Toric Hall algebras and infinite-dimensional Lie algebras

Joint work with Jaiung Jun

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Happy Birthday Dirk !

Main Idea:

- The Connes-Kreimer Hopf algebras of rooted trees and Feynman graphs, and many other combinatorial Hopf algebras arise as *Hall algebras*.
- Hall algebras have structure coefficients that count extensions in a category.
- I will describe a Hall algebra construction which attaches to a projective toric variety X_Σ a Hopf algebra $H^T_X \simeq \mathrm{U}(\mathfrak{n}^T_X)$.

Outline:

- Hall algebras of finitary abelian categories ("traditional setting")
- Hall algebras in the non-additive setting
- Monoid schemes
- The Hall algebra of T-sheaves on X_{Σ} and examples.

Hall algebras of finitary abelian categories ("traditional" setting)

Definition

An abelian (or exact) category $\mathcal{A}$ is called *finitary* if $\mathrm{Hom}(M, N)$ and $\mathrm{Ext}^1(M, N)$ are finite **sets** for any pair of objects $M, N \in \mathcal{A}$.

Example

- $\mathcal{A} = \mathrm{Rep}(Q, \mathbb{F}_q)$, where Q is a quiver.
- $\mathcal{A} = \mathrm{Coh}(X)$, where X is a projective variety over $\mathbb{F}_q$.

Given a finitary abelian category $\mathcal{A}$, we may define

$$H_{\mathcal{A}} := \{ f : Iso(\mathcal{A}) \rightarrow \mathbb{Q} | f \text{ has finite support } \}$$

with convolution product

$$f \bullet g([M]) = \sum_{N \subset M} f([M/N]) g([N])$$

It's easy to see that

$$\delta_{[M]} \bullet \delta_{[N]} = \sum g^K_{M,N} \delta_{[K]}$$

where

$$g^K_{M,N} = |\{L \subset K | L \simeq N, K/L \simeq M\}|$$

$g^K_{M,N} |Aut(M)||Aut(N)|$ counts the number of isomorphism classes of short exact sequences

$$0 \to N \to K \to M \to 0.$$

One can also consider a twist $\widetilde{H}_{\mathcal{A}}$ of $H_{\mathcal{A}}$ by the multiplicative Euler form

$$\langle M, N \rangle_m := \sqrt{\prod_{i=0}^{\infty} |\operatorname{Ext}^i(M,N)|^{(-1)^i}}$$

New multiplication:

$$f \star g([M]) = \sum_{N \subset M} := \langle M/N, N \rangle_m f([M/N]) g([N])$$

Theorem (Ringel, Green)

Let $\mathcal{A}$ be a finitary abelian category. Then $H_{\mathcal{A}}, \widetilde{H}_{\mathcal{A}}$ are associative algebras.

Bialgebra structures

When $\mathcal{A}$ is *hereditary* (*gldim*$(\mathcal{A}) \leq 1$), $\widetilde{H}_{\mathcal{A}}$ can be equipped with a co-product $\Delta_{\mathcal{A}}$ and antipode $S_{\mathcal{A}}$ (there are some subtleties here), such that $(H_{\mathcal{A}}, \Delta_{\mathcal{A}}, S_{\mathcal{A}})$ is a Hopf algebra.

Hall algebras of $\mathbb{F}_q$-linear finitary abelian categories are interesting quantum-group type objects.

Theorem (Ringel, Green)

Let Q be a quiver, with associated Kac-Moody algebra $\mathfrak{g}_Q$, There is an embedding

$$U^+_{\sqrt{q}}(\mathfrak{g}_Q) \hookrightarrow \widetilde{H}_{Rep(Q,\mathbb{F}_q)}$$

This is an isomorphism in types A, D, E.

Theorem (Kapranov, Kassel-Baumann)

There is an embedding

$$U^+_{\sqrt{q}}(\widehat{\mathfrak{sl}}_2) \hookrightarrow \widetilde{H}_{Coh(\mathbb{P}^1_{\mathbb{F}_q})}$$

- Work of Burban-Schiffmann relates Hall algebras of elliptic curves over $\mathbb{F}_q$ to spherical DAHA etc.
- Study of Hall algebras of coherent sheaves on smooth projective curves over $\mathbb{F}_q$ is closely related to the theory of automorphic forms over function fields (Kapranov)
- Extensive body of work by Kapranov, Schiffmann, Vasserot, and others.
- Little is known about the structure of these Hall algebras for higher genus curves and even less for higher-dimensional projective varieties X when $dim(X) > 1$.

