

# Connes-Kreimer Hopf Algebras :

## from Renormalisation to Tensor Models and Topological Recursion

Thomas Krajewski

Centre de Physique Théorique

Aix-Marseille University

`thomas.krajewski@cpt.univ-mrs.fr`

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## Summary

The Connes-Kreimer Hopf algebras of graphs have multiple aspects :

- **combinatorial** : formal power series indexed by graphs with substitution of one graph into an other
- **group theoretical** : group of characters and associated Lie algebra
- **analytic** : Birkhoff type factorisations

Summary of the talk :

- **Hopf algebras of trees and graphs**
- **Virasoro type constraints for random tensors**
- **Kontsevich-Soibelman approach to topological recursion**

Other occurrences : Polchinski's exact renormalisation (see arxiv 0806.4309), multiscale renormalisation (see arxiv 1211.4429), graph polynomials (see arxiv 1508.00814), ...

## Coproduct on graphs

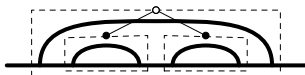
$\mathcal{H}_G$  free commutative algebra generated by some class of connected (irreducible) graphs with coproduct

$$\Delta\gamma = \gamma \otimes 1 + 1 \otimes \gamma + \sum_{\substack{\gamma_1, \dots, \gamma_n \\ \text{disjoint proper connected subgraphs}}} (\gamma_1 \cdots \gamma_n) \otimes \gamma / (\gamma_1 \cdots \gamma_n) \quad (1)$$

Contracted graph obtained by contracting in  $\gamma$  all subgraphs  $\gamma_1, \dots, \gamma_n$  to vertices, consistent if created type of vertex already present.

$$\Delta(\text{---}\overbrace{\text{---}}^{\text{---}}\text{---}) = \text{---}\overbrace{\text{---}}^{\text{---}}\text{---} \otimes 1 + 1 \otimes \text{---}\overbrace{\text{---}}^{\text{---}}\text{---} \quad (2)$$
$$+ 2 \text{---}\overbrace{\text{---}}^{\text{---}} \otimes \text{---}\overbrace{\text{---}}^{\text{---}} + (\text{---}\overbrace{\text{---}}^{\text{---}})^2 \otimes \text{---}\overbrace{\text{---}}^{\text{---}}$$

Hierarchy of subgraphs captured by a tree



## Coproduct on trees

$\mathcal{H}_T$  free commutative algebra generated by rooted trees with coproduct

$$\Delta T = T \otimes 1 + 1 \otimes T + \sum_{\text{admissible cut}} P^c(T) \otimes R^c(T) \quad (3)$$

Admissible cut : remove edges such that each path from the root to any leaf sees only one removed edge  $\rightarrow$  remaining part  $R^c(T)$  (containing the root) and pruned part  $P^c(T)$  (possibly disconnected).

$$\Delta \left( \begin{array}{c} \circ \\ | \\ \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right) = \begin{array}{c} \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \otimes 1 + 1 \otimes \begin{array}{c} \circ \\ | \\ \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} + 2 \circ \otimes \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + (\circ)^2 \otimes \begin{array}{c} \circ \\ | \\ \bullet \end{array} + \begin{array}{c} \circ \\ | \\ \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \otimes \circ \quad (4)$$

Ubiquity of rooted tree explained by universal property based on gathering trees along the root to get a new tree.

# Commutative graded Hopf algebra and group of characters

A commutative graded bialgebra  $\mathcal{H} = \bigoplus \mathcal{H}_n$  (with  $\mathcal{H}_0 = \mathbb{C}$  and counit  $\epsilon(x) = 0$  unless  $x \in \mathcal{H}_0$ )

$$\mathcal{H}_m \cdot \mathcal{H}_n \subset \mathcal{H}_{m+n}, \quad \Delta \mathcal{H}_n \subset \bigoplus_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q \quad (5)$$

is a Hopf algebra with antipode  $S$  defined by recursion.

