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Geometrical splitting and reduction of N-point Feynman diagrams

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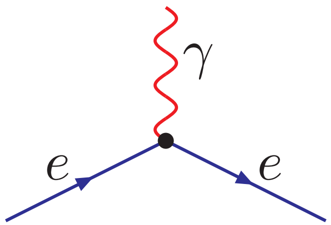
Algebraic Structures in Perturbative Quantum Field Theory

Conference in honor of Dirk Kreimer

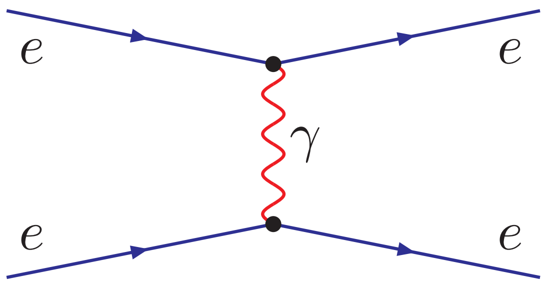
What is a Feynman diagram?

Convenient language to describe interaction of particles.

An example: Quantum ElectroDynamics (QED)



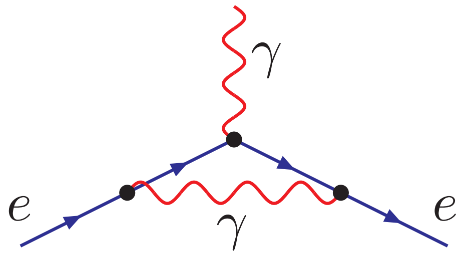
Elementary interaction vertex



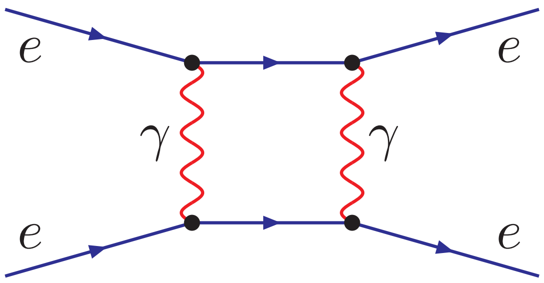
Interaction between two electrons

Higher-order contributions, loops

Next-order corrections: suppressed by $\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}$



1-loop 3-point function (triangle diagram)



1-loop 4-point function (box diagram)

Feynman rules

- Each vertex has a form $\mathcal{V}_{\rm indices}$, it is proportional to a coupling constant and may depend on the momenta of the particles.

- Each line connecting two interaction vertices (a propagator) has a form

$$\frac{\mathcal{P}_{\rm indices}}{p^2 - m^2 + {\rm i}0}$$

where $p = (E, \mathbf{p})$ is the 4d momentum
$\leftrightarrow$ Green function for D'Alembert-like equation.

- Each closed loop implies integration over the 4d momentum k flowing around this loop:

$$\int \frac{{\rm d}^4 k}{(2\pi)^4} \left\{ \cdots \right\}$$

But: Divergences $\Rightarrow$ Regularization needed

Dimensional regularization

One of the common tools used in loop calculations is *dimensional regularization*: the idea is to use the space-time dimension n as a regulator, $n = 4 - 2\varepsilon$ $(\varepsilon \to 0)$,

$$\int \mathrm{d}^4 k \left\{ \ldots \right\} \quad \Rightarrow \quad \int \mathrm{d}^n k \left\{ \ldots \right\}$$

