Resurgent Trans-series Analysis of Hopf Algebraic Renormalization

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Algebraic Structures in Perturbative Quantum Field Theory

celebrating Dirk Kreimer's 60^{th} birthday

IHES, November 19, 2020

M. Borinsky & GD, 2005.04265; M. Borinsky, GD, M. Meynig, 2020 to appear

O. Costin & GD, 1904.11593, 2003.07451, 2009.01962, ...

[DOE Division of High Energy Physics]

Motivation

- Kreimer-Connes:

[perturbative] QFT renormalisation $\longleftrightarrow$ Hopf algebra stucture

$\Rightarrow$ enables perturbative computations to very high order

- Écalle: resurgent asymptotics

[perturbative] series $\longrightarrow$ [perturbative + nonperturbative] transseries

$\Rightarrow$ nonperturbative physics encoded in perturbative physics

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IDEA: use resurgent trans-series to decode nonperturbative properties of QFT from their perturbative Hopf algebra structure

Resurgent Trans-Series

- Écalle: *resurgent functions* closed under all operations:

(Borel transform) + (analytic continuation) + (Laplace transform)

- common basic trans-series in QM & QFT applications:

$$f(x) \sim \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, x^p}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{1}{x}\right]\right)^k}_{\text{non-perturbative}} \underbrace{\left(\ln\left[x\right]\right)^l}_{\text{logarithm powers}}$$

- *transmonomial elements*: x, $e^{-\frac{1}{x}}$, $\ln(x)$, familiar in QFT
- new: analytic continuation encoded in trans-series
- new: trans-series coefficients $c_{k,l,p}$ are highly correlated
- explored in ODEs, PDEs, difference eqs., QM, matrix models, QFT, string theory, ...

"Resurgence"

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle

fluctuations about different singularities are <u>quantitatively related</u>

Resurgence in QFT ?

review: GD & M. Ünsal (1603.04924)

- $\exists$ many resurgence examples in matrix models and QM, but many results still await rigorous foundation
- renormalisation makes resurgence in quantum field theory more interesting
- recent progress for regularised QFTs and lattice QFT
- here: invoke Hopf algebra structure of perturbative QFT

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Q1: do the Dyson-Schwinger equations contain **all** the information (perturbative & non-perturbative) about a QFT ?

Q2: how can one decode the non-perturbative information ?

Q3: is there a natural Hopf algebraic formulation of the "bridge equations" which relate the perturbative and non-perturbative features ?

Nonlinear ODEs from Dyson-Schwinger Equations

Combinatoric explosion of renormalization tamed by Hopf algebra: 30-loop Padé-Borel resummation

D.J. Broadhurst [1], D. Kreimer [2]

Erwin Schrödinger Institute, A-1090 Wien, Austria

Physics Letters B 475 (2000) 63–70

Exact solutions of Dyson–Schwinger equations for iterated one-loop integrals and propagator-coupling duality

D.J. Broadhurst [a,1], D. Kreimer [b,2]

Nuclear Physics B 600 (2001) 403–422

An Étude in non-linear Dyson–Schwinger Equations*

Dirk Kreimer[a†]
Karen Yeats[b]

Nuclear Physics B (Proc. Suppl.) 160 (2006) 116–121

Nonlinear ODEs from Dyson-Schwinger Equations

- Broadhurst/Kreimer 1999/2000; Kreimer/Yeats 2006:

for certain QFTs the renormalisation group equations can be reduced to coupled nonlinear ODEs for the anomalous dimension in terms of the renormalised coupling

- resurgence is deeply understood for (nonlinear) ODEs (Écalle, Costin, Kruskal, Ramis, Sauzin, Fauvet, ...)
- so this is a natural place to start
- some paradigmatic cases: Wess-Zumino model (Bellon, Schaposnik, Clavier, 2008, 2016, 2018); 4 dim. Yukawa (Borinsky, GD, 2020); 6 dim. ϕ^3 theory (Bellon & Russo, 2020), (Borinsky, GD, Meynig, 2020)
- also related: Maiezza, Vasquez (2019, 2020)
- future goal: gauge theories

