\chi(\mathrm{Out}(F_n))$.

Kontsevich's argument

Kontsevich's argument

- We have the identity by Kontsevich (1992):

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \sum_{\text{connected graphs G}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut} G|} z^{\chi(G)}.$$

Kontsevich's argument

- We have the identity by Kontsevich (1992):

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \sum_{\text{connected graphs G}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut} G|} z^{\chi(G)}.$$

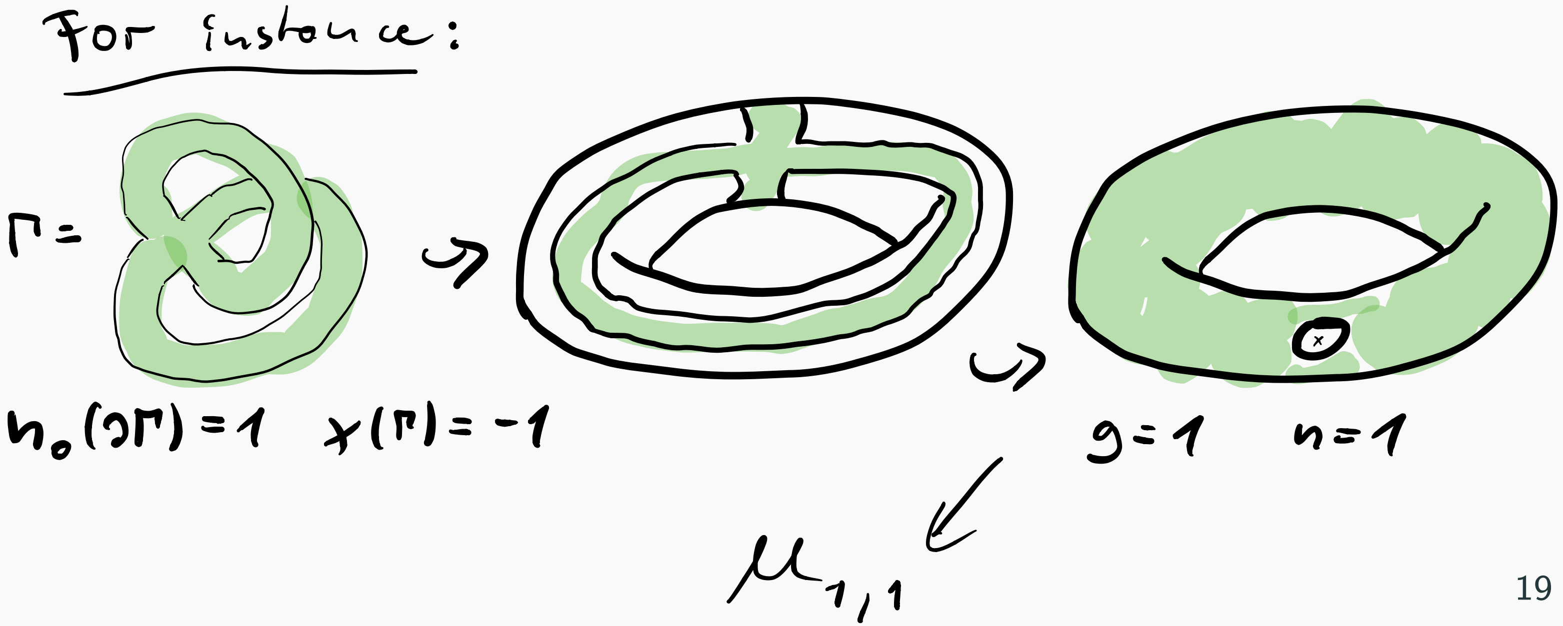
- Kontsevich proved this using a combinatorial model of $\mathcal{M}_{g,n}$ by Penner (1986) based on ribbon graphs.

Kontsevich's argument

- We have the identity by Kontsevich (1992):

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \sum_{\text{connected graphs G}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut} G|} z^{\chi(G)}.$$

- Kontsevich proved this using a combinatorial model of $\mathcal{M}_{g,n}$ by Penner (1986) based on ribbon graphs.



