Remarks on Feynman Amplitudes

S. Bloch

I. Feynman Motives

Family of deg $g+1$ hypersurfaces in $\mathbb{P}^{e-1}$ parametrized by masses + momenta. (For this talk, masses + momenta will be generic.)

For this talk, $g=2$

family of singular cubic hypersurfaces.

Kreimer, The master two-loop two point function; the General case. Physics letters B273 (1991).

Takeaways: (a) 2 loops are 'interesting physically. + algebraic geometers love cubic hypersurfaces.

(b) Focusing on generic parameters clarifies the geometry of the cubic.

II. Amplitude as a relative period

Absolute period: $X \xrightarrow{f} S/_{\mathbb{C}}$ smooth projective.

Gauss–Manin connection on $R^n f_*(\Omega^\bullet_{X/S})$
(Relative DR cohomology)

integrating over $\mathbb{Q}$-homology chains in the fibres $\rightsquigarrow$ Picard Fuchs DE.

Feynman amplitude satisfies inhomogeneous Picard Fuchs eqn. So it is not an absolute period.

Relative Periods

a. Very complicated:

1. integration chain not topologically closed (Bounds at ∞)
2. integration chain can meet polar locus.

Relative periods (cont.) Program of M. Kerr

Another sort of relative period associated to classes in Motivic Cohomology.

Ex. $X \subset \mathbb{P}^n$ ~~smooth~~ hypersurface.

Homogeneous coordinates $T_0, \dots, T_n$

$X^* = X - \bigcup_{i=0}^{n} \{T_i = 0\}$. Tame symbol $\left\{\frac{T_1}{T_0}, \dots, \frac{T_n}{T_0}\right\}$

represents class in $H^n_M(X^*, \mathbb{Z}(n))$

Special cases: X tempered, class lifts to $H^n_M(X, \mathbb{Z}(n))$.

cycle class $\in H^{n-1}(X, \mathbb{C}/\mathbb{Z}(n)) \longrightarrow H^{n-1}(X, \mathbb{R}(n-1))$

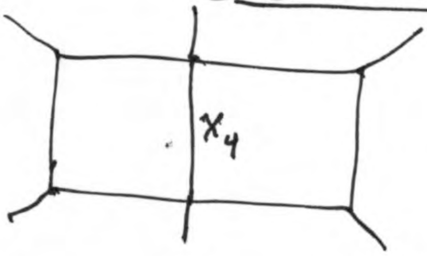
For $X = 3$ or 4 banana graph second Symanzik

Kerr relates this construction to Feynman Amplitude.

Consequence: Beilinson conjectures relate motivic cohomology classes to values of L-functions.

Problem (Open) Relate Feynman amplitudes to motivic cohomology for 2-loop graphs???

Double Box Motive



$X \subset \mathbb{P}^6$ cubic $\quad X: F = 0.$

Generic masses + momenta.

$X: F(x_1, \ldots, x_7) = 0$

$$F = Q(x_5, x_6, x_7)(x_1 + x_2 + \cdots + x_4) + Q'(x_1, x_2, x_3)(x_4 + x_5 + x_6 + x_7)$$

$$+ x_4 \cdot A$$

$$A = (x_1, x_2, x_3, x_4)\begin{pmatrix} a_{14} & a_{15} & a_{16} & a_{17} \\ a_{24} & \vdots & \vdots & \vdots \\ a_{34} & \vdots & \vdots & \vdots \\ 0 & a_{45} & a_{46} & a_{47} \end{pmatrix}\begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

(note F has no term in x_4^3)

Q, Q' smooth quadrics (conic curves in $\mathbb{P}^2$)

Singular locus $X_{sing} = C \amalg C'$

$C: x_1 = x_2 = x_3 = x_4 = Q = 0$; $C': x_4 = x_5 = x_6 = x_7 = Q' = 0.$

$\pi: V \longrightarrow \mathbb{P}^6$ blowup of $C \amalg C' \subset \mathbb{P}^6$

$Y \subset V$ strict transform of X.

<u>Prop.</u> $Y \to X$ Resolution of singularities. (generic parameters)

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & V \arrow[d, "\pi"] & E \amalg E' \arrow[l, hook'] \arrow[d] \\ X \arrow[r, hook] & \mathbb{P}^6 & C \amalg C' \arrow[l, hook'] \end{tikzcd}$$

$E \cap Y$, $E \cap Y'$ smooth (for general parameters)

Want to show $H^5(Y) \cong H^1(\mathcal{E})(-2)$, $\mathcal{E}$ an elliptic curve

Want to show $\boxed{F^3H^5(Y) \text{ dim } 1.}$

$H^*(V)$ gen. by algebraic Cycles $\Rightarrow F^4H^6(V) = 0$

$\therefore\ F^4H^6(V-Y) \cong F^3H^5(Y)$

<u>Deligne Thm.</u> $\Rightarrow$

$$F^4H^6(V-Y) \cong \mathbb{H}^2(V, \Omega^4_V(Y) \xrightarrow{d} \Omega^5_V(2Y) \xrightarrow{d} \Omega^6_V(3Y)).$$

Full computation is complicated.