Basic observation: the algebra structure on $H_{\mathcal{A}}$:

$$f \bullet g([M]) = \sum_{N \subset M} f([M/N])g([N])$$

does not use the fact that $\mathcal{A}$ is additive !

In fact, one can define Hall algebras of certain non-additive categories.

Proto-Exact categories

- A very flexible framework for working with Hall algebras is provided by **proto-exact** categories, due to Dyckerhoff-Kapranov.
- These are a (potentially non-additive) generalization of Quillen exact category.
- One can define algebraic K-theory of proto-exact categories via the Waldhausen construction.

Theorem (Dyckerhoff-Kapranov)

If $\mathcal{C}$ is a proto-exact category such that $\mathrm{Hom}(X, Y)$ and $\mathrm{Ext}_{\mathcal{C}}(X, Y)$ are finite sets $\forall X, Y \in \mathcal{C}$, then one can define an associative Hall algebra $H_{\mathcal{C}}$ as before (i.e. by counting short exact sequences).

Examples of proto-exact categories

Any exact or abelian category is proto-exact, but there are a number of non-additive examples, often of a "combinatorial" nature:

- Pointed sets $\mathcal{S}et_{\bullet}$
- If A is a monoid, the category of A-modules (pointed sets with A-action)
- $Rep(Q, \mathcal{S}et_{\bullet})$ where Q is a quiver.
- Pointed matroids
- **Rooted trees**
- **Feynman graphs**
- $Coh(X)$ - the category of coherent sheaves on a *monoid scheme* X.

Goal:

Let X be a projective variety over $\mathbb{F}_q$. We want to compute the "classical limit"

$$\lim_{q \to 1} H_{Coh(X)}$$

hoping it will shed some light on the structure of $H_{Coh(X)}$, especially when $dim(X) > 1$.

Philosphy of $\mathbb{F}_1$ - the "field" of one element

It's an old observation that many calculations performed over $\mathbb{F}_q$ have meaningful combinatorial limits as $q \to 1$

Example

$$|Gr(k,n)_{\mathbb{F}_q}| = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient (rational function in q). We have

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$$

This leads to the idea that "a pointed set is a vector space over $\mathbb{F}_1$".

Example

An observation of Tits is that if G is a simple algebraic group then

$$\lim_{q \to 1} |G(\mathbb{F}_q)| = |W(G)|$$

where $W(G)$ is the Weyl group of G.

This leads to the idea that "$G(\mathbb{F}_1) = W(G)$".

The "$\mathbb{F}_1$ dictionary" should go something like this:

- Vector spaces over $\mathbb{F}_1 \leftrightarrow \mathcal{S}et_{\bullet}$
- Algebra over $\mathbb{F}_1 \leftrightarrow$ monoid A
- $\mathbb{F}_1$-Algebra module $\leftrightarrow$ pointed set with A-action
- Scheme over $\mathbb{F}_1 \leftrightarrow$ monoid scheme

From this perspective, the proto-exact categories on the list below can be viewed as $\mathbb{F}_1$-linear:

- Pointed sets $\mathcal{S}et_{\bullet} = Vect_{\mathbb{F}_1}$
- If A is a monoid, the category of A-modules (pointed sets with A-action)
- $Rep(Q, Vect_{\mathbb{F}_1})$ where Q is a quiver.
- Pointed matroids
- Feynman graphs
- $Coh(X)$ - the category of coherent sheaves on a *monoid scheme* X.

When $\mathcal{C}$ is one of these categories, and finitary (Hom and Ext are finite), the Hall algebra $H_{\mathcal{C}}$ can be equipped with a simple co-commutative co-multiplication

$$\Delta : H_{\mathcal{C}} \mapsto H_{\mathcal{C}} \otimes H_{\mathcal{C}}$$

$$\Delta(f)(M, N) = f(M \vee N)$$

$(H_{\mathcal{C}}, \Delta)$ is a co-commutative bialgebra, and since it's graded (by $K_0^+(\mathcal{C})$) and connected, a co-commutative Hopf algebra.