Kontsevich's argument

- We have the identity by Kontsevich (1992):

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \sum_{\text{connected graphs G}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut} G|} z^{\chi(G)}.$$

- Kontsevich proved this using a combinatorial model of $\mathcal{M}_{g,n}$ by Penner (1986) based on ribbon graphs.
- The expression on the right hand side can be evaluated using a 'topological field theory':

$$\sum_{\text{connected graphs G}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut} G|} z^{\chi(G)} = \log\left(\frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)} dx\right)$$

$$= \sum_{k\geq 1} \frac{\zeta(-k)}{-k} z^{-k}$$

Kontsevich's argument

- We have the identity by Kontsevich (1992):

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \sum_{\text{connected graphs G}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut} G|} z^{\chi(G)}.$$

- Kontsevich proved this using a combinatorial model of $\mathcal{M}_{g,n}$ by Penner (1986) based on ribbon graphs.
- The expression on the right hand side can be evaluated using a 'topological field theory':

$$\sum_{\text{connected graphs G}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut} G|} z^{\chi(G)} = \log\left(\frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)} dx\right)$$

$$= \sum_{k \geq 1} \frac{\zeta(-k)}{-k} z^{-k}$$

- The formula for $\chi(\mathcal{M}_{g,n})$ follows via the short exact sequence

$$1 \to \pi_1(S_{g,n}) \to \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n} \to 1$$

Kontsevich's argument

- We have the identity by Kontsevich (1992):

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \sum_{\text{connected graphs G}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut} G|} z^{\chi(G)}.$$

- Kontsevich proved this using a combinatorial model of $\mathcal{M}_{g,n}$ by Penner (1986) based on ribbon graphs.
- The expression on the right hand side can be evaluated using a 'topological field theory':

$$\sum_{\text{connected graphs G}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut} G|} z^{\chi(G)} = \log\left(\frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)} dx\right)$$

$$= \sum_{k\geq 1} \frac{\zeta(-k)}{-k} z^{-k}$$

- The formula for $\chi(\mathcal{M}_{g,n})$ follows via the short exact sequence

$$1 \to \pi_1(S_{g,n}) \to \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n} \to 1$$

An algebraic viewpoint

An algebraic viewpoint

- Let $\mathcal{H}$ be the $\mathbb{Q}$-vector space spanned by a set of graphs:

$\mathcal{H} = \mathbb{Q} \underbrace{\emptyset}_{=:\mathbb{I}} + \mathbb{Q}$  $+ \mathbb{Q}$  $+ \dots$

An algebraic viewpoint

- Let $\mathcal{H}$ be the $\mathbb{Q}$-vector space spanned by a set of graphs:

$\mathcal{H} = \mathbb{Q}\underbrace{\;\emptyset\;}_{=:\mathbb{I}} + \mathbb{Q}$  $+ \mathbb{Q}$  $+ \ldots$

- Here: only connected graphs with 3-or-higher-valent vertices.

An algebraic viewpoint

- Let $\mathcal{H}$ be the $\mathbb{Q}$-vector space spanned by a set of graphs:

$\mathcal{H} = \mathbb{Q} \underbrace{\emptyset}_{=:\mathbb{I}} + \mathbb{Q}$  $+ \mathbb{Q}$  $+ \ldots$

- Here: only connected graphs with 3-or-higher-valent vertices.

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \phi(\mathcal{X})$$

where

$$\mathcal{X} := \sum_{G} \frac{G}{|\operatorname{Aut} G|} z^{\chi(G)} \in \mathcal{H}[[z^{-1}]]$$

and $\phi : \mathcal{H} \to \mathbb{Q}, G \to (-1)^{|V_G|}$

An algebraic viewpoint

- Let $\mathcal{H}$ be the $\mathbb{Q}$-vector space spanned by a set of graphs:

$\mathcal{H} = \mathbb{Q}\underbrace{\emptyset}_{=:\mathbb{I}} + \mathbb{Q}$  $+ \mathbb{Q}$  $+ \dots$

- Here: only connected graphs with 3-or-higher-valent vertices.

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \phi(\mathcal{X})$$

where

$$\mathcal{X} := \sum_{G} \frac{G}{|\operatorname{Aut} G|} z^{\chi(G)} \in \mathcal{H}[[z^{-1}]]$$

and $\phi : \mathcal{H} \to \mathbb{Q}$, $G \to (-1)^{|V_G|}$

$\Rightarrow$ ϕ is very simple and easy to handle via topological field theory.

- For $\mathrm{Out}(F_n)$, we find that

$$\sum_{n\geq 1} \chi(\mathrm{Out}(F_{n+1}))z^{-n} = \tau(\mathcal{X})$$

with $\mathcal{X}$ as before and

$$\tau : \mathcal{H} \to \mathbb{Q}, G \to \sum_{f\subset G} (-1)^{|E_f|}$$

where the sum is over all forests (acyclic subgraphs) of G.

