Spaces of graphs, tori and other flat Γ-complexes

Karen Vogtmann

Dirk Kreimer's birthday conference

I.H.E.S. (virtually)

18 November 2020

Spaces of graphs, tori and other flat Γ-complexes

Two universes - commutative and non-commutative

	Non-commutative	Commutative
free inhabitants	free groups	free abelian groups

In both cases the inhabitants exhibit a great deal of symmetry:

Any permutation of the generators induces an automorphism of the group.

Non-commutative	Partially commutative	Commutative
free groups	RAAGs	free abelian groups

Today I want to break that symmetry and talk about intermediate universes, ie partially commutative ones.

There is a zoo of free inhabitants of these universes
~. partially commutative groups
aka RAAGs

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Today I want to break that symmetry and talk about intermediate universes, ie partially commutative ones.

There is a zoo of free inhabitants of these universes

- partially commutative groups aka RAAGs

The broken symmetry results in a surprisingly rich set of subgroups – eg every hyperbolic 3-manifold group is (virtually) a RAAG.

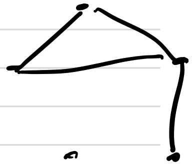
Non-commutative	Partially commutative	Commutative
free groups	RAAGs	free abelian groups

Each RAAG G has its own set of symmetries

visualization aid: graph Γ vertices = generators of Γ

edge between any two commuting generators

$G = A_\Gamma$



Any automorphism of G given by permuting generators is an automorphism of Γ

	Non-commutative	Partially commutative	Commutative
inhabitants	free groups F_n	RAAGs A_Γ	free abelian groups $\mathbb{Z}^n$
topological models	finite graphs: $\pi_1 = F_n$	Γ-complexes: $\pi_1 \cong A_\Gamma$	tori: $\pi_1 = \mathbb{Z}^n$
metric models	metric graphs	flat Γ-complexes	flat tori

Given a metric model X for G, together with an isomorphism
$$g : \pi_1 X \xrightarrow{\cong} G = A_\Gamma$$

$$\begin{tikzcd} \pi_1 X \arrow[r, "g"] \arrow[dr, "\alpha \circ g"'] & G \arrow[d, "\alpha"] \\ & G \end{tikzcd}$$

Can make a space by deforming the metric

$\alpha \in \mathrm{Aut}(G)$ acts on this space by changing the isomorphism (it does not change the metric on X)

$$\alpha \cdot (X, g) = (X, \alpha \circ g)$$

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metric models	metric graphs	flat Γ-complexes	flat tori
space of marked metric models	Outer space CV_n	$\mathcal{O}_\Gamma$ = "outer space for A_Γ"	symmetric space $SL_n\mathbb{R}/SO(n)$

CV_n is a space of marked graphs, $\mathcal{M}g_n = CV_n / Out(F_n)$ is just graphs

Theorem (Culler-V 1986) CV_n is contractible, the action is proper,

so $H^*(\mathcal{M}g_n; \mathbb{Q})$ is an invariant of $Out(F_n)$

CV_n and $\mathcal{M}g_n$ have variations $CV_{n,s}$, $\mathcal{M}g_{n,s}$ for graphs with leaves

They collect Feynman diagrams with fixed loop order and number of external leaves in a single geometric object.

Dirk, Marko Berghoff, Spencer Bloch and others have explored connections between the combinatorics and geometry of Outer space and the tools of perturbative quantum field theory

- parametric representation of Feynman integrals
- renormalization Hopf algebras
- Cutkosky rules

I don't know whether the partially commutative universe will also find applications in p.q.f.t., but if there are some then I think I'm talking to the right audience.

Piquing Dirk's interest has many benefits...

I met a lot of very interesting (and nice!) people:

Including Michi Borinsky, who used perturbative methods to finally solve a 30-year-old conjecture on the Euler characteristic of $\mathrm{Out}(F_n)$

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Commentarii Mathematici Helvetici

The Euler characteristic of $\mathrm{Out}(F_n)$

Michael Borinsky and Karen Vogtmann

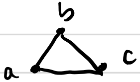
Abstract. We prove that the rational Euler characteristic of $\mathrm{Out}(F_n)$ is always negative and its asymptotic growth rate is $\Gamma(n - \frac{3}{2})/\sqrt{2\pi}\log^2 n$. This settles a 1987 conjecture of J. Smillie and the second author. We establish connections with the Lambert W function and the zeta function.