The Milnor-Moore theorem tells us that $H_{\mathcal{C}} \simeq U(\mathfrak{n}_{\mathcal{C}})$ where $\mathfrak{n}_{\mathcal{C}}$ is the Lie algebra of primitive elements, which correspond to δ_M, where M is indecomposable (M cannot be written non-trivially as $M = K \vee L$).

Examples of Hall algebras in the non-additive setting

- Let $\langle t \rangle$ be the free monoid on one generator t - i.e. $\langle t \rangle = \{0, 1, t, t^2, t^3 \cdots \}$. Then $H_{\langle t \rangle - mod}$ is isomorphic to the (dual of) the Connes-Kreimer Hopf algebra of rooted trees.
- Let Q be a quiver. Viewing the underlying un-oriented graph of Q as a Dynkin diagram, we obtain a Kac-Moody algebra

$$\mathfrak{g}_Q = \mathfrak{n}_Q^- \oplus \mathfrak{h}_Q \oplus \mathfrak{n}_Q^+.$$

Theorem (S)

$$H_{Rep(Q, Vect_{\mathbb{F}_1})} \simeq U(\mathfrak{n}_Q)/\mathcal{I}$$

where $\mathcal{I}$ is a certain ideal, which is trivial in type A.

- $H_{FeynmanGraphs}$ is isomorphic to the dual of Connes-Kreimer's Hopf algebra of Feynman graphs.
- $H_{pointedmatroids}$ is isomorphic to the dual of Schmitt's matroid-minor Hopf algebra

All of these categories have a (complicated) K-theory. For instance, $K_{\bullet}(Vect_{\mathbb{F}_1})$ computes the stable homotopy groups of spheres !

Monoid schemes (Deitmar, Soule, Kato, Connes-Consani ...)

- An ordinary scheme is obtained by gluing prime spectra of rings
- If A is a commutative monoid, we can define ideals and prime ideals in the obvious way ($I \subset A$ is an ideal if $aI \subset I$, $\forall a \in A$, $\mathfrak{p} \subset A$ is prime if it's proper and $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}, \text{ or } b \in \mathfrak{p}$).
- We can equip $Spec(A) := \{\mathfrak{p} \subset A | \mathfrak{p} \text{ is prime } \}$ with the Zariski topology as in the case of rings.
- We can glue affine monoid schemes $\{Spec(A_i)\}$ to get general monoid schemes $(X, \mathcal{O}_X)$. $\mathcal{O}_X$ is now a sheaf of commutative monoids.
- We think of monoid schemes as "schemes over $\mathbb{F}_1$".

Example

- If k is a field, $\mathbb{A}^1_k = Spec(k[t])$. $\mathbb{A}^1_{\mathbb{F}_1} = Spec(\langle t \rangle)$. $\mathbb{A}^1_{\mathbb{F}_1}$ has one closed point (t) and a generic point (0).
- Similarly, $\mathbb{A}^n_{\mathbb{F}_1} = Spec\langle t_1, \cdots, t_n \rangle$. Primes correspond to subsets of $\{t_1, \cdots, t_n\}$ - "coordinate subspaces".
- We have monoid inclusions:

$$\langle t \rangle \hookrightarrow \langle t, t^{-1} \rangle \hookleftarrow \langle t^{-1} \rangle.$$

Taking spectra, and denoting by $U_0 = Spec\,\langle t \rangle, U_\infty = Spec\,\langle t^{-1} \rangle$, we obtain the diagram

$$\mathbb{A}^1_{\mathbb{F}_1} \simeq U_0 \hookleftarrow U_0 \cap U_\infty \hookrightarrow U_\infty \simeq \mathbb{A}^1_{\mathbb{F}_1}.$$

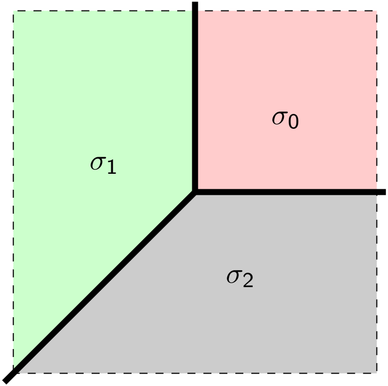
Gluing we get $\mathbb{P}^1_{\mathbb{F}_1}$. It has two closed points $0, \infty$, and a generic point.