- For $\mathrm{Out}(F_n)$, we find that

$$\sum_{n\geq 1} \chi(\mathrm{Out}(F_{n+1}))z^{-n} = \tau(\mathcal{X})$$

with $\mathcal{X}$ as before and

$$\tau : \mathcal{H} \to \mathbb{Q}, G \to \sum_{f\subset G} (-1)^{|E_f|}$$

where the sum is over all forests (acyclic subgraphs) of G.

$\Rightarrow$ Not directly approachable with a TFT...

- For $\mathrm{Out}(F_n)$, we find that

$$\sum_{n\geq 1} \chi(\mathrm{Out}(F_{n+1}))z^{-n} = \tau(\mathcal{X})$$

with $\mathcal{X}$ as before and

$$\tau : \mathcal{H} \to \mathbb{Q}, G \to \sum_{f\subset G} (-1)^{|E_f|}$$

where the sum is over all forests (acyclic subgraphs) of G.

$\Rightarrow$ Not directly approachable with a TFT...

- The necessary combinatorial model is the ‘forest collapse’ construction by Culler-Vogtmann (1986).

The Hopf algebra of graphs

The Hopf algebra of graphs

- With disjoint union of graphs $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}, G_1 \otimes G_2 \mapsto G_1 \uplus G_2$ as multiplication, the empty graph $\emptyset$ associated with the neutral element $\mathbb{I}$,

The Hopf algebra of graphs

- With disjoint union of graphs $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}, G_1 \otimes G_2 \mapsto G_1 \uplus G_2$ as multiplication, the empty graph $\emptyset$ associated with the neutral element $\mathbb{I}$,
- and the coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$,

$$\Delta : G \mapsto \sum_{\substack{g \subset G \\ \text{bridgeless } g}} g \otimes G/g,$$

where the sum is over all *bridgeless* subgraphs,

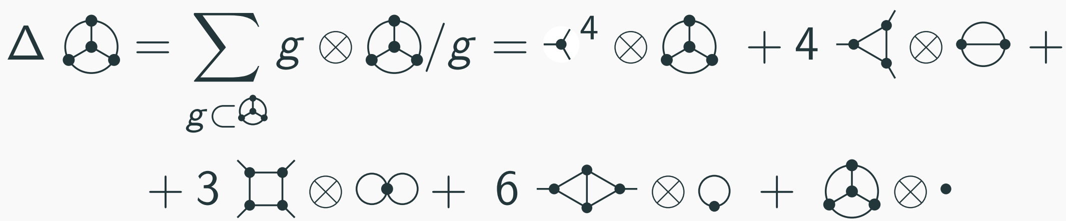
The Hopf algebra of graphs

- With disjoint union of graphs $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}, G_1 \otimes G_2 \mapsto G_1 \uplus G_2$ as multiplication, the empty graph $\emptyset$ associated with the neutral element $\mathbb{I}$,
- and the coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$,

$$\Delta : G \mapsto \sum_{\substack{g \subset G \\ \text{bridgeless } g}} g \otimes G/g,$$

where the sum is over all *bridgeless* subgraphs,

- the vector space $\mathcal{H}$ becomes the *core* Hopf algebra of graphs Kreimer (2009), which is closely related to the Hopf algebra of renormalization in quantum field theory.



- Characters, i.e. linear maps $\psi : \mathcal{H} \to \mathcal{A}$ which fulfill $\psi(\mathbb{I}) = \mathbb{I}_{\mathcal{A}}$ form a group under the convolution product,

$$\psi \star \mu = m \circ (\psi \otimes \mu) \circ \Delta$$

- Characters, i.e. linear maps $\psi : \mathcal{H} \to \mathcal{A}$ which fulfill $\psi(\mathbb{I}) = \mathbb{I}_{\mathcal{A}}$ form a group under the convolution product,

$$\psi \star \mu = m \circ (\psi \otimes \mu) \circ \Delta$$

Theorem MB-Vogtmann (2019)

The map ϕ associated to $\mathcal{M}_{g,n}$ and the map τ associated to $\mathrm{Out}(F_n)$ are mutually inverse elements under this group:

$$\tau = \phi^{\star -1} \qquad \phi = \tau^{\star -1}$$

- Characters, i.e. linear maps $\psi : \mathcal{H} \to \mathcal{A}$ which fulfill $\psi(\mathbb{I}) = \mathbb{I}_{\mathcal{A}}$ form a group under the convolution product,

$$\psi \star \mu = m \circ (\psi \otimes \mu) \circ \Delta$$

Theorem `MB-Vogtmann` (2019)

The map ϕ associated to $\mathcal{M}_{g,n}$ and the map τ associated to $\mathrm{Out}(F_n)$ are mutually inverse elements under this group:

$$\tau = \phi^{\star -1} \qquad \phi = \tau^{\star -1}$$

- That means τ is the *renormalized* version of ϕ.