$\mathcal{H}_G$  (with  $\#$  edges or  $\#$  cycles) and  $\mathcal{H}_T$  (with  $\#$  vertices) are commutative and graded

Characters  $\alpha$  (algebra morphisms  $\mathcal{H} \rightarrow$  commutative ring,  $\alpha(\gamma\gamma') = \alpha(\gamma)\alpha(\gamma')$ ) form a group  $G$  for the convolution product

$$\alpha * \beta = m \circ (\alpha \otimes \beta) \circ \Delta \quad (6)$$

with identity = counit and inverse  $\alpha^{-1} = \alpha \circ S$ .

Infinitesimal characters ( $\delta(\gamma\gamma') = \delta(\gamma)\epsilon(\gamma') + \epsilon(\gamma)\delta(\gamma')$ ) define the Lie algebra of  $G$ ,  $\alpha = \exp_* \delta$ .

# Renormalisation and characters of $\mathcal{H}_G$

Perturbative expansion of correlation functions on Feynman graphs

$$\frac{\int [D\phi] \phi(x_1) \cdots \phi(x_n) \exp -S[\phi]}{\int [D\phi] \exp -S[\phi]} = \sum_{\substack{\gamma \text{ Feynman graph} \\ \text{with } n \text{ external legs}}} \mathcal{A}_\gamma(x_1, \dots, x_n) \quad (7)$$

Dimensional regularisation :  $\mathcal{A}_\gamma(x_1, \dots, x_n)$  evaluated in complex dimension with poles in deviation  $z$  from physical dimension.

$\Rightarrow$  divergences canceled by counterterms  $S[\phi] \rightarrow S[\phi] + S_{CT}[\phi]$

Theorem (Connes & Kreimer, hep-th/9912092)

$$\phi_+ = \phi_- * \phi \quad \text{or equivalently} \quad \underbrace{\mathbb{C}[z, z^{-1}]}_{\phi} = \underbrace{\mathbb{C}[z^{-1}]}_{(\phi_-)^{-1}} * \underbrace{\mathbb{C}[z]}_{\phi_+} \quad (8)$$

- $\phi(\gamma)$  regularised amplitude (Laurent series) of  $A_\gamma$
- $\phi_-(\gamma)$  counterterm (pole part) for graph  $\gamma$
- $\phi_+(\gamma)$  renormalised amplitude (finite) of  $A_\gamma$

## Butcher's $B$ -series and rooted trees

Non linear operator  $X$  raised to the power of the tree

$$X^T(x) = \frac{1}{\#\text{Aut}(T)} \times \left( \begin{array}{l} \text{differentials of } X \text{ at } x \\ \text{composed from root to leaves} \end{array} \right) \quad (9)$$

Example :

$$X^{\bullet} = \frac{1}{2} X' [X'' [X, X]] = \frac{1}{2} \frac{\partial X^i}{\partial x^j} \frac{\partial^2 X^j}{\partial x^k \partial x^l} X^k X^l \quad (10)$$

Composition law :

$$\left( \sum_T \alpha(T) X^T \right) \circ \left( \sum_T \beta(T) X^T \right) = \left( \sum_T \beta * \alpha(T) X^T \right) \quad (11)$$

Geometric series :

$$x = x_0 + X(x) \Rightarrow x = [1 - X]^{-1}(x) = \sum_T X^T(x) \quad (12)$$

Initial motivation : composition of Runge-Kutta methods

## Random tensors

Natural generalisation of random matrices  $M_{ij} \rightarrow T_{ijk\dots}$  (rank  $r$  indices taking values  $i = 1, \dots, N$ )

$$\langle \mathcal{O}(T) \rangle = \frac{\int dT \exp -V_N(T) \mathcal{O}(T)}{\int dT \exp -V_N(T)} \quad (13)$$

Applications :

- Sum over random  $D$ -dimensional triangulations ( $T=(D-1)$  simplex).

$$\int_{\substack{\text{rank } D \\ \text{random tensor}}} dT \exp -V_N(T) = \sum_{\substack{\mathcal{T} \\ D\text{-dimensional triangulations}}} \mathcal{A}_{\mathcal{T}} \quad (14)$$

- random coupling constant for a  $N$ -vector model

$$\langle \exp -J_{i_1, \dots, i_r} \phi_{i_1} \cdots \phi_{i_r} \rangle_J \quad (15)$$

- Sachdev-Ye-Kitaev model without disorder (same large  $N$  limit)



