On the Enumerative Structures in QFT

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Outline

- **Chord diagrams**, and their asymptotics.
- Applications in **Quenched QED** and **Yukawa theory**.

Chord Diagrams

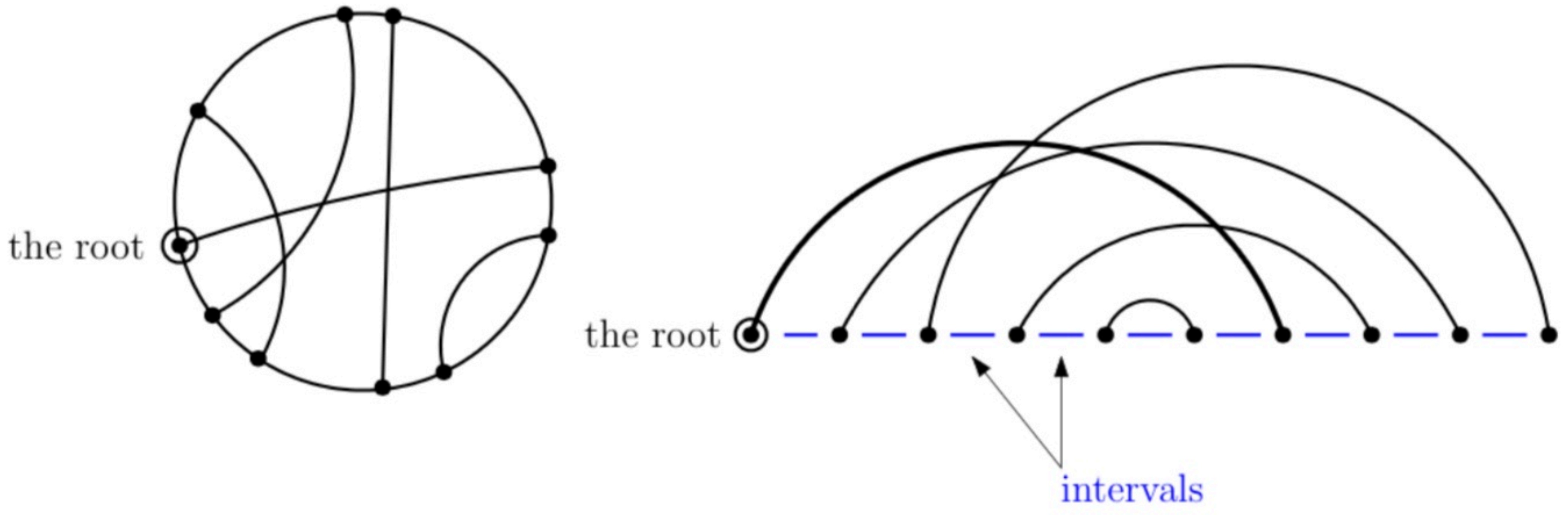
- The generating series for chord diagrams, counted by the number of chords, is denoted by $D(x)$. Note that there are $(2n-1)!!$ chord diagrams on n chords.

- $C(x)$ will denote the generating series for **connected chord diagrams,** and starts as $C(x) = x + x^2 + 4x^3 + 27x^4 + 248x^5 + \cdots$.

- If $D(x)$ is the generating series of all rooted chord diagrams then one can show that
$$D(x) = 1 + C(xD(x)^2).$$

- Another useful identity is that
$$2C(x)C'(x) = C(x)(1 + C(x)) - x\,.$$

- Below is a chord diagram and its **linear representation**



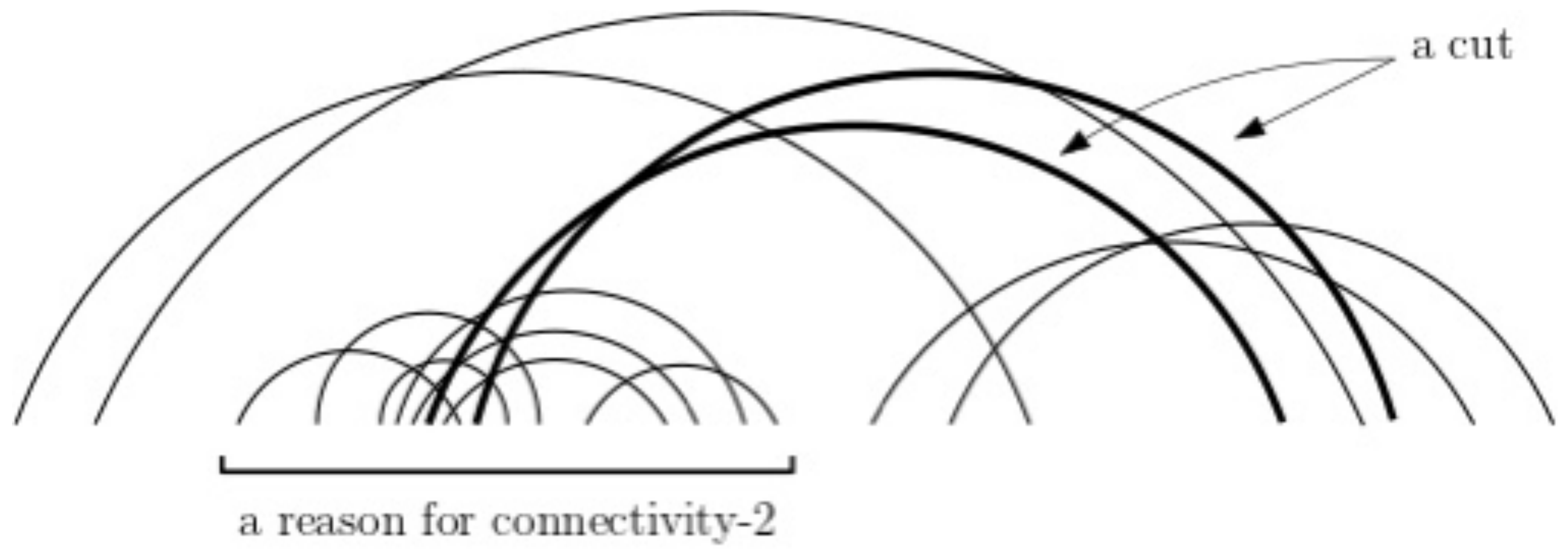
2-Connected Chord Diagrams

- **Definition:** A chord diagram on n chords is k-**connected** if **no** set S of **consecutive** endpoints with size $|S| < 2n - k$ is paired with **less** than k endpoints out of S.

- The generating series for **2-connected** chord diagrams will be denoted here by $C_{\geq 2}(x)$, **whereas** the generating series of **connectivity-1** diagrams will be denoted by $C_1(x)$. In particular, $C(x) = C_1(x) + C_{\geq 2}(x)$.

- The first terms read as $C_{\geq 2}(x) = x^2 + x^3 + 7x^4 + 63x^5 + 729x^6 + \cdots$.

- The diagram below is **2-connected** but not **3-connected**. A **cut**, and a **reason for connectivity-2** are illustrated.



- In **[Kleit]**, D. Kleitman gives an argument that for large n, the **proportion** of k-connected chord diagrams approaches e^{-k}.

- As mentioned earlier, in **[MBor1]** it is shown that $\left(\mathscr{A}^2_{\frac{1}{2}}C\right)(x) = \dfrac{x}{\sqrt{2\pi C(x)}}\, e^{-\frac{1}{2x}(2C(x)+C(x)^2)}$. This is then used to get **asymptotic** information for connected chord diagrams. The e^{-1} from Kleitman's result stands for the first term in the obtained asymptotic expansion.

- It is tempting to try doing the same thing for higher connectivity chord diagrams, we will focus on **2-connected** chord diagrams. The major **difficulty** lies in obtaining a functional equation that relates for example 2-connected chord diagrams to connected chord diagrams.

- The situation is resolved by the next result ...