- Recall that $\chi(\mathcal{M}_{g,n})$ is *explicitly* encoded by a TFT:

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \phi(\mathcal{X}) = \log\left(\frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)} dx\right)$$

- Recall that $\chi(\mathcal{M}_{g,n})$ is *explicitly* encoded by a TFT:

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \phi(\mathcal{X}) = \log\left(\frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)} dx\right)$$

- The duality between ϕ and τ implies that $\chi(\mathrm{Out}(F_n))$ is encoded by the *renormalization* of the same TFT:

$$0 = \log\left(\frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)+\frac{x}{2}+T(-ze^x)} dx\right)$$

where $T(z) = \sum_{n\geq 1} \chi(\mathrm{Out}(F_{n+1}))z^{-n}$.

- Recall that $\chi(\mathcal{M}_{g,n})$ is *explicitly* encoded by a TFT:

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \phi(\mathcal{X}) = \log\left(\frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)} dx\right)$$

- The duality between ϕ and τ implies that $\chi(\mathrm{Out}(F_n))$ is encoded by the *renormalization* of the same TFT:

$$0 = \log\left(\frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)+\frac{x}{2}+T(-ze^x)} dx\right)$$

where $T(z) = \sum_{n\geq 1} \chi(\mathrm{Out}(F_{n+1}))z^{-n}$.

- This TFT encodes the statement of Theorem 2 and gives an *implicit* encoding of the numbers $\chi(\mathrm{Out}(F_n))$.

Outlook: The naive Euler characteristic

Outlook: The naive Euler characteristic

The ‘naive’ Euler characteristic

$$\widetilde{\chi}(\operatorname{Out} F_n) = \sum_k (-1)^k \dim H_k(\operatorname{Out} F_n; \mathbb{Q})$$

is harder to analyse than the rational Euler characteristic.

Outlook: The naive Euler characteristic

The ‘naive’ Euler characteristic

$$\widetilde{\chi}(\mathrm{Out}\,F_n) = \sum_k (-1)^k \dim H_k(\mathrm{Out}\,F_n; \mathbb{Q})$$

is harder to analyse than the rational Euler characteristic.

$$\widetilde{\chi}(\mathrm{Out}\,F_n) = \sum_{\langle\sigma\rangle} \chi(C_\sigma)$$

sum over conjugacy elements of finite order in $\mathrm{Out}\,F_n$ and C_σ is the centralizer corresponding σ Brown (1982).

Outlook: The naive Euler characteristic

The 'naive' Euler characteristic

$$\widetilde{\chi}(\mathrm{Out}\, F_n) = \sum_k (-1)^k \dim H_k(\mathrm{Out}\, F_n; \mathbb{Q})$$

is harder to analyse than the rational Euler characteristic.

$$\widetilde{\chi}(\mathrm{Out}\, F_n) = \sum_{\langle \sigma \rangle} \chi(C_\sigma)$$

sum over conjugacy elements of finite order in $\mathrm{Out}\, F_n$ and C_σ is the centralizer corresponding σ Brown (1982).

⇒ Preliminary investigations on C_σ indicate that

$$\lim_{n \to \infty} \frac{\widetilde{\chi}(\mathrm{Out}\, F_n)}{\chi(\mathrm{Out}\, F_n)} = c > 0$$

A missing piece:

complex	rational: χ	integral: e
associative/$\mathcal{M}_{g,n}$	Harer, Zagier 1986	Harer, Zagier 1986
commutative	Kontsevich 1993	Willwacher, Živković 2
Lie/Out(F_n)	Kontsevich 1993	?

Lie/Out(F_n) integral case $e(\mathrm{Out}(F_n))$ only known for $n \leq 11$.
Thanks to a supercomputer calculation by Morita 2014.

Missing Euler characteristic of the Lie case

Theorem MB, Vogtmann 2020 (in preparation)

$$\prod_{n\geq 1}\left(\frac{1}{1-z^{-n}}\right)^{e(\mathrm{Out}(F_{n+1}))} =$$

$$\left(\prod_{k\geq 1}\int \frac{\mathrm{d}\, x_k}{\sqrt{2\pi k/z^k}}\right) e^{\sum_{k\geq 1}\frac{z^k}{k}\left(c_k-\frac{c_{2k}}{2}+\frac{c_k^2}{2}-\frac{x_k^2}{2}-(1+c_k)\sum_{j\geq 1}\frac{\mu(j)}{j}\log(1+c_{jk})\right)}$$

where $c_{2k} = x_{2k} + z^{-k}$ and $c_{2k-1} = x_{2k-1}$ for all $k \geq 1$.