## Invariant interactions

$T_{ijk\dots}$  real or complex with possible index symmetries with potential invariant under  $O(N)$  or  $U(N)$  transformations

$$T_{ijk\dots} \quad \rightarrow \quad T'_{ijk\dots} = O_i^{i'} O_j^{j'} O_k^{k'} \dots T_{i'j'k'\dots} \quad (16)$$

$$V_N(T) = \frac{N^{r-1}}{2t} T^2 + \sum_{\gamma \text{ valence } D \text{ graph}} \frac{x_\gamma N^{s_\gamma}}{\#\text{Aut } \gamma} (T \dots T)_\gamma \quad (17)$$


with

- $(T \dots T)_\gamma$  invariant polynomial in components of  $T_{ijk\dots}$  obtained by assigning tensors to vertices and contract indices along the edges.
- $x_\gamma$  coupling constant, also used to generate expectation values by derivation
- $s_\gamma$  scaling power in  $N$  chosen so that a large  $N$  limit exists, in many cases  $s_\gamma = r - 1$


## Examples of invariant interactions

- Dipole (scaling  $N^2$ ) 


$$(T \cdots T)_\gamma = \sum_{1 \leq i, j, k \leq N} T_{ijk} T_{ijk}$$

- Quartic melon (scaling  $N^2$ ) 

$$(T \cdots T)_\gamma = \sum_{1 \leq i, j, k, l, m, n \leq N} T_{ijk} T_{ijl} T_{mnk} T_{mnl}$$

- Tetrahedron (scaling  $N^{5/2}$ ) 

$$(T \cdots T)_\gamma = \sum_{1 \leq i, j, k, l, m, n \leq N} T_{ijk} T_{klm} T_{mjn} T_{nli}$$

- Trace for a random matrix (scaling  $N$ ) 

$$\text{tr } M^n = \sum_{1 \leq i_1, \dots, i_n \leq N} M_{i_1 i_2} \cdots M_{i_{n-1} i_n} M_{i_n i_1}$$

## Complex non symmetric tensors

Complex tensors without any index permutational symmetry

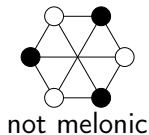
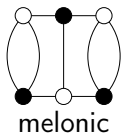
Graph invariants : black and white  $T$  and  $\bar{T}$  distinguished by black and white vertices, edges join only black and white (quadratic  $\bar{T}T$ ) vertices with label  $1, 2, \dots, r$  each vertex (place of index)

Theorem (Bonzom, Gurau & Rivasseau arxiv 1202.3637)

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \frac{\int d\bar{T} dT (T \cdots T)_\gamma \exp \{-N^{r-1} V(\bar{T}, T)\}}{\int d\bar{T} dT \exp \{-N^{r-1} V(\bar{T}, T)\}} \quad \text{finite} \quad (18)$$

with graph  $\gamma$  and all graphs in  $V$  connected

Limit dominated by melonic graphs : for each vertex  $v$  there is a conjugate vertex  $\bar{v}$  such that removing  $v$  and  $\bar{v}$  leads to exactly  $r$  connected components



## Change of variable in the partition function

Infinitesimal change of variable in the partition function integral

$$T \rightarrow T' = T + \epsilon \underbrace{(T \cdots T)_{\gamma \setminus v}}_{\delta T} \quad (19)$$

with  $\gamma \setminus v$  obtained by removing vertex  $v$  ( $r$  external legs)

$(T \cdots T)_{\gamma}$ 
 $(T \cdots T)_{\gamma \setminus v}$   
 $\sum_{l,m,n} T_{ijl} \bar{T}_{lmn} T_{mnk}$

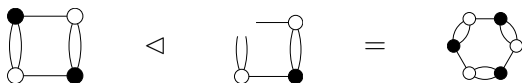
Invariance of the partition function :

$$0 = \delta Z = \int dT \left\{ \underbrace{\text{Tr} \frac{\partial(\delta T)}{\partial T}}_{\text{Jacobian}} - \underbrace{\frac{\partial V_N(T)}{\partial T} \delta T}_{\text{variation of integrand}} \right\} \exp -V_N(T) \quad (20)$$