- **Proposition 2.2:** The following functional relation holds for connected and 2-connected chord diagrams:

$$C = \frac{C^2}{x} - C_{\geq 2}\left(\frac{C^2}{x}\right).$$

Factorially Divergent Power Series

- **Notation and definition** [MBor1]**:** For real numbers α and β with $\alpha > 0$, $\mathbb{R}[[x]]_{\beta}^{\alpha}$ will denote the set of all formal power series f for which there exists a sequence $(c_k^f)_{k\in\mathbb{N}}$ of real numbers such that
$$f_n = \sum_{k=0}^{R-1} c_k^f \Gamma_{\beta}^{\alpha}(n-k) + \mathcal{O}(\Gamma_{\beta}^{\alpha}(n-R)), \text{ for all } R \in \mathbb{N}_0 .$$

- The modified gamma function above is defined to be $\Gamma_{\beta}^{\alpha}(n) := \alpha^{n+\beta}\Gamma(n+\beta)$, where $\Gamma(z) = \int_0^{\infty} x^{z-1}e^{-x}dx$ for $\mathrm{Re}(z) > 0$ is the standard gamma function.

- Excluding the “factorial divergence” captured by the gamma functions, we will focus on the corresponding coefficients (c_k).

- The idea in **[MBor1]** was to encode this characteristic information in the form of a power series via the map

$$(\mathscr{A}^{\alpha}_{\beta} f)(x) = \sum_{k=0}^{\infty} c^f_k x^k \,,$$

  defined over $\mathbb{R}[[x]]^{\alpha}_{\beta}$. Then it is shown in **[MBor1]** that $\mathbb{R}[[x]]^{\alpha}_{\beta}$ is a **subring** of $\mathbb{R}[[x]]$ and that the map $\mathscr{A}^{\alpha}_{\beta}$ is a **derivation**.

- Such maps are referred to as **alien derivatives** in the context of resurgence theory.

- In **[MBor1]** it is shown that $\left(\mathscr{A}^2_{\frac{1}{2}}C\right)(x) = \dfrac{x}{\sqrt{2\pi C(x)}}\, e^{-\frac{1}{2x}(2C(x)+C(x)^2)}$. This is then used to get the **asymptotic** information of connected chord diagrams.

Asymptotic Analysis

- By the properties of factorially divergent power series and alien derivatives we can verify that $C_{\geq 2}(x) \in \mathbb{R}[[x]]^{2}_{\frac{1}{2}} \subset \mathbb{R}[[x]]^{2}_{\frac{3}{2}}$.

- Applying the alien derivative $\mathscr{A}^{2}_{\frac{3}{2}}$ to the functional equation in **Proposition 2.2**, we eventually get

$$\left(\mathscr{A}^{2}_{\frac{1}{2}} C_{\geq 2}\right)(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{x^2}{\left(\frac{C_{\geq 2}}{(1 - C_{\geq 2}/x)}\right)} \cdot e^{\frac{-1}{2x}\left[\left(\frac{1}{(1 - C_{\geq 2}/x)} + x\right)^2 - 1\right]}.$$

- If we set $S(x) = 1/\left(1 - \frac{C_{\geq 2}(x)}{x}\right)$, the series of sequences of 2-connected chord diagrams counted by one less chord, then we can write

$$\left(\mathscr{A}^{2}_{\frac{1}{2}} C_{\geq 2}\right)(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{x^2}{C_{\geq 2} S} \cdot e^{\frac{-1}{2x}\left[(S + x)^2 - 1\right]}.$$

- By the definition of the map $\mathscr{A}^2_{\frac{1}{2}}$ one can work out the details to find that

$$
\begin{aligned}
(C_{\geq 2})_n &= \sum_{k=0}^{R-1} [x^k]\left(\mathscr{A}^2_{\frac{1}{2}} C_{\geq 2}\right)(x) \cdot \Gamma^2_{\frac{1}{2}}(n-k) + \mathscr{O}(\Gamma^2_{\frac{1}{2}}(n-R)) \\
&= \sqrt{2\pi}\ \sum_{k=0}^{R-1} [x^k]\left(\mathscr{A}^2_{\frac{1}{2}} C_{\geq 2}\right)(x) \cdot (2(n-k)-1)!! + \mathscr{O}((2(n-R)-1)!!) \\
&= e^{-2}(2n-1)!!\Bigg(1 - \frac{6}{2n-1} - \frac{4}{(2n-3)(2n-1)} - \frac{218}{3}\frac{1}{(2n-5)(2n-3)(2n-1)} - \\
&\qquad - \frac{890}{(2n-7)(2n-5)(2n-3)(2n-1)} - \frac{196838}{15}\frac{1}{(2n-9)\cdots(2n-1)} - \ldots\Bigg).
\end{aligned}
$$

- The result by Kleitman **[Kleit]** corresponds to the first term in this expansion. By the above approach, any precision can be achieved and an arbitrary number of terms can be produced.

- This also shows that a randomly chosen chord diagram on n chords is 2-connected with a probability of

$$\frac{1}{e^2}\left(1-\frac{3}{n}\right)+\mathcal{O}(1/n^2)\,.$$

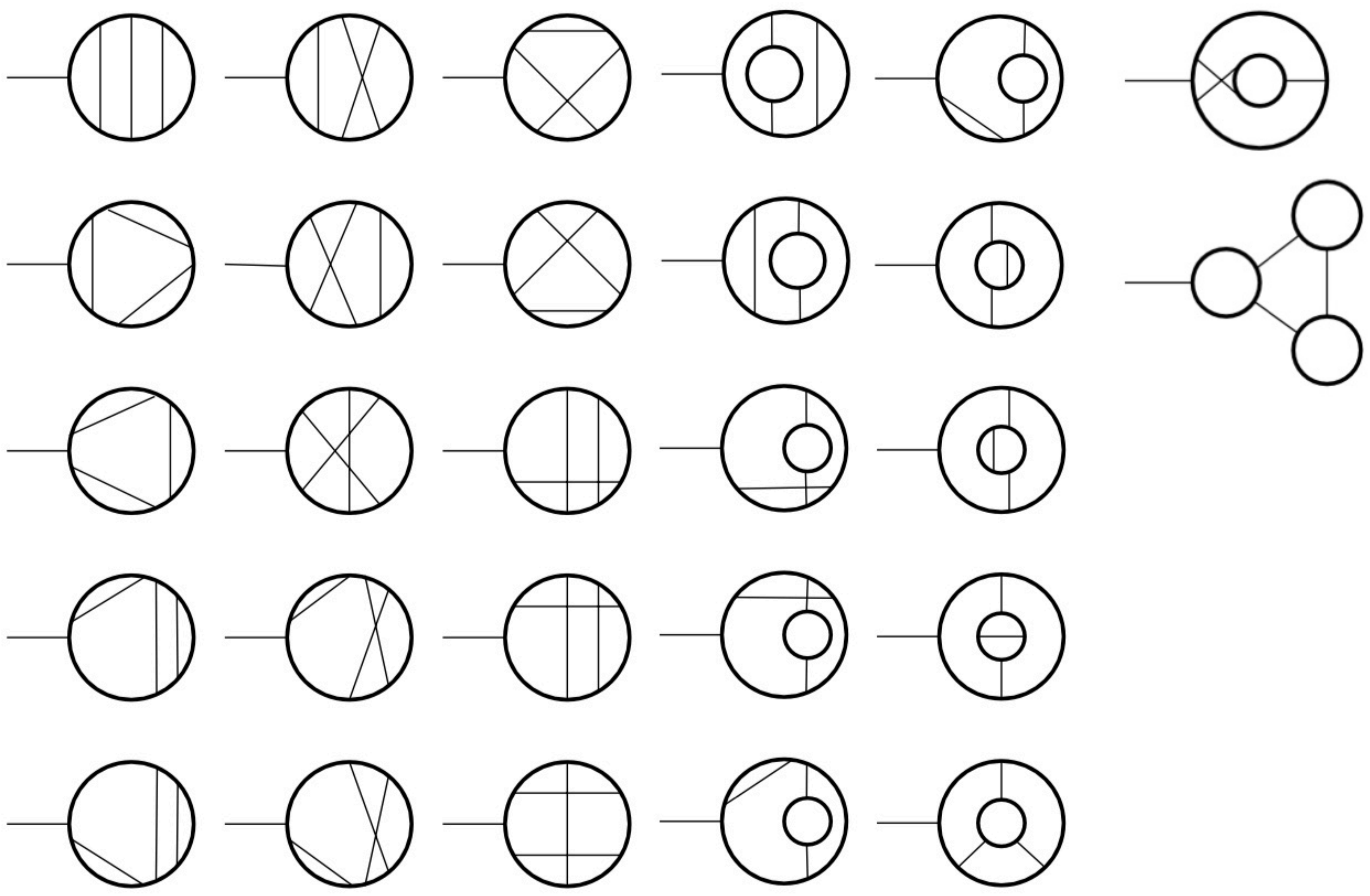
Zero-dimensional QQED and Yukawa Theories