$\Rightarrow$ 'Explicit' formula for $e(\mathrm{Out}(F_n))$.

(Can be 'easily' computed up to $n = 40$ vs 11 known values.)

Contributions and open questions

Short summary:

Contributions and open questions

Short summary:

- $\chi(\mathrm{Out}(F_n)) \neq 0$

Contributions and open questions

Short summary:

- $\chi(\mathrm{Out}(F_n)) \neq 0$

Open questions:

- The rapid growth of $\chi(\mathrm{Out}(F_n))$ indicates that there is much unstable homology.

Contributions and open questions

Short summary:

- $\chi(\mathrm{Out}(F_n)) \neq 0$

Open questions:

- The rapid growth of $\chi(\mathrm{Out}(F_n))$ indicates that there is much unstable homology.
  Preliminary investigations into the $\widetilde{\chi}(\mathrm{Out}(F_n))$ support this.

Contributions and open questions

Short summary:

- $\chi(\mathrm{Out}(F_n)) \neq 0$

Open questions:

- The rapid growth of $\chi(\mathrm{Out}(F_n))$ indicates that there is much unstable homology.
  Preliminary investigations into the $\widetilde{\chi}(\mathrm{Out}(F_n))$ support this.
  What generates it?

Contributions and open questions

Short summary:

- $\chi(\mathrm{Out}(F_n)) \neq 0$

Open questions:

- The rapid growth of $\chi(\mathrm{Out}(F_n))$ indicates that there is much unstable homology.
  Preliminary investigations into the $\widetilde{\chi}(\mathrm{Out}(F_n))$ support this.
  What generates it?
- The TFT analysis indicates a non-trivial 'duality' between $\mathrm{MCG}(S_g)$ and $\mathrm{Out}(F_n)$. Obvious candidate: Koszul duality (?)

Contributions and open questions

Short summary:

- $\chi(\mathrm{Out}(F_n)) \neq 0$

Open questions:

- The rapid growth of $\chi(\mathrm{Out}(F_n))$ indicates that there is much unstable homology.
  Preliminary investigations into the $\widetilde{\chi}(\mathrm{Out}(F_n))$ support this.
  What generates it?
- The TFT analysis indicates a non-trivial 'duality' between $\mathrm{MCG}(S_g)$ and $\mathrm{Out}(F_n)$. . . Koszul duality (?)
- Can renormalized TFT arguments also be used for other groups and for finer invariants? For instance RAAGs or explicit homology groups.

Bonus: Sketch of Kontsevich's TFT proof of the Harer-Zagier formula

Step 1 of Kontsevich's proof

Generalize from $\mathcal{M}_g$ to $\mathcal{M}_{g,n}$, the moduli space of surfaces of genus g and n punctures.

Step 1 of Kontsevich's proof

Generalize from $\mathcal{M}_g$ to $\mathcal{M}_{g,n}$, the moduli space of surfaces of genus g and n punctures.

We can 'forget one puncture':

$$\mathrm{MCG}(S_{g,n+1}) \to \mathrm{MCG}(S_{g,n})$$

Step 1 of Kontsevich's proof

Generalize from $\mathcal{M}_g$ to $\mathcal{M}_{g,n}$, the moduli space of surfaces of genus g and n punctures.

We can 'forget one puncture':

$$1 \to \pi_1(S_{g,n}) \to \mathrm{MCG}(S_{g,n+1}) \to \mathrm{MCG}(S_{g,n}) \to 1$$

$$\Rightarrow \chi(\mathrm{MCG}(S_{g,n+1})) = \chi(\mathcal{M}_{g,n+1}) = \chi(\pi_1(S_{g,n}))\,\chi(\mathcal{M}_{g,n})$$

Step 1 of Kontsevich's proof

Generalize from $\mathcal{M}_g$ to $\mathcal{M}_{g,n}$, the moduli space of surfaces of genus g and n punctures.