From fans to monoid schemes

A *fan* $\Sigma \subset \mathbb{R}^n$ (in the sense of toric geometry) gives rise to a monoid scheme X_Σ

- Each cone $\sigma \in \Sigma$ yields a monoid S_σ.
- The fan gives gluing data for the $Spec(S_\sigma)$'s as in the construction of toric varieties.

The projective plane $\mathbb{P}^2_{\mathbb{F}_1}$, as a monoid scheme, arises from the following fan:



The fan Σ for $\mathbb{P}^2$

From σ_i, one obtains the following three affine monoid schemes:

1. $X_{\sigma_0} = \mathrm{Spec}(\langle x_1, x_2 \rangle) = \mathbb{A}^2_{\mathbb{F}_1}$,
2. $X_{\sigma_1} = \mathrm{Spec}(\langle x_1^{-1}, x_1^{-1} x_2 \rangle) = \mathbb{A}^2_{\mathbb{F}_1}$,
3. $X_{\sigma_2} = \mathrm{Spec}(\langle x_1 x_2^{-1}, x_2^{-1} \rangle) = \mathbb{A}^2_{\mathbb{F}_1}$.

which can be glued to form $\mathbb{P}^2_{\mathbb{F}_1}$.

Coherent sheaves

- If A is a commutative monoid, and M is an A-module, then we can form a quasicoherent sheaf $\widetilde{M}$ on $\operatorname{Spec} A$ as in the case of rings (localization works the same way).
- Such $\widetilde{M}$'s can be glued on an affine cover to yield quasicoherent sheaves on a monoid scheme X (for coherent we would take the M's to be finitely generated).

Proposition

Let X be a monoid scheme. The categories $Qcoh(X)$, $Coh(X)$ are proto-exact. So is the category $Coh(X)_Z$ of sheaves with prescribed set (resp. scheme)-theoretic support $Z \subset X$.

Problem: finitarity fails in the monoid scheme setting

- Even when Σ is the fan of a smooth projective toric variety, and $\mathcal{F}, \mathcal{F}' \in Coh(X_\Sigma)$, we may have $|\operatorname{Ext}(\mathcal{F}, \mathcal{F}')| = \infty$.
- There is therefore no way to define $H_{Coh(X_\Sigma)}$.

T-sheaves

To resolve the problem of infinite Ext's, we pass to a sub-category $Coh^T(X_\Sigma)$ of $Coh(X_\Sigma)$ - the category of T-sheaves.

On $Spec(S_\sigma)$, a T-sheaf corresponds to an S_σ-module M such that

1. M admits an S_σ-grading.
2. For $m, m' \in M$, and $s \in S_\sigma$,

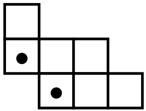
$$sm = sm' \neq 0 \Leftrightarrow m = m'$$

Theorem

An indecomposable T-sheaf on $\mathbb{A}^n_{\mathbb{F}_1} \simeq Spec(\langle x_1, \cdots, x_n \rangle)$ corresponds to a (possibly infinite) connected n-dimensional skew shape (convex connected sub-poset of $\mathbb{Z}^n_{\geq 0}$).

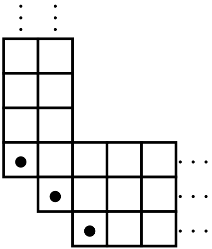
Example - torsion T-sheaf on $\mathbb{A}^2$ supported at the origin

Let $n = 2$. To the skew shape $\mathcal{S}$ below, we can associate a module $M_{\mathcal{S}}$ over the monoid $\langle x_1, x_2 \rangle$. x_1 (resp. x_2) act on $M_{\mathcal{S}}$ by moving one box to the right (resp. one box up) until reaching the edge of the diagram, and 0 beyond that. A minimal set of generators for $M_{\mathcal{S}}$ is indicated by the black dots:

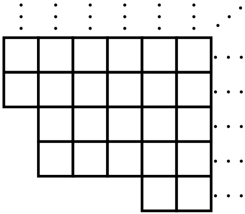


Note that $\mathfrak{m}^3 \cdot M_{\mathcal{S}} = 0$, where $\mathfrak{m}$ is the maximal ideal (x_1, x_2). $\widetilde{M}_{\mathcal{S}}$ is therefore a torsion sheaf supported at the origin in $\mathbb{A}^2_{\mathbb{F}_1}$.