Γ-complexes

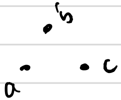
Γ = simplicial graph

A_Γ = associated RAAG

eg

$\Gamma =$ 

$A_\Gamma = \mathbb{Z}^3$

$\Gamma =$ 

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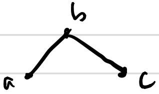
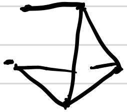
$\Gamma =$ 

$A_\Gamma = F_2 \times \mathbb{Z}$

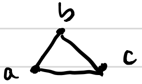
$\Gamma =$ 
$A_\Gamma = A_\Gamma$

Γ-complexes

Γ = simplicial graph

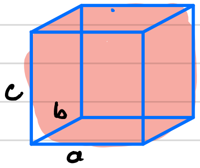
A_Γ = associated RAAG

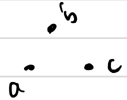
Space with $\pi_1 \cong A_\Gamma$

$\Gamma =$ 

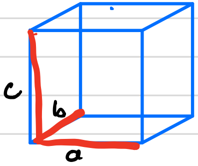
$A_\Gamma = \mathbb{Z}^3$ → $T^3 = S^1 \times S^1 \times S^1$

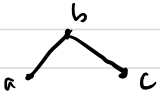
(identify opposite sides)



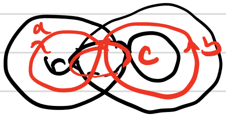
$\Gamma =$ 

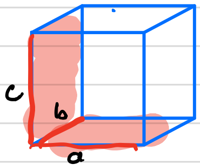
$A_\Gamma = F_3$ → $R_3 = S^1 \vee S^1 \vee S^1$

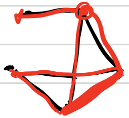


$\Gamma =$ 

$A_\Gamma = \underline{F_2} \times \underline{\mathbb{Z}}$ → $R_2 \times S^1$





$\Gamma =$ 
$A_\Gamma = A_\Gamma$ → Subcomplex of T^5 consisting of a torus for each clique

Definition. The **Salvetti complex**

S_Γ is the subcomplex of T^n consisting of the k-tori spanned by the k-cliques in Γ.

S_Γ is the most basic example of a Γ-complex

It is a **cube complex**: made of cubes glued together along faces.

Remark

To get a complex with $\pi_1 \cong A_\Gamma$, we only need the 2-skeleton of S_Γ (a 2-torus for each edge)

Using k-tori for all k-cliques guarantees that S_Γ has a metric of non-positive curvature (Gromov)

ie $\widetilde{S_\Gamma}$ is CAT(0)

(geodesic triangles are at least as thin as Euclidean ones)

Recall:

An isomorphism $\pi_1 S_\Gamma \xrightarrow{g} A_\Gamma$

is called a marking

$\mathrm{Aut}(A_\Gamma)$ acts on the set of marked Salvettis by changing the marking

$$\begin{tikzcd}
\pi_1 S_\Gamma \arrow[r, "g"] \arrow[dr, "\alpha\circ g"'] & A_\Gamma \arrow[d, "\alpha"] \\
& A_\Gamma
\end{tikzcd}$$

$$\alpha(S_\Gamma, g) = (S_\Gamma, \alpha\circ g)$$

If I don't want to specify a basepoint for S_Γ, I only get an action of $\mathrm{Out}(A_\Gamma)$

I want to make a space of marked Γ-complexes

(and prove this space has properties that are useful for group theory)

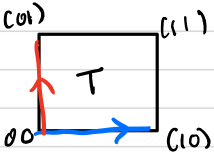
Clues are given by the cases

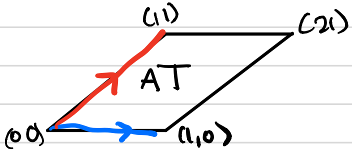
$A_\Gamma = \mathbb{Z}^n$ and $A_\Gamma = F_n$

$A_\Gamma = \mathbb{Z}^n$

$S_\Gamma = T^n = \mathbb{R}^n/\Lambda$, Λ a lattice

This gives T^n a flat metric





$$A = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \in GL_2\mathbb{Z} = Out(\mathbb{Z}^2)$$

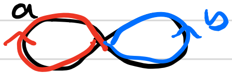
blue loop and red loop are basis for $\pi_1 T \cong \mathbb{Z}^2$

Same flat torus, different loops

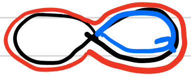
= different markings

$A_\Gamma = F_n$

$S_\Gamma = R_n$, all edges equal lengths.