Double n-gons

$X \subset \mathbb{P}^{2n-2}$ $\quad X : Q'(x_1, \dots, x_{n-1})(x_n + \dots + x_{2n-1})$

$$+ Q(x_{n+1}, \dots, x_{2n-1}) \cdot (x_1 + \dots + x_n) + x_n Q = 0.$$

Q, Q' general quadrics in $n-1$ variables.

Same picture

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & V \arrow[d, "\pi"] & \supset E \amalg E' \arrow[d] \\ X \arrow[r, hook] & \mathbb{P}^{2n-2} & \supset C \amalg C' \end{tikzcd}$$

$\tilde{\Omega}^{2n-2}_{\mathbb{P}^{2n-2}}((n-1)X) \subset \Omega^{2n-2}_{\mathbb{P}^{2n-2}}((n-1)X)$

Sections which pull back to

$\Omega^{2n-2}_V((n-1)Y) \subset \Omega^{2n-2}_V((n-1)\pi^* X)$.

(i.e. pullback has no poles along $E \amalg E'$)

<u>Prop.</u> $H^0(\tilde{\Omega}^{2n-2}_{\mathbb{P}^{2n-2}}((n-1)X)) \cong \mathbb{C}$.

<u>Ex.</u> $H^0(\Omega^6_V(3Y)) = \mathbb{C}$ for the double box.

Rmk. The situation for non-planar 2 loop graphs is more complicated.

General case $(p,q,r) = 2$ loop graph

 subdivide into p, q, r edges

$x_1, \dots, x_p, x_{p+1}, \dots, x_{p+q}, x_{p+q+1}, \dots, x_{p+q+r}$

Assume $p, q, r \geq 2$:

choose quadrics $Q_{(p)}(x_1, \dots, x_p)$, $Q_{(q)}(x_{p+1}, \dots, x_{p+q})$, $Q_{(r)}(x_{p+q+1}, \dots, x_{p+q+r})$
(General coefficients)

$F_{(p)} := Q_{(p)} \cdot (x_{p+1} + \dots + x_{p+q+r})$

$F_{(q)} := Q_{(q)} \cdot (x_1 + \dots + x_p + x_{p+q+1} + \dots + x_{p+q+r})$

$F_{(r)} := Q_{(r)} \cdot (x_1 + \dots + x_p + x_{p+1} + \dots + x_{p+q})$

$$F_{(p,q,r)} := \sum a_{ijk} x_i x_j x_k \,; \quad 1 \leq i \leq p, \; p+1 \leq j \leq p+q, \; p+q+1 \leq k \leq p+q+r$$

$$F = F_{(p)} + F_{(q)} + F_{(r)} + F_{(p,q,r)}$$

Non-intersecting Linear spaces: $L_p: x_{p+1} = \cdots = x_{p+q+r} = 0$

$L_q: x_1 = \cdots = x_p = x_{p+q+1} = \cdots = x_{p+q+r} = 0$

$L_r: x_1 = \cdots = x_{p+q} = 0$

Smooth disjoint quadrics

$R_p = L_p \cap \{Q_{(p)} = 0\}$; $R_q = L_q \cap \{Q_{(q)} = 0\}$; $R_r = L_r \cap \{Q_{(r)} = 0\}$.

$X: F = 0 \quad X_{sing} = R_p \amalg R_q \amalg R_r$

(*) $\pi: V \longrightarrow \mathbb{P}^{p+q+r-1}$ blowup of $\mathbb{P}^{p+q+r-1}$ along $R_p \amalg R_q \amalg R_r$

Y = strict transform of X to V

Then Y is smooth.

Special cases: $r = 1$; $p, q \geq 2$

$$F = Q_{(p)}(x_1, \ldots, x_p)(x_{p+1} + \cdots + x_{p+q+1}) + Q_{(q)}(x_{p+1}, \ldots, x_{p+q})(x_1 + \cdots + x_p + x_{p+q+1})$$
$$+ c\, x_{p+q+1}^2 \cdot (x_1 + \cdots + x_{p+q}) + x_{p+q+1}\Big(\sum_{\substack{1 \leq i \leq p \\ p+1 \leq j \leq p+q}} a_{ij} x_i x_j\Big)$$

$q = r = 1$, $p \geq 2$

$$F = Q_{(p)}(x_1, \ldots, x_p)(x_{p+1} + x_{p+2}) + c_{p+1} x_{p+1}^2 (x_1 + \cdots + x_p + x_{p+2})$$
$$+ c_{p+2} x_{p+2}^2 (x_1 + \cdots + x_p + x_{p+1}) + x_{p+1} x_{p+2} \cdot \sum_{i=1}^{p} a_i x_i.$$

(*) holds when $r = 1$, taking $R_r = \phi$.