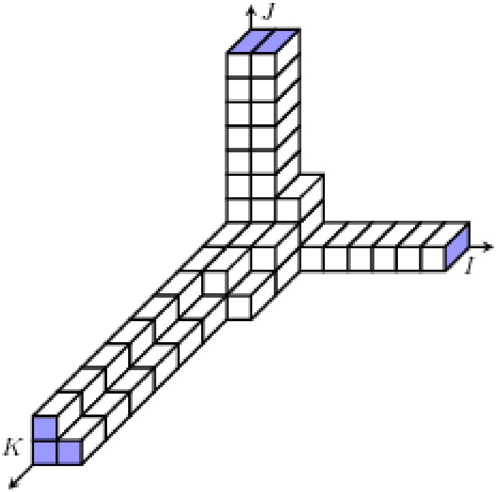
Example - torsion T-sheaf supported on the union of coordinate axes in $\mathbb{A}^2$



Example - torsion-free T-sheaf on $\mathbb{A}^2$



Example - torsion sheaf on $\mathbb{A}^3$ supported on union of coordinate axes



- If X_Σ is smooth, projective, and n-dimensional, then for each maximal cone $\sigma \in \Sigma$, $Spec(S_\sigma) \simeq \mathbb{A}^n$. A T-sheaf on X_Σ can therefore be thought of as being glued together from n-dimensional skew shapes.

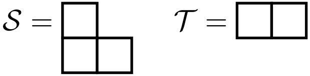
Theorem (J - S)

Let Σ be the fan of projective toric variety. Then the category $Coh^T(X_\Sigma)$ of coherent T-sheaves on X_Σ has the structure of finitary proto-abelian category. Its Hall algebra $H_{Coh(X_\Sigma)^T}$ has the structure of a co-commutative Hopf algebra isomorphic to $U(\mathfrak{n}_X)$, where $\mathfrak{n}_X$ has as basis the indecomposable coherent T-sheaves on X_Σ.

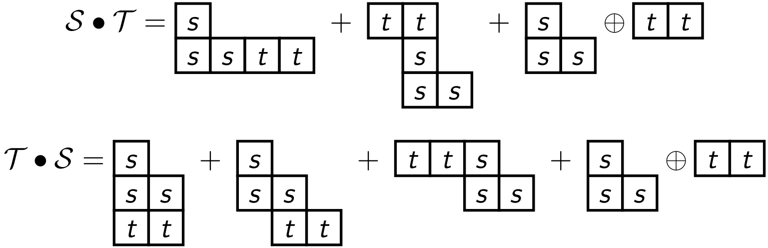
The theorem also holds for categories $Coh^T(X_\Sigma)_Z$ of T-sheaves supported in a closed subset/subscheme $Z \subset X$..

Example - multiplying two torsion sheaves in Hall algebra $H_{Coh^T(\mathbb{A}^2)_0}$

Let $n = 2$. By abuse of notation, we identify the skew shape λ with the delta-function of the corresponding coherent sheaf. Let



We have



where for each skew shape we have indicated which boxes correspond to S and T.

By identifying connected, finite, n-dimensional skew shapes with $Coh^T(\mathbb{A}^n)_0$, we obtain a Lie bracket on these, defined by

$$[\mathcal{S}, \mathcal{T}] = \mathcal{S} \bullet \mathcal{T} - \mathcal{T} \bullet \mathcal{S}$$

This Lie algebra has all structure constants $\pm 1, 0$.