Resurgence for Nonlinear ODEs (O. Costin, 1998)

n^{th} order nonlinear ODE: $\mathbf{y}' = \mathbf{f}(z, \mathbf{y}) \quad , \quad z \in \mathbb{C} \quad , \quad \mathbf{y} \in \mathbb{C}^n$

- normal form:

$$\mathbf{y}' = \mathbf{f}_0(z) - \hat{\Lambda}\mathbf{y} - \frac{1}{z}\hat{B}\mathbf{y} + \mathbf{g}(z, \mathbf{y})$$

$\Rightarrow$ resurgent trans-series

$$\mathbf{y} = \mathbf{y}_0 + \sum_{\mathbf{k}} \sigma_1^{k_1} \dots \sigma_n^{k_n} \, e^{-(\mathbf{k}\cdot\lambda)z} \, z^{-\mathbf{k}\cdot\beta} \sum_{l} a_{\mathbf{k};l} \, z^{-l}$$

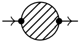
- discontinuities of singularities of Borel transform $\mathcal{B}\mathbf{y}_{\mathbf{k}}$ at $\mathbb{N}\,\lambda_j$ are all related by a network of algebraic "bridge equations"

> *"... a surprising upshot: given one formal solution, (generically) an n-parameter family of solutions can be constructed out of it"*

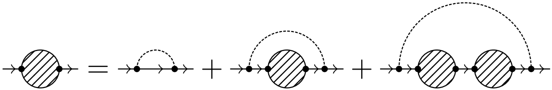
4 dimensional massless Yukawa theory

Broadhurst/Kreimer, 1999

- renormalised fermion self-energy

$$\Sigma(q) :=$$  $$= \not{q}\, \Sigma(q^2)$$

- Dyson-Schwinger equation



$+ \cdots -$ subtractions

- anomalous dimension $\gamma(\alpha)$ ($\alpha \equiv$ renormalised coupling):

$$\gamma(\alpha) = \frac{d}{d \ln q^2} \ln \left(1 - \Sigma(q^2)\right) \Bigg|_{q^2=\mu^2}$$

- renormalisation group $\Rightarrow$ non-linear ODE

$$2\gamma = -\alpha - \gamma^2 + 2\alpha\, \gamma \frac{d}{d\alpha}\gamma$$

Perturbative Solution (rescale: $\gamma(\alpha) := 2\,C\left(-\frac{\alpha}{4}\right)$)

$$\left[C(x)\left(2\,x\,\frac{d}{dx}-1\right)-1\right]C(x)=-x$$

- perturbative solution: $C(x)=\sum_{n=1}^{\infty} C_n x^n$ (OEIS: A000699)

$C_n = [1, 1, 4, 27, 248, 2830, 38232, 593859, 10401712, 202601898, \dots]$

- combinatorics: generating function for "connected chord diagrams"

- large order asymptotics

$$C_n \sim e^{-1}\,\frac{2^{n+\frac{1}{2}}\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{2\pi}}\left(1-\frac{\frac{5}{2}}{2\left(n-\frac{1}{2}\right)}-\frac{\frac{43}{8}}{2^2\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)}-\dots\right)$$

Perturbative Solution (rescale: $\gamma(\alpha) := 2\, C\left(-\frac{\alpha}{4}\right)$)

$$\left[C(x)\left(2\,x\,\frac{d}{dx}-1\right)-1\right]C(x) = -x$$

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- missing boundary condition parameter ?

Écalle: formal series → trans-series : $\qquad C(x) = \sum_{k=0}^{\infty} \sigma^k\, C^{(k)}(x)$

Trans-series Solution

M. Borinsky & GD, 2005.04265

- expand $C(x) = C^{(0)}(x) + \sigma\, C^{(1)}(c) + \sigma^2\, C^{(2)}(x) + \dots$
- $C^{(0)}(x) =$ previous formal perturbative series solution
- linear inhomogeneous equations for $C^{(k)}(x)$ for $k \geq 1$

$$C^{(1)}(x) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{x}}{C^{(0)}(x)} \exp\left[-\frac{\left(C^{(0)}(x)+1\right)^2}{2x}\right]$$

$$\sim \frac{e^{-1/(2x)}}{\sqrt{x}} \frac{e^{-1}}{\sqrt{2\pi}} \left[1 - \frac{5}{2}\, x - \frac{43}{8}\, x^2 - \frac{579}{16}\, x^3 - \dots\right]$$