Yukawa Theory

- We will be interested in interpreting the Yukawa theory diagrams in terms of connected chord diagrams.
- In **[MBor3]**, M. Borinsky calculated the observables in the table below and studied their asymptotics using singularity analysis.

		$\hbar^0$	$\hbar^1$	$\hbar^2$	$\hbar^3$	$\hbar^4$	$\hbar^5$	$\hbar^6$
1	$\partial^0_{\phi_c}(\partial_{\psi_c}\partial_{\bar{\psi}_c})^0 G^{\text{Yuk}}\big\|_{\phi_c=\psi_c=0}$	0	0	1/2	1	9/2	31	283
2	$\partial^1_{\phi_c}(\partial_{\psi_c}\partial_{\bar{\psi}_c})^0 G^{\text{Yuk}}\big\|_{\phi_c=\psi_c=0}$	0	1	1	4	27	248	2830
3	$\partial^2_{\phi_c}(\partial_{\psi_c}\partial_{\bar{\psi}_c})^0 G^{\text{Yuk}}\big\|_{\phi_c=\psi_c=0}$	-1	1	3	20	189	2232	31130
4	$\partial^0_{\phi_c}(\partial_{\psi_c}\partial_{\bar{\psi}_c})^1 G^{\text{Yuk}}\big\|_{\phi_c=\psi_c=0}$	-1	1	3	20	189	2232	31130
5	$\partial^1_{\phi_c}(\partial_{\psi_c}\partial_{\bar{\psi}_c})^1 G^{\text{Yuk}}\big\|_{\phi_c=\psi_c=0}$	1	1	9	100	1323	20088	342430

- For example $[\hbar^4]\partial^1_{\phi_c}(\partial_{\psi_c}\partial_{\bar{\psi}_c})^0 G^{\text{Yuk}}\Big|_{\phi_c=\psi_c=0}$ counts **27** 1PI Yukawa **tadpole** graphs with **loop number** 4. The fact that this agrees with **connected chord** diagrams with 4 **chords** is no coincidence as we shall see. Below are the 27 tadpoles:



- **Theorem 3.2:** The number of Yukawa **1PI tadpole** graphs with **loop** number n is equal to the number of **connected** chord diagrams on n **chords**. In other words

$$[\hbar^n]\partial^1_{\phi_c}(\partial_{\psi_c}\partial_{\bar{\psi}_c})^0 G^{\mathsf{Yuk}}\Big|_{\phi_c=\psi_c=0} = C_n\,.$$

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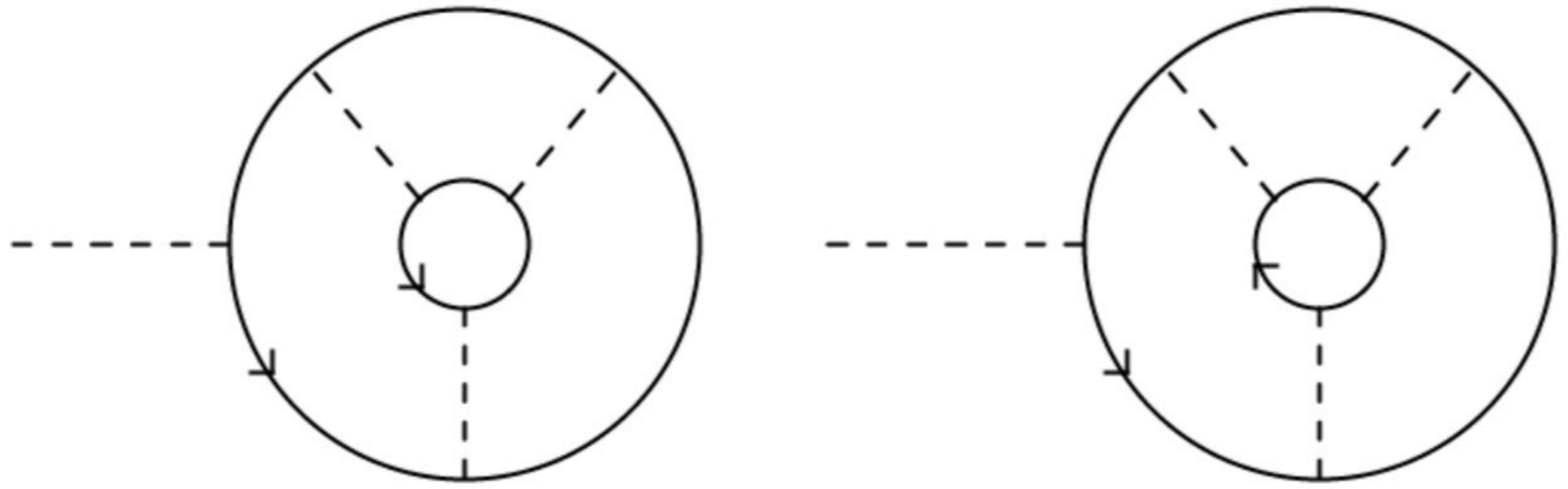
Sketch of proof:

- We will show that the the generating series $T(x)$ for the class $\mathscr{U}_{10}$ of **1PI Yukawa theory tadpoles**, counted by the **loop number** is equal to $C(x)$ by showing that $2xT(x)T'(x) = T(x)^2 + T(x) - x,$ the **recurrence** for connected chord diagrams.

- The first step is to set a standard for the graphs so that we no longer worry about the direction of the fermion loops: we assume that all fermion loops are **counter-clockwise**, compensated with crossings of boson edges.