G



αG

$$\alpha \begin{cases} a \mapsto ab \\ b \mapsto b \end{cases} \in Out(F_2)$$

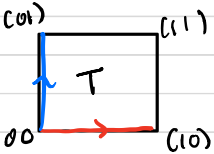
blue loop and red loop are basis for $\pi_1 G \cong F_2$

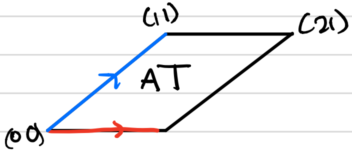
Same metric graph different loops

= different markings

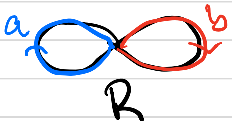
Connecting the dots:

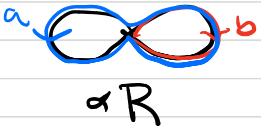
Tori





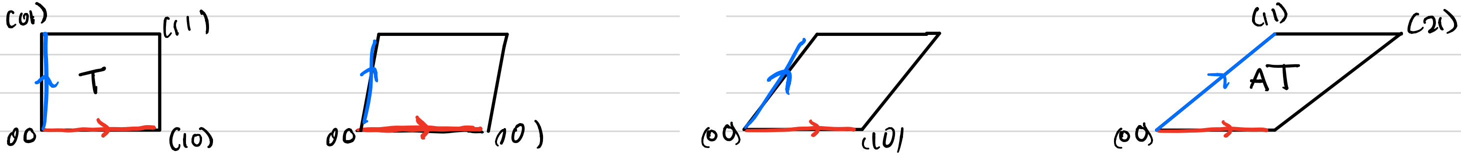
Roses





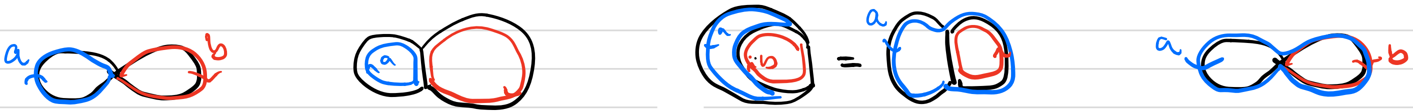
Connecting the dots:

Tori



skew gradually. On the way the tori have different metrics but are all flat tori with volume 1

Roses



insert a new edge, then collapse an old edge, adjusting lengths to preserve volume 1.

The space of marked flat tori with volume 1 is contractible

The space of marked metric graphs with volume 1 and 1 or 2 vertices is connected

To get a contractible space you need to allow graphs with up to $2n-2$ vertices

In general, a Γ-complex is a cube complex such that

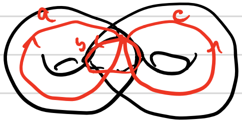
- A standard collapsing operation (hyperplane collapse) gives the Salvetti

If $A_\Gamma = F_n$, a Γ-complex is a graph with $\pi_1 \cong F_n$ and no separating edges (1-particle irreducible)

The collapsing operation collapses a maximal tree.

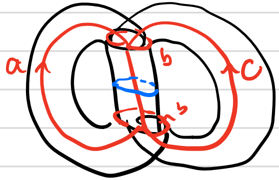
Example

Γ-complexes for $\Gamma =$ a — b — c

$S_\Gamma =$ 

Salvetti

↑ collapse central cylinder

$S_\Gamma^P =$ 

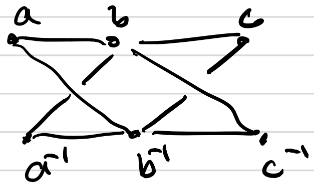
collapsing (any) cylinder of S_Γ^P to its waist circle recovers S_Γ