# Insertion and contraction

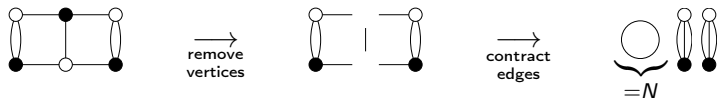
- potential : insert  $\gamma \setminus v$  at vertices of graphs in  $V_N(T)$

$$\frac{\partial (T \cdots T)_{\gamma'}(T \cdots T)_{\gamma \setminus v}}{\partial T} \quad (21)$$



- Jacobian : remove an other vertex  $v'$  in  $\gamma$  (derivative) and contract pairs of half-edges between  $v$  and  $v'$  (trace).

$$\frac{\partial (\delta T)}{\partial T} = (T \cdots T)_{\gamma/v'} \quad (22)$$



# Virasoro type constraints for random tensors

Tensor model partition function obey the constraints

$$L_\gamma Z = 0 \quad (23)$$

with linear differential operator of order at most  $r = \text{rank of tensor}$

$$L_\gamma = -\frac{\partial}{\partial x_\gamma} + t \prod_{\substack{\gamma_i \\ \text{connected components of } \gamma/vv'}} \frac{\partial}{\partial x_{\gamma_i}} + t \sum_{\substack{\gamma' \\ v' \text{ vertex of } \gamma'}} x_{\gamma'} \frac{\partial}{\partial x_{\gamma' \triangleleft_{v'} (\gamma \setminus v)}} \quad (24)$$

Correct scaling in  $N$  recovered by  $t \rightarrow \frac{t}{N^{r-1}}$  and  $x_\gamma \rightarrow N^{s_\gamma} x_\gamma$   
 $\Rightarrow$  leading order contribution for melonic graphs

Representation of the Lie algebra of infinitesimal characters

$$\left[ \sum_{\gamma} \delta(\gamma) L_\gamma, \sum_{\gamma} \delta'(\gamma) L_\gamma \right] = \sum_{\gamma} [\delta, \delta'](\gamma) L_\gamma \quad (25)$$

with  $[\delta, \delta'] = \delta * \delta' - \delta' * \delta$  (insertion of one graph into the other)

## Quantum Airy structures

Quadratic differential operators

$$L_i = -\hbar\partial_i + \frac{1}{2}A_{ijk}x^jx^k + \hbar B_{ij}^kx^j\partial_k + \frac{\hbar^2}{2}C_i^{jk}\partial_j\partial_k + \hbar D_i \quad (26)$$

obeying a Lie Algebra relation

$$[L_i, L_j] = f_{ij}^k L_k \quad (27)$$

Theorem (Kontsevich & Soibelman, [arxiv.1701.09137](https://arxiv.org/abs/1701.09137))

There is a unique power series  $S = \sum_{g \geq 1, n \geq 3} S_{g,n} \hbar^g x^n$  (with  $\deg(\hbar) = 2\deg(x)$ ) such that

$$\exp -\frac{S}{\hbar} L_i \exp \frac{S}{\hbar} = 0. \quad (28)$$

Non linear equation for  $S$  to be solved recursively

$$\partial_i S = \frac{1}{2}A_{ijk}x^jx^k + B_{ij}^kx^j\partial_k S + \frac{1}{2}C_i^{jk}\partial_j S\partial_k S + \frac{\hbar}{2}C_i^{jk}\partial_{jk}^2 S + \hbar D_i \quad (29)$$

starting with  $S = 0 + \frac{1}{6}A_{ijk}x^ix^jx^k + \hbar D_i x^i + \dots$

## General Virasoro type constraints

Higher order operators to deal with constraints in tensor models

$$L_i = \hbar \frac{\partial}{\partial x_i} + t K_i \quad (30)$$

with differential operators (finite sum or formal power series)

$$K_i = \sum_n \hbar^n A_{i,i_1 \dots i_n}(x) \frac{\partial^n}{\partial x_{i_1} \dots \partial x_{i_n}} \quad (31)$$

obeying a Lie algebra

$$[L_i, L_j] = f_{ij}^k L_k \quad (32)$$

### Theorem

*Given  $S_0$ , there is a power series  $S$  (unique up to an additive constant) such that (as formal power series in  $t$ )*