- resurgence: $C^{(1)}(x)$ expressed in terms of $C^{(0)}(x)$

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- resurgence: $C^{(1)}(x)$ expressed in terms of $C^{(0)}(x)$
- characteristic signature of resurgent structure:

$$C_n^{(0)} \sim e^{-1} \frac{2^{n+\frac{1}{2}}\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{2\pi}} \left(1 - \frac{\frac{5}{2}}{2\left(n-\frac{1}{2}\right)} - \frac{\frac{43}{8}}{2^2\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)} - \dots\right)$$

- combinatorics of $C_n^{(1)}$: Mahmoud & Yeats, 2020

Resurgent structure

- large order asymptotics of $C_n^{(1)}$ coefficients

$$C_n^{(1)} \sim -2e^{-2}\,\frac{2^{n+\frac{3}{2}}\,\Gamma\left(n+\frac{3}{2}\right)}{2\pi}\left(1-\frac{5}{2\left(n+\frac{1}{2}\right)}-\frac{\frac{11}{2}}{2^2\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}-\ldots\right)$$

- next nonperturbative solution ($\xi(x)\equiv\frac{1}{\sqrt{x}}\,e^{-1/(2x)}$):

$$C^{(2)}(x)\sim\xi(x)^2\,\frac{e^{-2}}{2\pi}\left[\frac{1}{x}-5-\frac{11}{2}x-\frac{97}{2}x^2-\ldots\right]$$

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- continues to all orders $\Rightarrow$ all-orders summation

$$C(x)=\left[\exp\left(\sigma\,\xi(x)f(x,y)\,\frac{\partial}{\partial y}\right)\cdot y\right]_{y=C^{(0)}(x)}$$

generating function : $\quad f(x,y)\equiv\frac{1}{\sqrt{2\pi}}\frac{x}{y}\exp\left[-\frac{1}{2x}\,y(y+2)\right]$

- also follows from Borinsky's alien derivative on the ring of formal power series

Resurgence in the 4 dimensional massless Yukawa Model

- trans-series: the (asymptotic) perturbative solution to the nonlinear ODE for the anomalous dimension can be extended to a trans-series which resums all nonperturbative orders

- non-perturbative terms $C^{(k)}(x)$ $(k \geq 1)$ $\longleftrightarrow$ singularities of the Borel transform of the perturbative series

- resurgence: all non-perturbative terms are expressed explicitly in terms of the original formal series $C^{(0)}(x)$

fluctuations about different singularities are quantitatively related

Resurgence in the 6 dim. Scalar ϕ^3 Theory

- physically more interesting quantum field theory

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{g}{3!}\,\phi^3 \qquad , \qquad \alpha := \frac{g^2}{(4\pi)^3}$$

- asymptotically free; $d = 6$ critical dimension; Lipatov instanton; renormalons; $\rightarrow$ non-perturbative physics

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- Broadhurst/Kreimer: 3rd order ODE (with quartic nonlinearity) for anomalous dimension (rescale $\alpha = 3x$)

$$\left[C\left(2\,x\,\frac{d}{dx} - 1\right) - 1\right]\left[C\left(2\,x\,\frac{d}{dx} - 1\right) - 2\right]\left[C\left(2\,x\,\frac{d}{dx} - 1\right) - 3\right] C = 3x$$

- perturbative solution: $C(x) = 6\sum_{n=1}^{\infty}\left(-\frac{1}{12}\right)^n C_n x^n$

$$C_n = \{1, 11, 376, 20241, 1427156, 121639250, 12007003824, \dots\}$$

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- no known combinatorial interpretation of C_n: OEIS A051862

Trans-series Analysis

Borinsky, GD, Meynig, 2020, to appear

- Broadhurst/Kreimer: $C_n \sim 12^n\,\Gamma\,(n+2)$
- with more data

$$C_n \sim 12^n\,\Gamma\left(n+\frac{23}{12}\right)\left(1-\frac{3\,\frac{97}{144}}{\left(n+\frac{11}{12}\right)}-\frac{3^2\,\frac{53917}{124416}}{\left(n+\frac{11}{12}\right)\left(n-\frac{1}{12}\right)}-\dots\right)+\dots$$

- there are 3 "missing" b.c. parameters !