Original



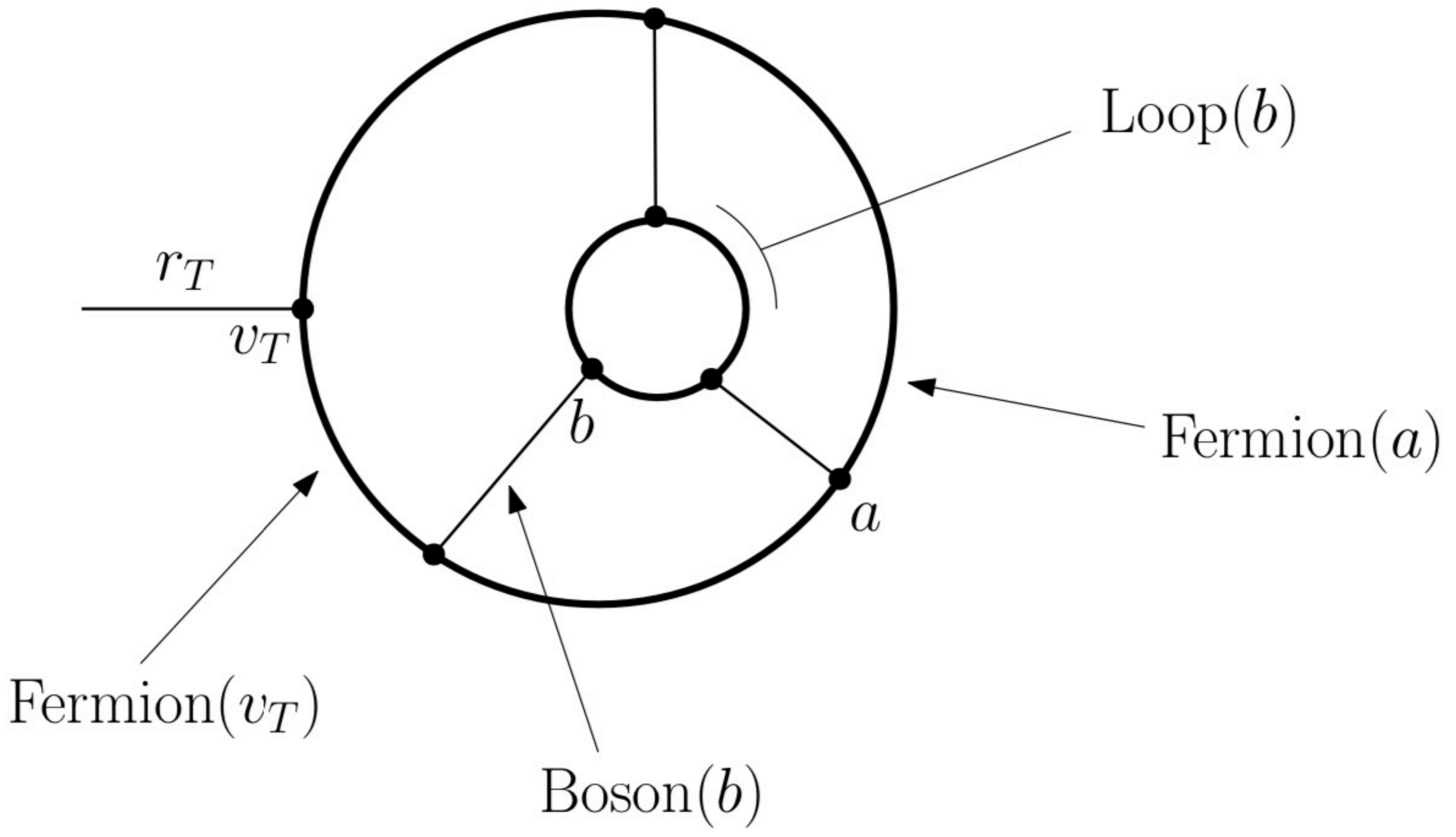
Two tadpoles may differ due to the relative orientation of fermion loops.

Standard

- Then we can prove that, in these graphs, the number of independent cycles is the **same** as the number of all boson edges. So, our generating series $T(x)$ now **counts** with the number of **boson** edges.

- We devise a **reversible** algorithm to get a bijection $\Psi : (\mathscr{U}_{10} \times \mathscr{U}^{\bullet}_{10}) \longrightarrow (\mathscr{U}_{10} \times \mathscr{U}_{10}) \bigcup (\mathscr{U}_{10} - \{\mathscr{X}\})$, which is what we need to satisfy the recurrence.

- We will use the following notation:



Algorithm Ψ: $(\mathcal{U}_{10} \times \mathcal{U}^{\bullet}_{10}) \longrightarrow (\mathcal{U}_{10} \times \mathcal{U}_{10}) \bigcup (\mathcal{U}_{10} - \{\mathcal{X}\})$

Input: $(T_1, (T_2, d)) \in (\mathcal{U}_{10} \times \mathcal{U}^{\bullet}_{10})$, with notation as described above.

(a) If $d = u_2$ just **return** (T_1, T_2).

(b) If $d \neq u_2$, do the following:

Move (counter-clockwise) along Loop(v_1) in T_1, determine Fermion(v_1) and let w be the first vertex met on the loop. Note that w may be v_1 itself.

1. If $w = v_1$, i.e. T_1 contains no internal boson edges, **return** the tadpole T obtained as follows:

   (i) Insert vertex v_1 together with the leg r_1 into Fermion(d) in T_2 by making a subdivision of Fermion(d).

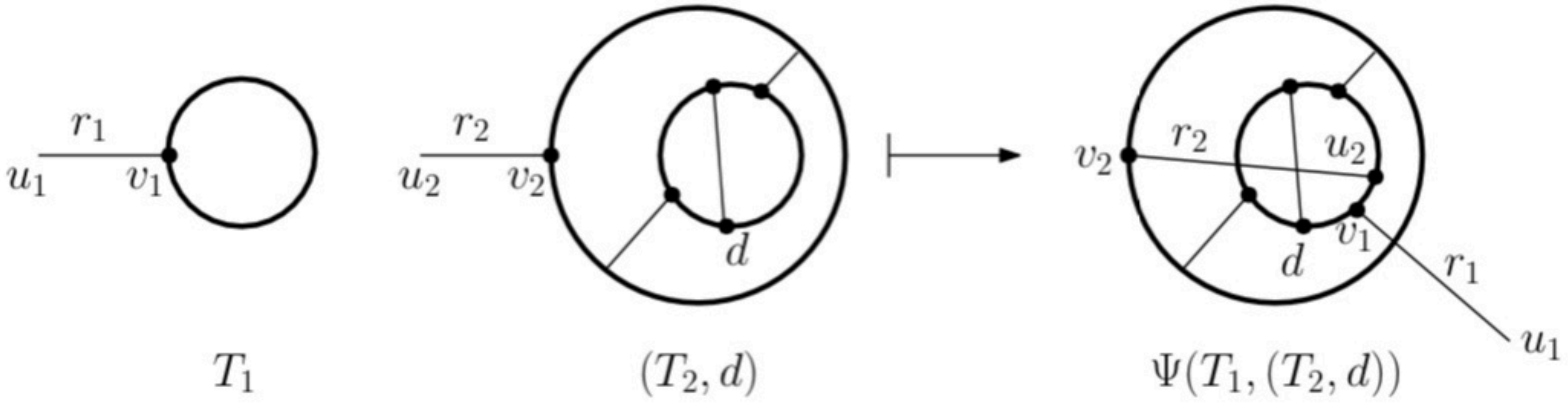
   (ii) Insert u_2 into the new Fermion(v_1) on Loop(d).