We can 'forget one puncture':

$$1 \to \pi_1(S_{g,n}) \to \mathrm{MCG}(S_{g,n+1}) \to \mathrm{MCG}(S_{g,n}) \to 1$$

$$\Rightarrow \chi(\mathrm{MCG}(S_{g,n+1})) = \chi(\mathcal{M}_{g,n+1}) = \underbrace{\chi(\pi_1(S_{g,n}))}_{=2-2g-n} \chi(\mathcal{M}_{g,n})$$

Step 2 of Kontsevich's proof

- Use a combinatorial model for $\mathcal{M}_{g,n}$

Step 2 of Kontsevich's proof

- Use a combinatorial model for $\mathcal{M}_{g,n}$

$\Rightarrow$ Ribbon graphs Penner (1986)

Step 2 of Kontsevich's proof

- Use a combinatorial model for $\mathcal{M}_{g,n}$

$\Rightarrow$ Ribbon graphs Penner (1986)

Every point in $\mathcal{M}_{g,n}$ can be associated with a ribbon graph Γ such that

- Γ has n boundary components: $h_0(\partial\Gamma) = n$
- $\chi(\Gamma) = |V_\Gamma| - |E_\Gamma| = 2 - 2g - n$.

Step 2 of Kontsevich's proof

- Use a combinatorial model for $\mathcal{M}_{g,n}$

$\Rightarrow$ Ribbon graphs Penner (1986)

Every point in $\mathcal{M}_{g,n}$ can be associated with a ribbon graph Γ such that

- Γ has n boundary components: $h_0(\partial\Gamma) = n$
- $\chi(\Gamma) = |V_\Gamma| - |E_\Gamma| = 2 - 2g - n$.

$\Rightarrow$ Γ can be interpreted as a surface of genus g with n punctures.

Step 2 of Kontsevich's proof

- Use a combinatorial model for $\mathcal{M}_{g,n}$

$\Rightarrow$ Ribbon graphs Penner (1986)

Every point in $\mathcal{M}_{g,n}$ can be associated with a ribbon graph Γ such that

- Γ has n boundary components: $h_0(\partial\Gamma) = n$
- $\chi(\Gamma) = |V_\Gamma| - |E_\Gamma| = 2 - 2g - n$.

$\Rightarrow$ Γ can be interpreted as a surface of genus g with n punctures.

Step 3 of Kontsevich's proof

$$\chi(\mathcal{M}_{g,n}) = \sum_{\sigma} \frac{(-1)^{\mathsf{dim}(\sigma)}}{|\,\mathsf{Stab}(\sigma)|}$$

Step 3 of Kontsevich's proof

$$\chi(\mathcal{M}_{g,n}) = \sum_{\sigma} \frac{(-1)^{\mathsf{dim}(\sigma)}}{|\,\mathsf{Stab}(\sigma)|}$$

Step 3 of Kontsevich's proof

$$\chi(\mathcal{M}_{g,n}) = \sum_{\sigma} \frac{(-1)^{\dim(\sigma)}}{|\operatorname{Stab}(\sigma)|}$$

$$= \sum_{\substack{\Gamma \\ h_0(\partial\Gamma)=n \\ \chi(\Gamma)=2-2g-n}} \frac{(-1)^{|V_\Gamma|}}{|\operatorname{Aut}\Gamma|}$$

Step 3 of Kontsevich's proof

$$\chi(\mathcal{M}_{g,n}) = \sum_{\sigma} \frac{(-1)^{\mathrm{dim}(\sigma)}}{|\,\mathrm{Stab}(\sigma)|} = \sum_{\substack{\Gamma \\ h_0(\partial\Gamma)=n \\ \chi(\Gamma)=2-2g-n}} \frac{(-1)^{|V_\Gamma|}}{|\,\mathrm{Aut}\,\Gamma|}$$

Used by Penner (1988) to calculate $\chi(\mathcal{M}_g)$ with Matrix models.

Step 4 of Kontsevich's proof

Kontsevich's simplification:

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n}$$

Step 4 of Kontsevich's proof

Kontsevich's simplification:

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \sum_{g,n} \sum_{\substack{\text{ribbon graphs } \Gamma \\ h_0(\partial\Gamma)=n \\ \chi(\Gamma)=2-2g-n}} \frac{(-1)^{|V_\Gamma|}}{|\operatorname{Aut}\Gamma|} \frac{1}{n!} z^{\chi(\Gamma)}$$

Step 4 of Kontsevich's proof

Kontsevich's simplification:

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \sum_{g,n} \sum_{\substack{\text{ribbon graphs } \Gamma \\ h_0(\partial\Gamma)=n \\ \chi(\Gamma)=2-2g-n}} \frac{(-1)^{|V_\Gamma|}}{|\operatorname{Aut}\Gamma|} \frac{1}{n!} z^{\chi(\Gamma)}$$

$$= \sum_{\text{graphs G}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut} G|} z^{\chi(G)}$$

Step 4 of Kontsevich's proof

Kontsevich's simplification:

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \sum_{g,n} \sum_{\substack{\text{ribbon graphs } \Gamma \\ h_0(\partial\Gamma)=n \\ \chi(\Gamma)=2-2g-n}} \frac{(-1)^{|V_\Gamma|}}{|\operatorname{Aut}\Gamma|} \frac{1}{n!} z^{\chi(\Gamma)}$$

$$= \sum_{\text{graphs G}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut} G|} z^{\chi(G)}$$

This is the perturbative series of a simple TFT:

$$= \log\left(\frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)} dx\right)$$

Step 4 of Kontsevich's proof

Kontsevich's simplification:

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \sum_{g,n} \sum_{\substack{\text{ribbon graphs } \Gamma \\ h_0(\partial\Gamma)=n \\ \chi(\Gamma)=2-2g-n}} \frac{(-1)^{|V_\Gamma|}}{|\operatorname{Aut}\Gamma|} \frac{1}{n!} z^{\chi(\Gamma)}$$

$$= \sum_{\text{graphs G}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut} G|} z^{\chi(G)}$$

This is the perturbative series of a simple TFT:

$$= \log\left(\frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)} dx\right)$$

Evaluation is classic (Stirling/Euler-Maclaurin formulas)

$$= \sum_{k\geq 1} \frac{\zeta(-k)}{-k} z^{-k}$$

Last step of Kontsevich's proof

$$\sum_{\substack{g,n \\ 2-2g-n=k}} \frac{\chi(\mathcal{M}_{g,n})}{n!} = \frac{B_{k+1}}{k(k+1)}$$

Last step of Kontsevich's proof

$$\sum_{\substack{g,n \\ 2-2g-n=k}} \frac{\chi(\mathcal{M}_{g,n})}{n!} = \frac{B_{k+1}}{k(k+1)}$$

$\Rightarrow$ recover Harer-Zagier formula using the identity

$$\chi(\mathcal{M}_{g,n+1}) = (2-2g-n)\chi(\mathcal{M}_{g,n})$$

Analogous proof strategy for $\chi(\mathrm{Out}(F_n))$ using renormalized TFTs

Step 1

Generalize from $\mathrm{Out}(F_n)$ to $A_{n,s}$ and from $\mathcal{O}_n$ to $\mathcal{O}_{n,s}$, Outer space of graphs of rank n and s legs.

`Contant, Kassabov, Vogtmann (2011)`

Step 1

Generalize from $\mathrm{Out}(F_n)$ to $A_{n,s}$ and from $\mathcal{O}_n$ to $\mathcal{O}_{n,s}$, Outer space of graphs of rank n and s legs.

`Contant, Kassabov, Vogtmann (2011)`

Forgetting a leg gives the short exact sequence of groups

$$1 \to F_n \to A_{n,s} \to A_{n,s-1} \to 1$$

Step 2

- Use a combinatorial model for $\mathcal{G}_{n,s}$

Step 2

- Use a combinatorial model for $\mathcal{G}_{n,s}$

$\Rightarrow$ graphs with a forest Smillie-Vogtmann (1987):

Step 2

- Use a combinatorial model for $\mathcal{G}_{n,s}$

$\Rightarrow$ graphs with a forest Smillie-Vogtmann (1987):

A point in $\mathcal{G}_{n,s}$ can be associated with a pair of a graph G and a forest $f \subset G$.

Step 2

- Use a combinatorial model for $\mathcal{G}_{n,s}$

$\Rightarrow$ graphs with a forest Smillie-Vogtmann (1987):

A point in $\mathcal{G}_{n,s}$ can be associated with a pair of a graph G and a forest $f \subset G$.

$$(G, f)$$

Step 3

$$\chi(A_{n,s}) = \sum_{\sigma} \frac{(-1)^{\dim(\sigma)}}{|\operatorname{Stab}(\sigma)|}$$

Step 3

$$\chi(A_{n,s}) = \sum_{\sigma} \frac{(-1)^{\dim(\sigma)}}{|\operatorname{Stab}(\sigma)|}$$

Step 3

$$\chi(A_{n,s}) = \sum_{\sigma} \frac{(-1)^{\dim(\sigma)}}{|\,\mathrm{Stab}(\sigma)|}$$

$$= \sum_{\substack{\text{graphs } G \\ \text{with } s \text{ legs} \\ \mathrm{rank}(\pi_1(G))=n}} \sum_{\text{forests } f \subset G} \frac{(-1)^{|E_f|}}{|\,\mathrm{Aut}\, G|}$$

Step 4

Renormalized TFT interpretation MB-Vogtmann (2019):

$$\chi(A_{n,s}) = \sum_{\substack{\text{graphs } G \\ \text{with } s \text{ legs} \\ \text{rank}(\pi_1(G))=n}} \frac{1}{|\operatorname{Aut} G|} \sum_{\text{forests } f\subset G} (-1)^{|E_f|}$$