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- there are 3 "missing" b.c. parameters !
- transseries ansatz: $C(x) \sim x^\beta \, e^{-\lambda/x} \, (1 + \dots)$

$$\lambda = 1 \qquad \& \qquad \beta = -\frac{23}{12}$$
$$\lambda = 2 \qquad \& \qquad \beta = +\frac{1}{6}$$
$$\lambda = 3 \qquad \& \qquad \beta = -\frac{11}{4}$$

$\Rightarrow$ three <u>resonant</u> Borel singularities at $t = -1, -2, -3$

Trans-series Analysis: full three-term trans-series

$$C(x) \sim C_{\rm pert}(x) \quad + \quad \sum_{k=1}^{\infty} \sigma_{[1]}^k \left(\frac{e^{-\frac{1}{x}}}{x^{23/12}} \right)^k \sum_{n=0}^{\infty} C_{[1],n}^{(k)} x^n$$

$$+ \quad \sum_{k=1}^{\infty} \sigma_{[2]}^k \left(\frac{e^{-\frac{2}{x}}}{x^{-1/6}} \right)^k \sum_{n=0}^{\infty} C_{[2],n}^{(k)} x^n$$

$$+ \quad \sum_{k=1}^{\infty} \sigma_{[3]}^k \left(\frac{e^{-\frac{3}{x}}}{x^{11/4}} \right)^k \sum_{n=0}^{\infty} C_{[3],n}^{(k)} x^n + \dots$$

- compute fluctuation coefficients from ODE: e.g. $C_{[1],n}^{(k=1)}$

$$C_{[1],n}^{(k=1)} = \left\{ 1, \frac{97}{144}, \frac{53917}{124416}, \dots \right\}$$

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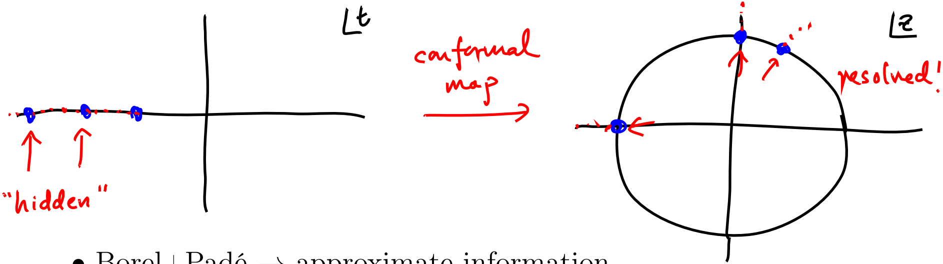
$$C_{[1],n}^{(k=1)} = \left\{1, \frac{97}{144}, \frac{53917}{124416}, \dots\right\}$$

- resurgence relation:

$$C_n^{\rm pert} \sim 12^n\, \Gamma\left(n + \frac{23}{12}\right) \left(1 - \frac{3\,\frac{97}{144}}{\left(n + \frac{11}{12}\right)} - \frac{3^2\,\frac{53917}{124416}}{\left(n + \frac{11}{12}\right)\left(n - \frac{1}{12}\right)} - \dots\right)$$

Borel Analysis of Perturbative Series

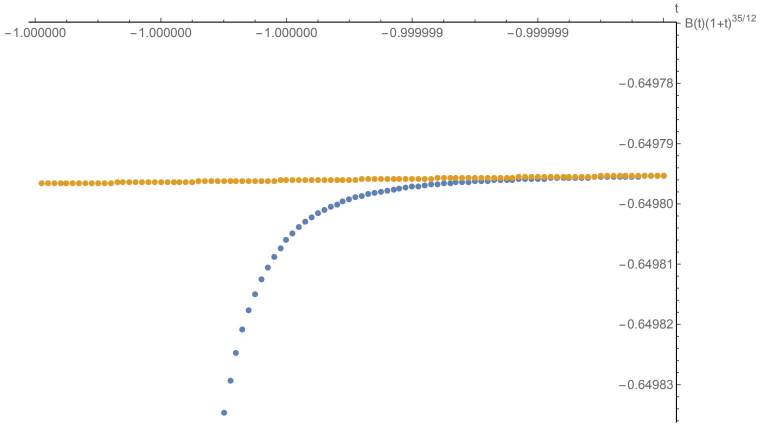
- non-perturbative information = location and nature of Borel singularities, and associated Stokes constants $S_{[j]}$
- decoding this non-perturbative information from a finite amount of perturbative data requires new methods: Borel-Padé & conformal/uniformizing maps (Costin, GD: 2003.07451, 2009.01962)