There is a combinatorial description of a Γ-complex in terms of partitions:

Form a graph $\Gamma^\pm$ with vertices x, x^{-1} for $x \in V(\Gamma)$

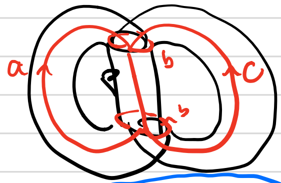
connect x to y if $[x,y]=1$ but $x \neq y^{-1}$

in our example S_Γ^P:

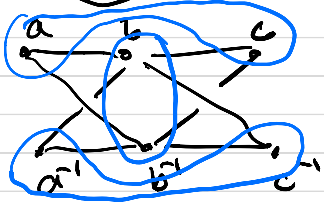
$\Gamma^\pm$ 

P is a partition of $V(\Gamma^\pm)$ into 3 sets, determined by the new edge P

$S_\Gamma^{\mathcal{P}} =$



$\Gamma^\pm$



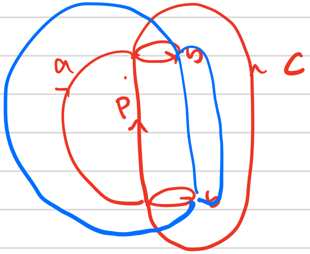
$\mathcal{P}$ is a partition of $V(\Gamma^\pm)$ into 3 sets, determined by the new edge P

$\operatorname{lk} P = \{b, b^{-1}\}$ half-edges at both vertices

$P = \{a, c\}$ half-edges at one vertex

$\bar{P} = \{a^{-1}, c^{-1}\}$ half-edges at other vertex

Given $\mathcal{P} = \{P \mid \bar{P} \mid \operatorname{lk} P\}$ satisfying some basic rules, I can reconstruct $S^{\mathcal{P}}$:



Rules for $\mathcal{P}$:

- There is some $a \in P$ with $a^{-1} \in \bar{P}$
- x, y in same component of $\Gamma^\pm - \operatorname{st} a$ $\Rightarrow$ x, y in same side of $\mathcal{P}$
- $x \in P,\ x^{-1} \in \bar{P} \Rightarrow \operatorname{lk} x \subset \operatorname{lk} P$

There is also a notion of compatibility

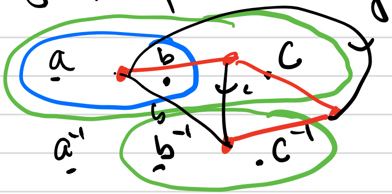
Given a set of compatible partitions $\mathcal{P}_1, \dots, \mathcal{P}_k$

I can construct a NPC cube complex with k vertices

Collapsing hyperplanes $\longleftrightarrow$ removing partitions.

collapsing all $\mathcal{P}_i$ gives back S_Γ

For $A_\Gamma = F_n$, $\mathcal{P} = \{P \mid \bar{P} \mid \phi\}$ is just a partition of $V(\Gamma^{\pm})$



$\mathcal{P}, Q$ compatible if and only if they can be drawn with non-intersecting circles

The dual tree is a maximal tree in a graph X

To get a space, want to put metrics on Γ-complexes

"Cubes" are isometric to Euclidean parallelotopes

To retain enough control to prove things, want metric to be be flat (locally CAT(0))

Definition A point in $\mathcal{O}_\Gamma$ is a locally CAT(0) metric space X isometric to some flat Γ-complex, together with an isomorphism

$$\pi_1 X \xrightarrow{g} A_\Gamma$$

Theorem (Bregman-Charney-V 2020)

$\mathcal{O}_\Gamma$ is contractible and the action of $\mathrm{Out}(A_\Gamma)$ is proper.

Depends on earlier theorem

Theorem (Charney-Stambach-V, 2017)

The geometric realization of the poset of Γ-complexes with untwisted markings is contractible

Adding the metric information and arbitrary markings was harder than we anticipated...

Issues to be dealt with in proof

1. Combinatorics of untwisted Γ-complexes

Σ_Γ

2. Straightening a twisted Γ-complex

T_Γ

3. Determining all possible parallelotope decompositions of Y compatible with the marking.

Θ_Γ


HAPPY
BIRTHDAY