2. If $w \neq v_1$, **return** the tadpole T obtained as follows:

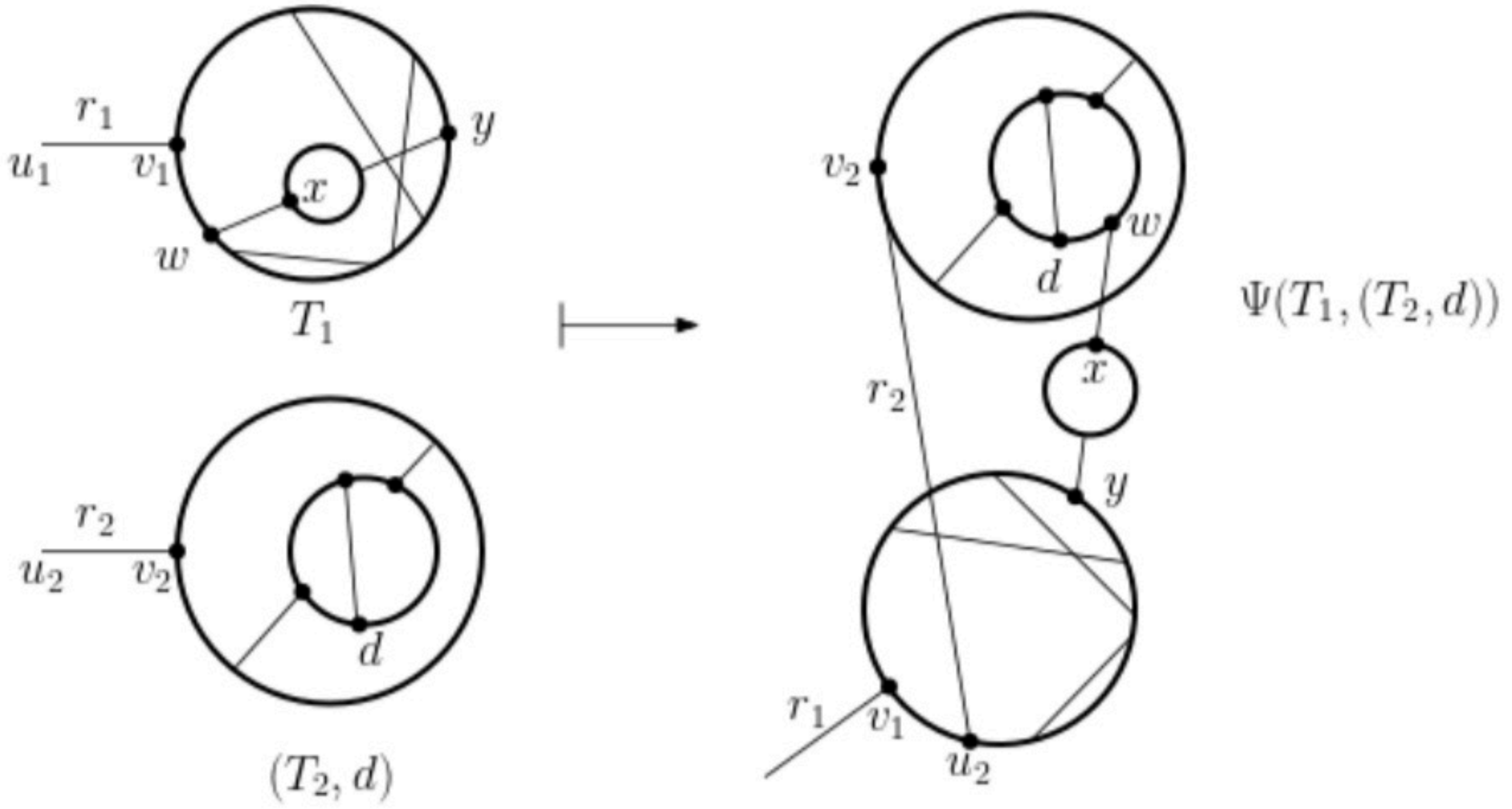
   (i) Insert u_2 into Fermion(v_1) in T_1.

   (ii) Detach w from Loop(v_1) and insert it into Fermion(d) in T_2.

- Examples:

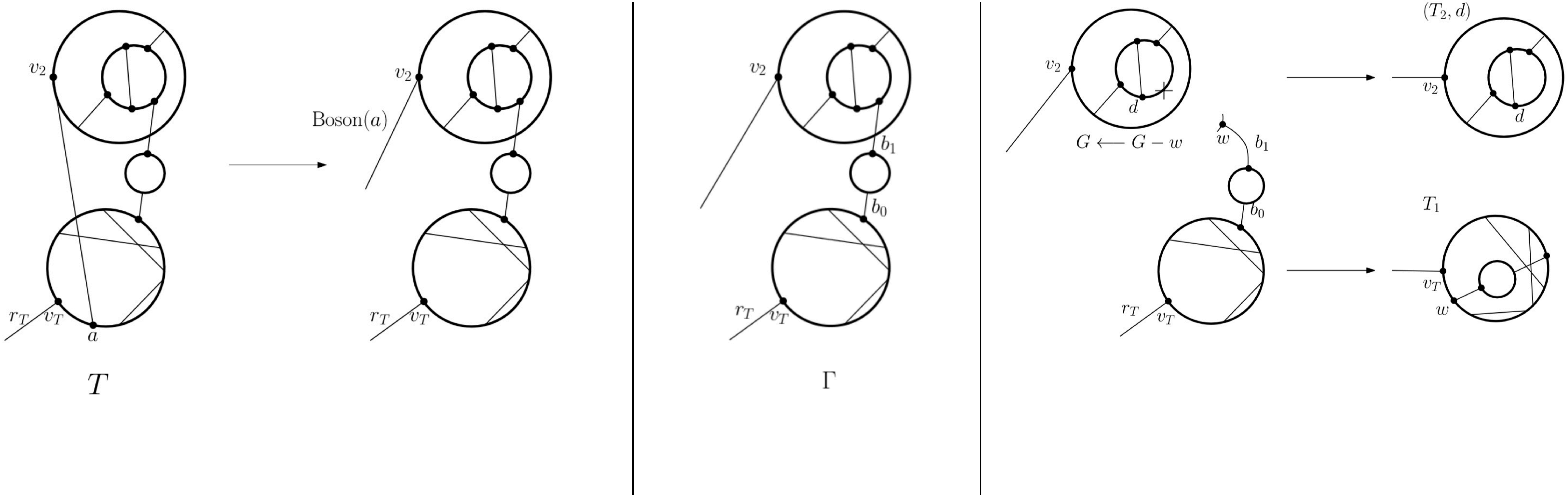


r_1
u_1
v_1
r_2
u_2
v_2
d
v_2
r_2
u_2
d
v_1
r_1
u_1
T_1
(T_2, d)
$\Psi(T_1, (T_2, d))$



r_1
u_1
v_1
x
y
w
T_1
r_2
u_2
v_2
d
(T_2, d)
v_2
d
w
$\Psi(T_1, (T_2, d))$
x
r_2
y
r_1
v_1
u_2

- This algorithm is **reversible**, why? The idea involves repeating a search for bridges in the component not containing the root till we determine d.