$$\exp -\frac{S}{\hbar} L_i \exp \frac{S}{\hbar} = \partial_i S_0. \quad (33)$$



## Proof in two steps

- Non linear equation for the 1-form  $\Omega_i = \partial_i S$

$$\Omega = \Omega_0 + tX[\Omega] \quad (34)$$

solved by rooted trees

$$\Omega = (1 - X)^{-1}[\Omega_0] \quad (35)$$

$$= \sum_{T \text{ rooted trees}} X^T[\Omega_0] \quad (36)$$

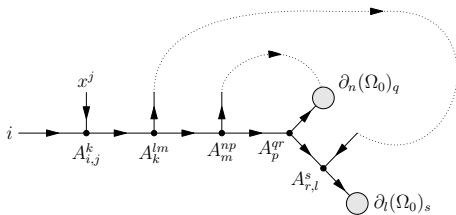
- Exactness of  $\Omega$  (induction relying on Lie algebra structure)  
 $\Rightarrow$  existence of  $S$  such that  $\Omega_i = \partial_i S$ ,

$$\partial_i \Omega_j - \partial_j \Omega_i = 0 \quad \Rightarrow \quad S(x) = \int_0^1 ds x^i \Omega_i(sx). \quad (37)$$

## Graphical expansion

Perturbative solution based on trees with extra structure

- vertices decorated with tensors  $A_{i_1 \dots i_m}^{j_1 \dots j_m}$  (lower indices incoming contracted with  $x$ , upper indices outgoing contracted with  $\delta$ )
- rooted trees oriented from root to "big" leaves (insertion of  $\partial_i S_0 =$  starting point of recursion)
- extra "small" incoming leaves at vertices contracted with  $x^j$
- "loop" edges oriented from vertices to leaves (derivatives)
- power series  $t^{\#\text{vertices}} \hbar^{\#\text{loops}}$



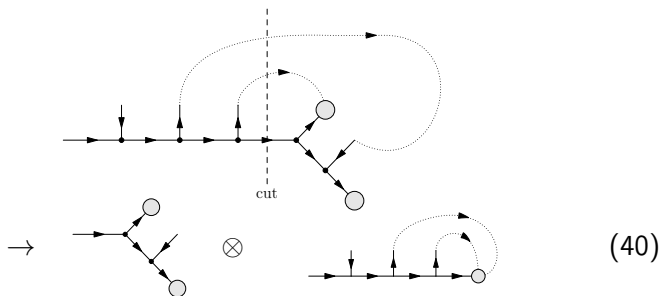
Associated amplitude :

$$A_{i,j}^k x^j A_k^{lm} A_m^{np} A_p^{qr} \partial_n \Omega_q A_{r,l}^s \partial_l(\Omega)_s \quad (38)$$

## Hopf algebra structure

Free commutative algebra generated by trees with loops with coproduct defined by admissible cuts on the tree edges followed by contraction of the pruned subtrees into big vertices in the trunk

$$\Delta T = T \otimes 1 + 1 \otimes T + \sum_{\text{admissible cut}} P^c(T) \otimes R^c(T) / P^c(T) \quad (39)$$



Describes substitution  $S \rightarrow S' \rightarrow S''$  in iterated computations

## Perturbative vs topological (semi-classical) expansion

- Perturbative expansion in powers of  $t$

Solution as a formal power series in  $t =$  perturbative expansion of a free energy  $F$  ( $\leftrightarrow S$ ) with  $Z(x) = \exp[-N^{r-1}F(x)]$  in terms of ordinary Feynman (ribbon) graphs with  $t^{\#(\text{edges})}$

- Topological expansion in powers of  $\hbar$  and  $x$

For  $r = 2$  (random matrices), topological expansion in inverse powers of  $N$ , initiated by disk ( $g = 0, n = 1$ ) and cylinder ( $g = 0, n = 2$ )  $\Rightarrow$  higher topologies

$$N^2 \frac{\partial F}{\partial x} = \sum_{g \geq 0, n \geq 1} n F_{g,n} x^{n-1} N^{2-2g-n}. \quad (41)$$

Based on analytic properties of amplitudes in a complex variable  $z$  (possible relation to a Birkhoff decomposition). Possible approach for random tensors  $r \geq 3$ .