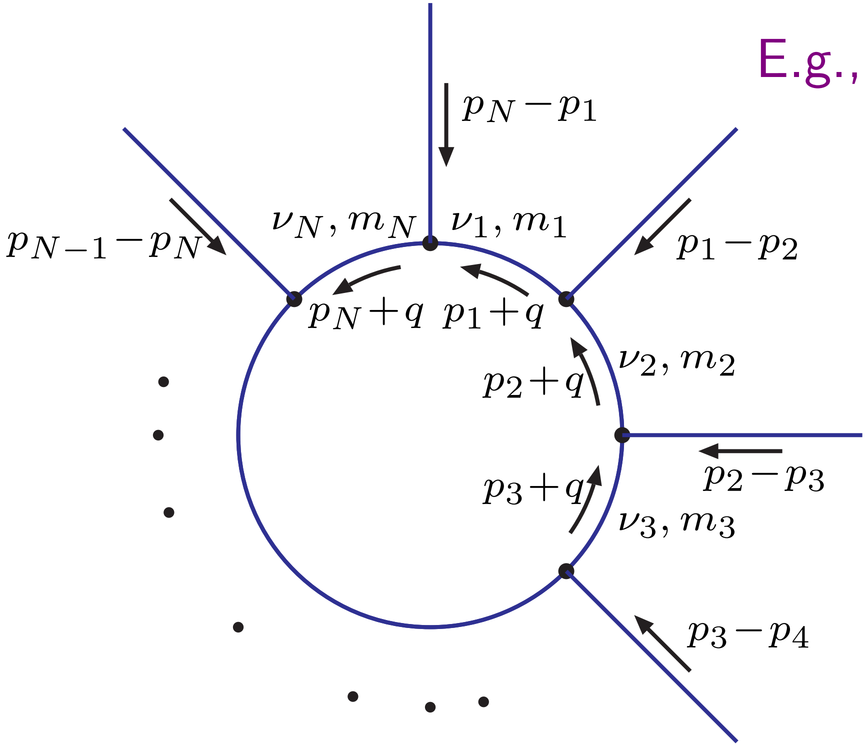
G. 'tHooft and M. Veltman, Nucl. Phys. **B44** (1972) 189;
C.G. Bollini and J.J. Giambiagi, Nuovo Cimento **12B** (1972) 20;
J.F. Ashmore, Lett. Nuovo Cim. **4** (1972) 289;
G.M. Cicuta and E. Montaldi, Lett. Nuovo Cim. **4** (1972) 329.

Then singularities appear as $1/\varepsilon$ poles. Simple example: a tadpole,

$$\int \frac{\mathrm{d}^n k}{[k^2 - m^2]^\nu} = \mathrm{i}^{1-2\nu} \pi^{n/2} \frac{\Gamma(\nu - n/2)}{\Gamma(\nu)} (m^2)^{n/2-\nu}$$

For $\nu = 2$: $\Gamma(2 - n/2) = \Gamma(\varepsilon) \sim 1/\varepsilon$

One-loop N-point function $J^{(N)}(n;\nu_1,\ldots,\nu_N)$



E.g., $M \to N - M$ processes
$(M < N)$

Depends on
$\frac{1}{2}N(N-1)$ invariants $k_{jl}^2 = (p_j - p_l)^2$
and N masses m_i

$$J^{(N)}\left(n;\nu_1,\ldots,\nu_N|\{k_{jl}^2\};\{m_i\}\right) \equiv \int \frac{\mathrm{d}^n q}{\left[(p_1+q)^2 - m_1^2\right]^{\nu_1} \cdots \left[(p_N+q)^2 - m_N^2\right]^{\nu_N}}$$

How Moscow, Hobart and Mainz got connected through Dirk

The story began in early 1990s. I lived in Moscow and worked on N-point Feynman diagrams, using Mellin–Barnes representations, hypergeometric functions, etc.

E.E. Boos and A.I.D., Theor. Math. Phys. **89** (1991) 1052.
A.I.D., J. Math. Phys. **32** (1991) 1052; J. Math. Phys. **33** (1992) 358.

Dirk worked in Mainz (a.k.a. Mayence) and studied similar Feynman diagrams, using another type of hypergeometric functions, the so-called Carlson R-functions.