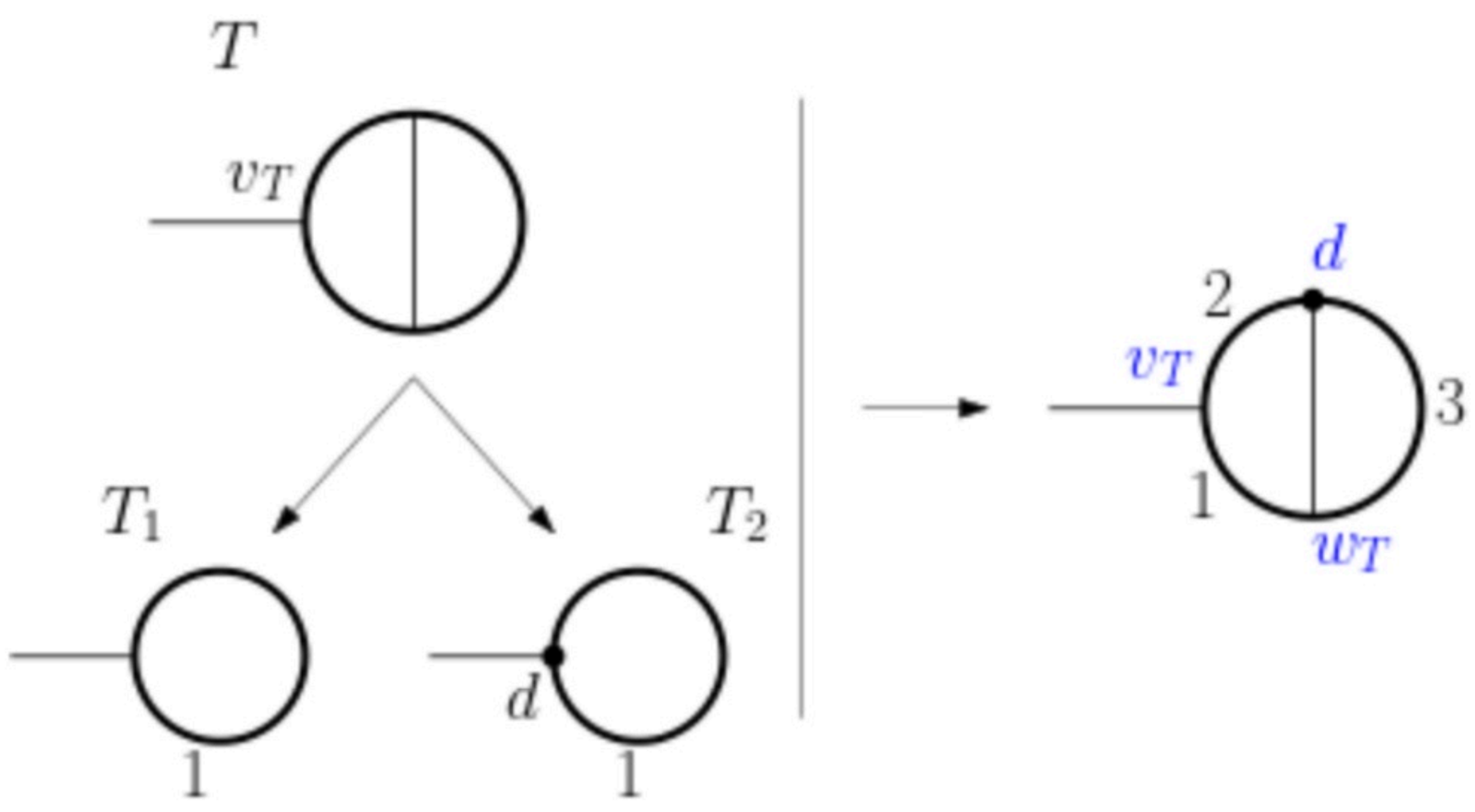


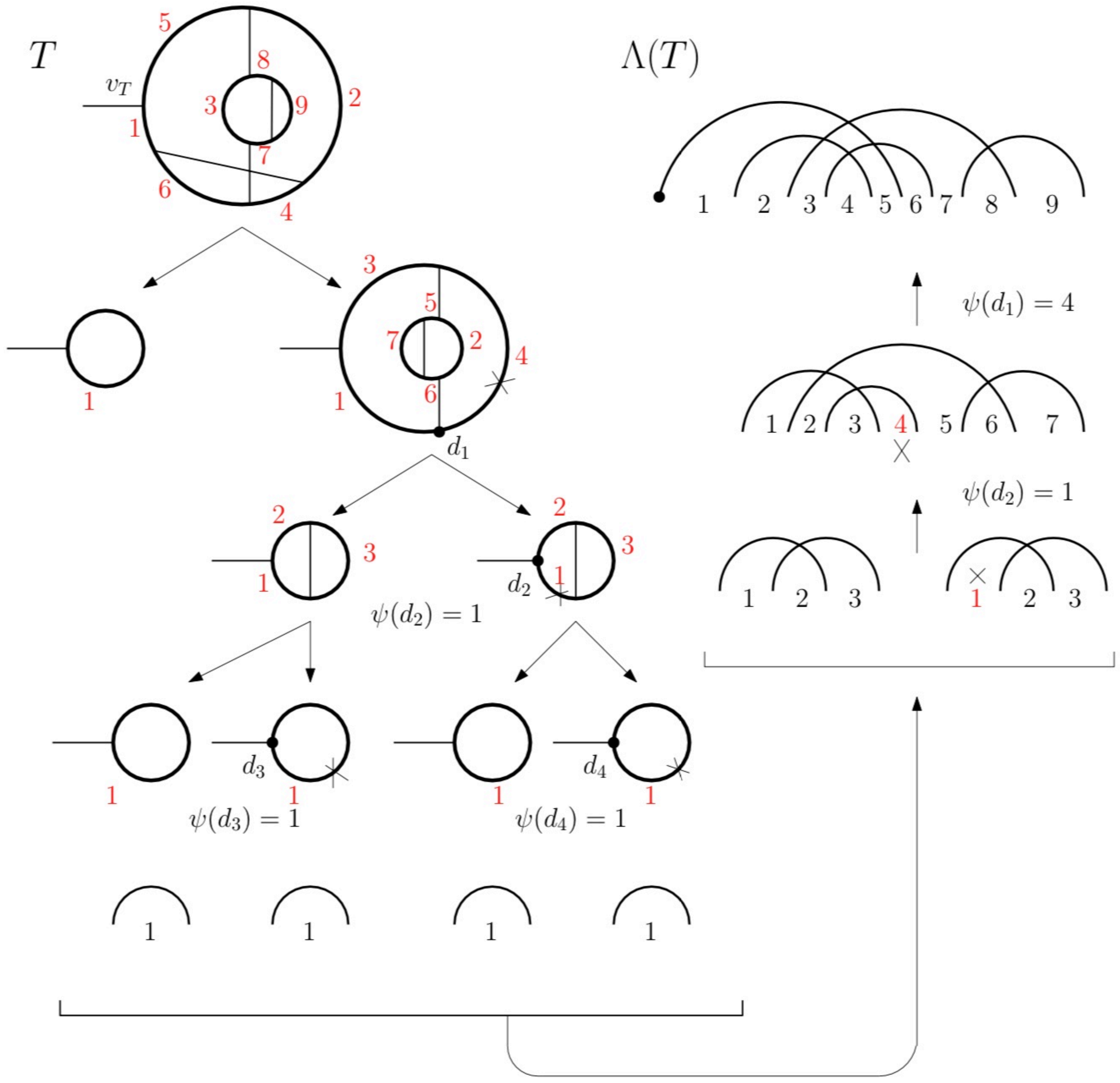
- **Theorem 3.3:** A bijection $\Lambda : \mathscr{U}_{10} \longrightarrow \mathscr{C}$ can be defined recursively as follows:

$$\Lambda(T) = \nabla^{-1}\big(\Lambda(T_1), (\Lambda(T_2), \psi(d))\big),$$

where $\Psi^{-1}(T) = (T_1, (T_2, d))$.

- The function $\psi : V(T) \to \mathbb{N}$ is an order defined to simulate the interval linear ordering for chord diagrams in a way that respects root share decomposition. Here is an example:





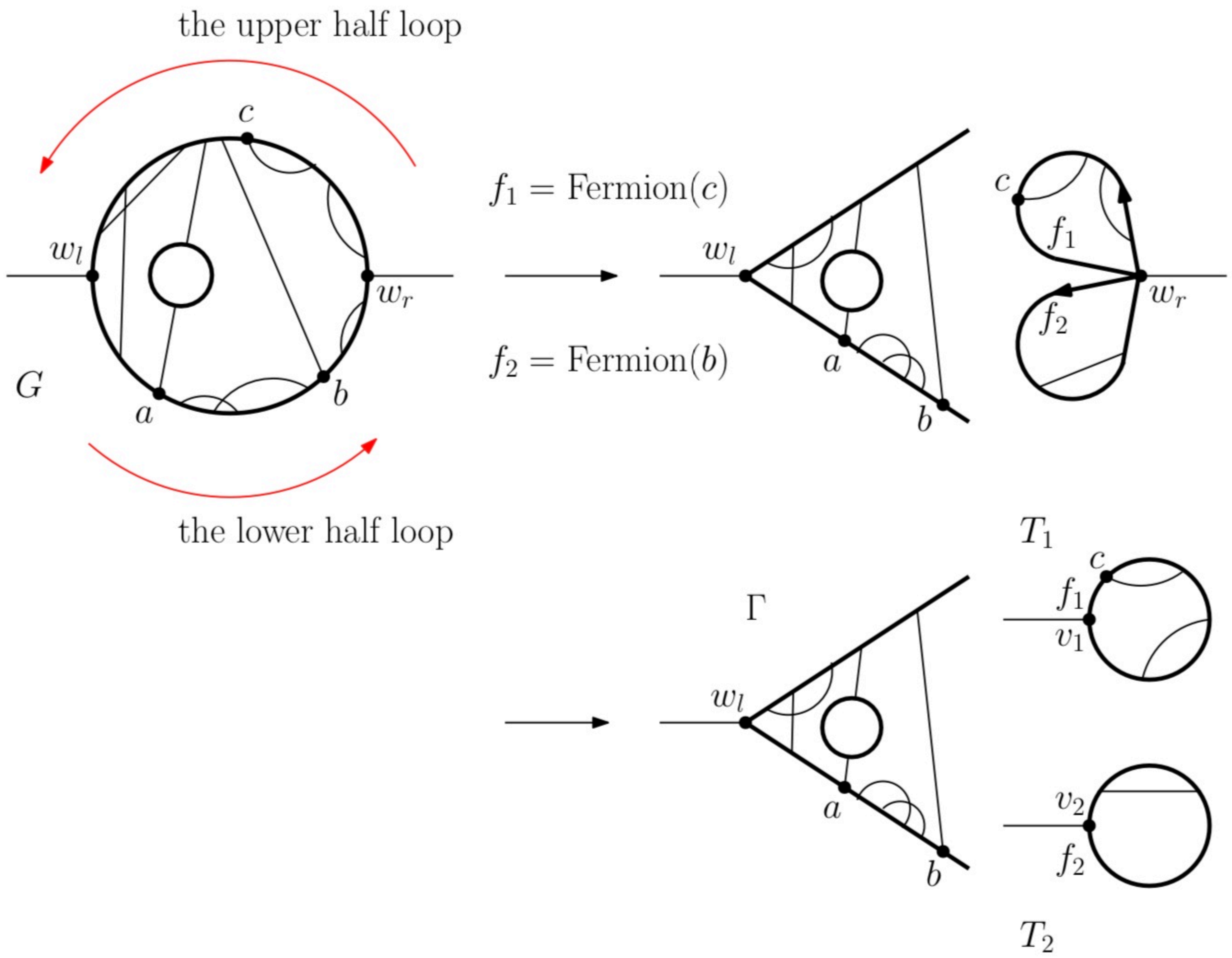
T
v_T
$\Lambda(T)$
d_1
d_2
d_3
d_4
$\psi(d_1) = 4$
$\psi(d_2) = 1$
$\psi(d_2) = 1$
$\psi(d_3) = 1$
$\psi(d_4) = 1$