Example: $Coh^T(\mathbb{P}^1)$

Theorem

$H_{Coh^T(\mathbb{P}^1)} \simeq U(\mathfrak{gl}_2^+[t,t^{-1}])$ *where*

$$\mathfrak{gl}_2^+[t,t^{-1}] \simeq \begin{pmatrix} a(t) & b(t) \\ 0 & c(t) \end{pmatrix}$$

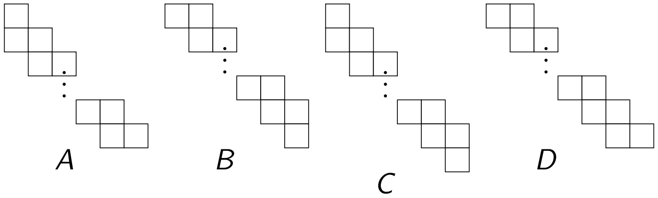
with $a(t), c(t) \in \mathbb{Q}[t], b(t) \in \mathbb{Q}[t,t^{-1}]$.

(classical limit of Kapranov's and Baumann-Kassel's result). Here

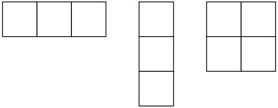
$$\mathcal{T}_{0,r} \to \begin{pmatrix} t^r & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{T}_{\infty,s} \to \begin{pmatrix} 0 & 0 \\ 0 & -t^s \end{pmatrix}, \mathcal{O}(n) \to \begin{pmatrix} 0 & t^n \\ 0 & 0 \end{pmatrix}$$

Example: Second formal neighborhood of 0 in $\mathbb{A}^2$

Consider the sub-scheme $Y = Spec(\langle x_1, x_2 \rangle / \mathfrak{m}^2)$ of $\mathbb{A}^2$, where $\mathfrak{m} = (x_1, x_2)$. Indecomposable T-sheaves on Y are one of the following types:



These are skew shapes not containing one of the following "disallowed" diagrams:



Theorem

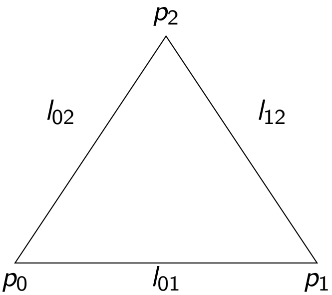
$H_{Coh^T(Y)} \simeq U(\mathfrak{k})$, where $\mathfrak{k}$ is the Lie subalgebra of $\mathfrak{gl}_2[t]$:

$$\begin{pmatrix} d(t) & a(t) \\ b(t) & c(t) \end{pmatrix}$$

where $a(t), b(t)$ are odd polynomials, with $\deg(a(t)) \geq 3, \deg(b(t)) \geq 1$, and $c(t), d(t)$ are even polynomials with $\deg(c(t)) \geq 2, \deg(d(t)) \geq 4$.

Example - $Coh^T(\mathbb{P}^2)$

We can visualize $\mathbb{P}^2$ as follows:



where p_i are torus fixed-points, and the l's the torus-fixed $\mathbb{P}^1$'s connecting them.

- We can classify all indecomposable T-sheaves on $\mathbb{P}^2$. These are (roughly) of three types:
  1. These are point sheaves supported at p_i, $i = 0, 1, 2$ - each generates a copy of $Coh^T(\mathbb{A}^2)_0$.
  2. Sheaves supported along the triangle of $\mathbb{P}^1$'s
  3. Torsion-free sheaves of rank 1.
- The Lie algebra $\mathfrak{n}_{\mathbb{P}^2}$ is very large, and seems difficult to relate to anything explicit.

Let $\mathfrak{p}$ denote the Lie subalgebra of $\mathfrak{n}_{\mathbb{P}^2}$ generated by $\mathcal{O}_{ij}(n)$, where the latter is the degree n line bundle supported on l_{ij}. By looking at how $\mathfrak{p}$ acts on vector bundles on $\mathbb{P}^2$, can show there is a surjection

$$\mathfrak{p} \twoheadrightarrow \mathfrak{gl}^{-}_{\infty}$$

where $\mathfrak{gl}_{\infty}$ is the Lie algebra of infinite matrices $E_{i,j}, i,j \in \mathbb{Z}$, and $\mathfrak{gl}^{-}_{\infty}$ is the lower-triangular part, where $i > j$.