- Borel+Padé → approximate information
- Borel+conformal map+Padé → resurgent singularities
- Borel+uniformizing map+Padé → optimal (2009.01962)

Borel Analysis of Fluctuation Series

- uniformization map in Borel plane enables (optimal) high precision extraction of Stokes constants:

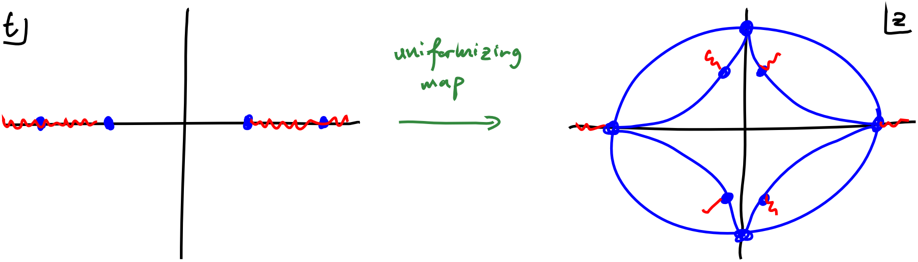


- conformal map [blue]; uniformizing map [gold]

Borel Analysis of Fluctuation Series

- uniformized Borel analysis → large order growth
- fluctuations about $t = -2$ have interference terms

$$C_{[2],n}^{(k=1)} \sim -0.350382 \cdot (-3)^{-n}\,\Gamma\left(n+\frac{35}{12}\right) + 2.70724 \cdot (-3)^{-n}\,\Gamma\left(n+\frac{23}{12}\right) + 2.17179 \cdot 3^{-n}\Gamma\left(n+\frac{25}{12}\right) + 4.38855 \cdot 3^{-n}\Gamma\left(n+\frac{13}{12}\right)$$



Borel Analysis of Fluctuation Series

- uniformized Borel analysis $\rightarrow$ large order growth
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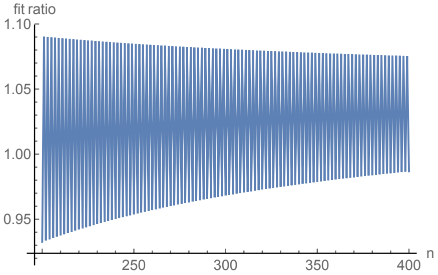

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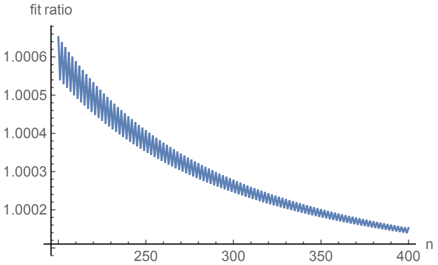

Resurgence in the 6 dimensional Scalar ϕ^3 Theory

- richer non-perturbative structure than Yukawa model
- 3rd order ODE with 4th order non-linearity
- 3 different non-perturbative structures, with different fluctuation powers
- large order/low order resurgence relations
- non-perturbative terms expressed in terms of formal perturbative series

Conclusions

perturbative Hopf algebra renormalisation

resurgent $\Downarrow$ analysis

non-perturbative completion

- does there exist a "natural" Hopf algebraic non-perturbative (trans-series) structure ?
- functional relation & Borinsky's "alien derivation" ?
- multi-component fields ? (Gracey, 2015; Giombi et al ...)
- relation with instantons and renormalons ?
- other renormalisation schemes ?
- 2d σ models, Chern-Simons, SUSY, QED, QCD, ... ?

My Real Conclusion

Happy Birthday Dirk !

(and thank you for your inspirational scientific ideas and leadership)