- As a consequence of the previous, it also follows that the green function for **vacuum** diagrams satisfies $[\hbar^{n+1}]\partial^0_{\phi_c}(\partial_{\psi_c}\partial_{\bar{\psi}_c})^0 G^{\mathsf{Yuk}}(\hbar, \phi_c, \psi_c)\Big|_{\phi_c=\psi_c=0} = [x^n]\dfrac{C(x)^2}{2x}$.

- Also for the graphs with **2** external bosons

$$[\hbar^n]\partial^2_{\phi_c}(\partial_{\psi_c}\partial_{\bar{\psi}_c})^0 G^{\mathsf{Yuk}}(\hbar, \phi_c, \psi_c)\Big|_{\phi_c=\psi_c=0} = [x^n]\ \frac{C(x)^2}{x}\ \left[\frac{C_{\geq 2}(t)}{t^2}\Big|_{t=C(x)^2/x}\right],$$ or equivalently

$$U_{20}(x) = C(x)^2\ \left[\frac{C_{\geq 2}(t)}{t^2}\Big|_{t=C(x)^2/x}\right].$$

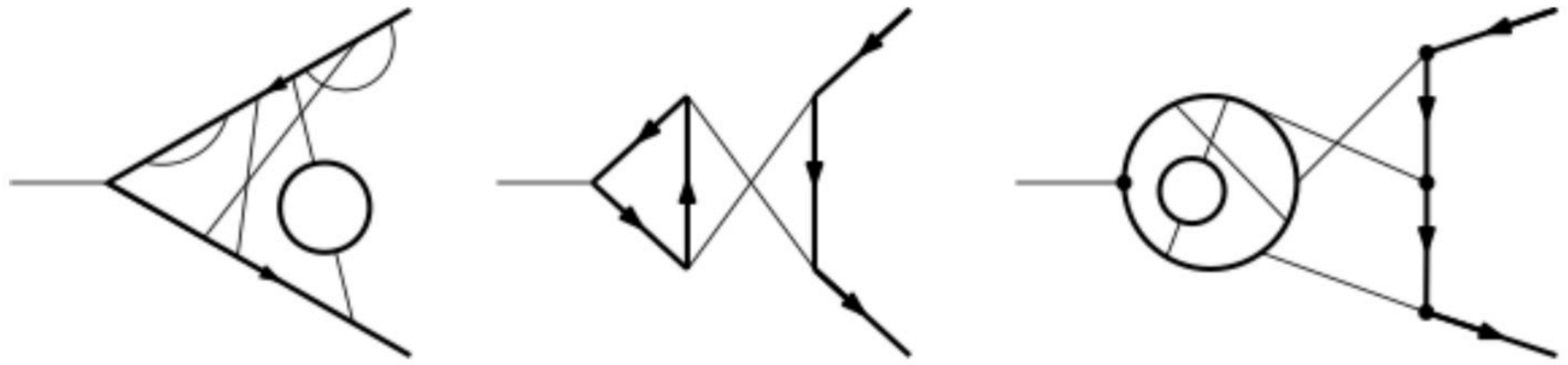


the upper half loop
c
w_l
w_r
G
a
b
the lower half loop
f_1 = Fermion(c)
f_2 = Fermion(b)
w_l
a
b
c
f_1
f_2
w_r
T_1
Γ
w_l
a
b
c
f_1
v_1
v_2
f_2
T_2

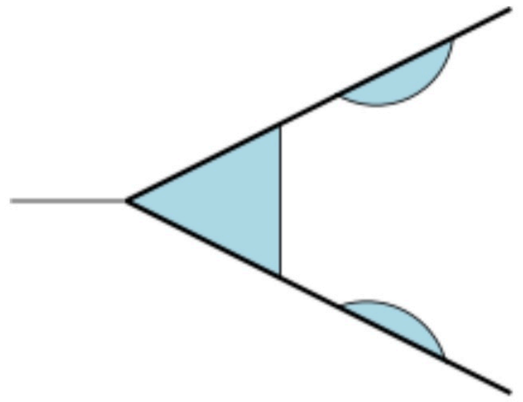
Yukawa Graphs $\partial^1_{\phi_c}(\partial_{\psi_c}\partial_{\bar{\psi}_c})^1 G^{\text{Yuk}}(\hbar, \phi_c, \psi_c)\Big|_{\phi_c=\psi_c=0}$

- **Theorem 3.4:** Let $\mathscr{U}_{11}$ be the class of Yukawa 1PI graphs generated by $\partial^1_{\phi_c}(\partial_{\psi_c}\partial_{\bar{\psi}_c})^1 G^{\text{Yuk}}(\hbar, \phi_c, \psi_c)\Big|_{\phi_c=\psi_c=0}$ and let $U_{11}(x)$ be their generating series, counted by the number of all boson edges. Then $U_{11}(x) = x \left. \frac{C_{\geq 2}(t)}{t^2} \right|_{t=C(x)^2/x}$.