Step 4

Renormalized TFT interpretation MB-Vogtmann (2019):

$$\chi(A_{n,s}) = \sum_{\substack{\text{graphs } G \\ \text{with } s \text{ legs} \\ \text{rank}(\pi_1(G))=n}} \frac{1}{|\operatorname{Aut} G|} \underbrace{\sum_{\text{forests } f\subset G} (-1)^{|E_f|}}_{=:\tau(G)}$$

Step 4

Renormalized TFT interpretation MB-Vogtmann (2019):

$$\chi(A_{n,s}) = \sum_{\substack{\text{graphs } G \\ \text{with } s \text{ legs} \\ \text{rank}(\pi_1(G))=n}} \frac{1}{|\operatorname{Aut} G|} \underbrace{\sum_{\text{forests } f \subset G} (-1)^{|E_f|}}_{=:\tau(G)}$$

τ fulfills the identities $\tau(\emptyset) = 1$ and

$$\sum_{\substack{g \subset G \\ g \text{ bridgeless}}} \tau(g)(-1)^{|E_{G/g}|} = 0 \qquad \text{for all } G \neq \emptyset$$

Step 4

Renormalized TFT interpretation MB-Vogtmann (2019):

$$\chi(A_{n,s}) = \sum_{\substack{\text{graphs } G \\ \text{with } s \text{ legs} \\ \text{rank}(\pi_1(G))=n}} \frac{1}{|\operatorname{Aut} G|} \underbrace{\sum_{\text{forests } f \subset G} (-1)^{|E_f|}}_{=:\tau(G)}$$

τ fulfills the identities $\tau(\emptyset) = 1$ and

$$\sum_{\substack{g \subset G \\ g \text{ bridgeless}}} \tau(g)(-1)^{|E_{G/g}|} = 0 \qquad \text{for all } G \neq \emptyset$$

$\Rightarrow$ τ is an inverse of a character in a Connes-Kreimer-type renormalization Hopf algebra. Connes-Kreimer (2001)

Step 4

Renormalized TFT interpretation MB-Vogtmann (2019):

$$\chi(A_{n,s}) = \sum_{\substack{\text{graphs } G \\ \text{with } s \text{ legs} \\ \text{rank}(\pi_1(G))=n}} \frac{1}{|\text{Aut } G|} \underbrace{\sum_{\text{forests } f \subset G} (-1)^{|E_f|}}_{=:\tau(G)}$$

τ fulfills the identities $\tau(\emptyset) = 1$ and

$$\sum_{\substack{g \subset G \\ g \text{ bridgeless}}} \tau(g)(-1)^{|E_{G/g}|} = 0 \qquad \text{for all } G \neq \emptyset$$

$\Rightarrow$ τ is an inverse of a character in a Connes-Kreimer-type renormalization Hopf algebra. Connes-Kreimer (2001)

The group invariants $\chi(A_{n,s})$ are encoded in a renormalized TFT.

TFT evaluation

Let $$T(z,x) = \sum_{n,s\geq 0} \chi(A_{n,s}) z^{1-n} \frac{x^s}{s!}$$

TFT evaluation

Let $$T(z,x) = \sum_{n,s\geq 0} \chi(A_{n,s}) z^{1-n} \frac{x^s}{s!}$$

then $$1 = \frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{T(z,x)} dx$$

TFT evaluation

Let $$T(z,x) = \sum_{n,s\geq 0} \chi(A_{n,s}) z^{1-n} \frac{x^s}{s!}$$

then $$1 = \frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{T(z,x)} dx$$

Using the short exact sequence, $1 \to F_n \to A_{n,s} \to A_{n,s-1} \to 1$ results in the action

$$1 = \frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)+\frac{x}{2}+T(-ze^x)} dx$$

where $T(z) = \sum_{n\geq 1} \chi(\mathrm{Out}(F_{n+1})) z^{-n}$.

TFT evaluation

Let $$T(z,x) = \sum_{n,s\geq 0} \chi(A_{n,s}) z^{1-n} \frac{x^s}{s!}$$

then $$1 = \frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{T(z,x)} dx$$

Using the short exact sequence, $1 \to F_n \to A_{n,s} \to A_{n,s-1} \to 1$ results in the action

$$1 = \frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)+\frac{x}{2}+T(-ze^x)} dx$$

where $T(z) = \sum_{n\geq 1} \chi(\mathrm{Out}(F_{n+1})) z^{-n}$.

This gives the implicit result in Theorem B.