Yukawa Graphs $\partial^1_{\phi_c}(\partial_{\psi_c}\partial_{\bar{\psi}_c})^1 G^{\text{Yuk}}(\hbar, \phi_c, \psi_c)\Big|_{\phi_c=\psi_c=0}$

- Examples: 

- Note also that the following is not considered as it is **not** 1PI:

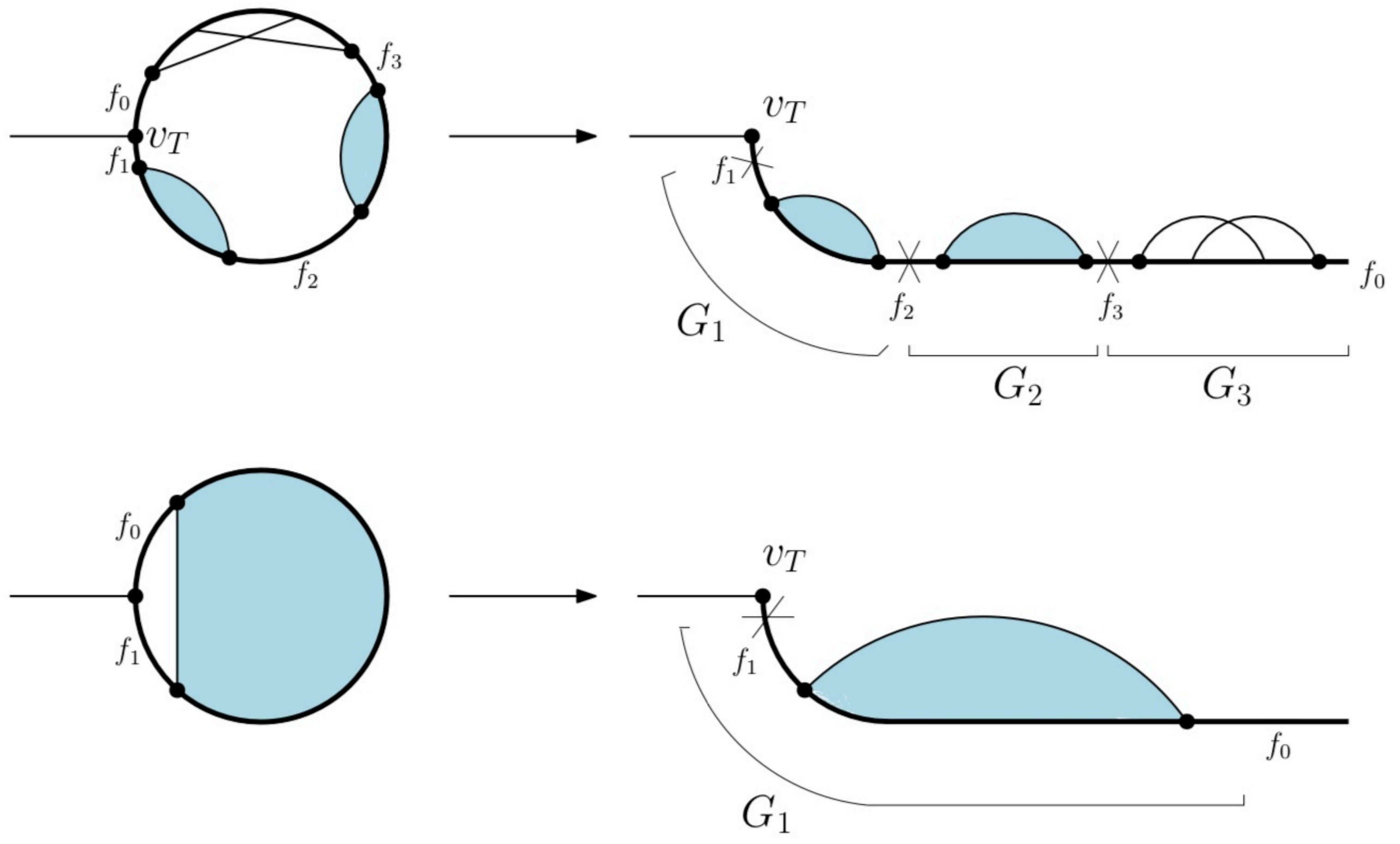


- Again one can prove that the loop number for these graphs is **equal** to the number of internal boson edges.

Yukawa Graphs $\partial^0_{\phi_c}(\partial_{\psi_c}\partial_{\bar{\psi}_c})^1 G^{\text{Yuk}}(\hbar, \phi_c, \psi_c)\Big|_{\phi_c=\psi_c=0}$

- **Theorem 3.5:** A Yukawa 1PI **tadpole** graph can be **decomposed** as a boson leg together with a **list** of graphs from $\mathcal{U}_{01}$. In particular, on the level of generating functions we will have $T(x) = \dfrac{x}{1 - U_{01}(x)}$. Then

$$U_{01}(x) = 2xC'(x) - C(x) = C(x)^2 \left[\frac{C_{\geq 2}(t)}{t^2}\Bigg|_{t=C(x)^2/x} \right] = U_{20}(x)\,.$$



- **Theorem 3.1:** The generating series $z_{\phi_c|\psi_c|^2}(\hbar_{\text{ren}})$ and $z_{|\psi_c|^2}(\hbar_{\text{ren}})$ count **2-connected** chord diagrams. More precisely, $[\hbar_{\text{ren}}^{n-1}]\, z_{\phi_c|\psi_c|^2}(\hbar_{\text{ren}}) = [\hbar_{\text{ren}}^{n}]\, z_{|\psi_c|^2}(\hbar_{\text{ren}}) = [x^n]\, C_{\geq 2}(x)\,.$

Sketch of proof:

- First note that, by the previous slides, $z_{\phi_c|\psi_c|^2}(\hbar_{\text{ren}})$ counts 1PI **primitive** QED graphs with **no** fermion loops. The power of $\hbar_{\text{ren}}$ is the loop number.
- One shows that the loop number for such graphs is the same as the number of internal photon edges.
- Finally, we will need to prove that **subdivergences** exactly correspond to reasons for connectivity-1.

- By obtaining the previous relations, we can readily get the asymptotic information using what we know about chord diagrams.

References

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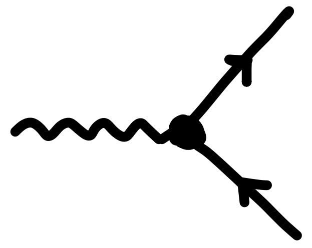
Thank you!

Quenched QED

- We will calculate the asymptotics of $z_{\phi_c|\psi_c|^2}(\hbar_{\text{ren}})$ in **quenched QED** using the results we have about **2-connected** chord diagrams and **without** any reference to singularity analysis.

- The partition function for QED-type theories is generally of the form $Z(\hbar, j, \eta) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2} + jx + \frac{|\eta|^2}{1-x} + \hbar \log \frac{1}{1-x}\right)} dx$.

  Quenched QED is an approximation of QED where **fermion loops** are **not** present. So the term $\hbar \log \frac{1}{1-x}$ does not appear in the partition function.

- The theory has a unique vertex type, which is cubic: 

- In this case the partition function becomes an **ordinary** integral.
- The amplitude for every diagram is 1. The **Feynman rules** are represented by the **character** ϕ that sends $\Gamma \mapsto \hbar^{\ell(\Gamma)}$.
- The benefit of this model lies in the interpretation of observables as **combinatorial generating functions,** and has been extensively studied since the 1950's.
- We will set (where the sign is for edge-type residues):

$$X^r = 1 \pm \sum_{\substack{\text{1PI graphs } \Gamma \\ \text{with residue } r}} \frac{1}{\text{Sym } \Gamma} \Gamma.$$

- It can be shown that (see **[MBor3]**), in this case, the **counter-terms** $z_r := S^\phi\{X^r\}$ satisfy

$$z_r(\hbar_{\text{ren}}) = \frac{1}{g^r(\hbar(\hbar_{\text{ren}}))} \ ,$$

where $g^r(\hbar)$ is the proper green function for residue r, and $\hbar_{\text{ren}}$ is called the **renormalized expansion parameter**, which we need not define here.

- **Theorem ([MBor2]):** In a theory with a **cubic** vertex-type, the numeric coefficients of $z_r(\hbar_{\text{ren}})$ count the number of **primitive** diagrams if r is vertex-type.

- As is customary in QFT, to move to the **quantum effective action** G, which generates **1PI** diagrams, one takes the **Legendre** transform of the free energy $W(\hbar, j)$:
$$G(\hbar, \varphi_c) := W - j\varphi_c,$$
where $\varphi_c := \partial_j W$. The coefficients $[\varphi_c^n]G$ are the (proper) Green functions.

- The order of the derivative $\partial^n_{\varphi_c} G|_{\varphi_c=0} = [\varphi_c^n]G$ determines the number of external legs.

- If **residue** r is the k external legs residue, then $g^r = \partial^k_{\varphi_c} G|_{\varphi_c=0}$.