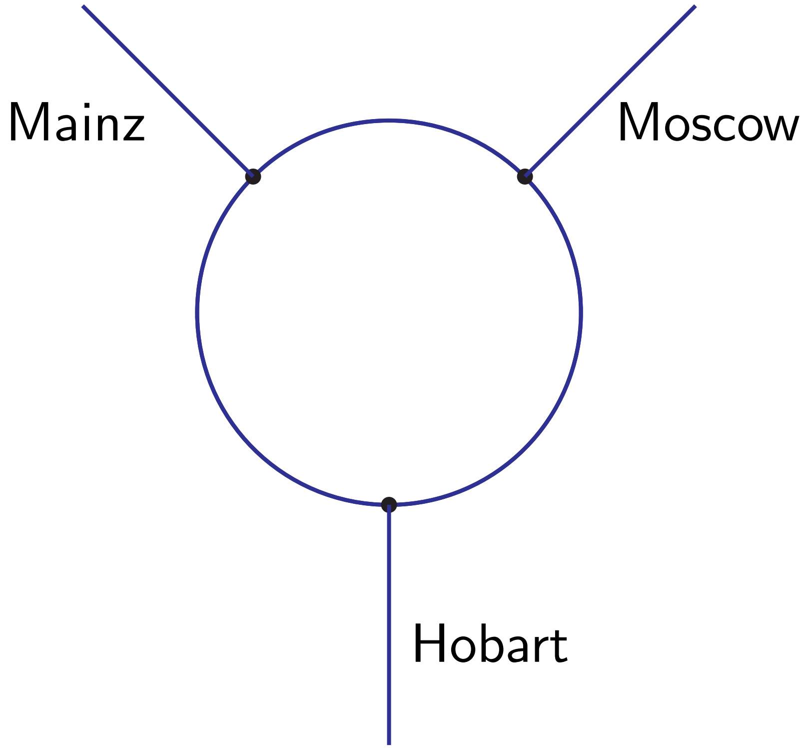
D.K., Z. Phys. **C54** (1992) 667; Int. J. Mod. Phys. **A8** (1993) 1797.
L. Brücher, J. Franzkowski and D.K., Mod. Phys. Lett. **A9** (1994) 2335.

In 1992 Dirk invited me to give a seminar at University of Mainz – this was our first meeting. Later on, in 1993–1995 I was a postdoc in Bergen (Norway), whereas Dirk went to Hobart (Tasmania, Australia), where he worked with Bob Delbourgo. Then Dirk returned to Europe, and in 1996 I was able to continue research work with Bob within the same project – this is how we started our work on geometrical approach to Feynman diagrams.

A.I.D. and R. Delbourgo, J. Math. Phys. **39** (1998) 4299; J. Phys. **A37** (2004) 4871

...and I also spent four years in Mainz in 1998–2001... Of course, we had a lot of communications with Dirk!

How Moscow, Hobart and Mainz got connected through Dirk (continued)



Feynman parameters

Parametric representation for the one-loop N-point function in n dimensions ($\nu_i = 1$):

$$J^{(N)}(n; 1, \ldots, 1) = \mathrm{i}^{1-n} \pi^{n/2}\, \Gamma\left(N - \frac{n}{2}\right) \underbrace{\int\limits_0^1 \ldots \int\limits_0^1}_{N} \frac{\left(\prod \mathrm{d}\alpha_i\right) \cdot \delta\left(\sum \alpha_i - 1\right)}{\left[\sum\sum\limits_{j<l} \alpha_j \alpha_l k_{jl}^2 - \sum \alpha_i m_i^2\right]^{N-n/2}}$$

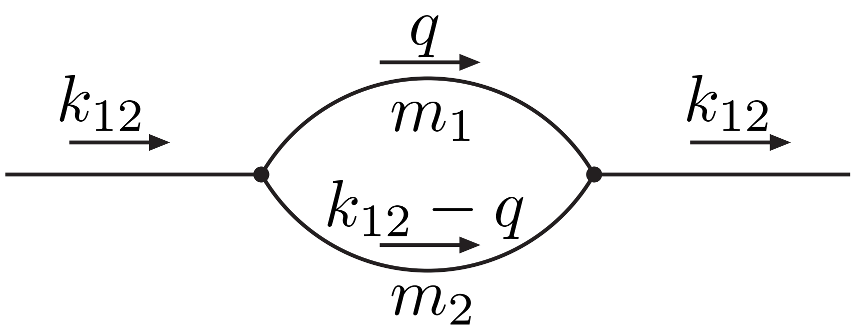
By using $\sum \alpha_i = 1$ we can make the quadratic form homogeneous in α_i:

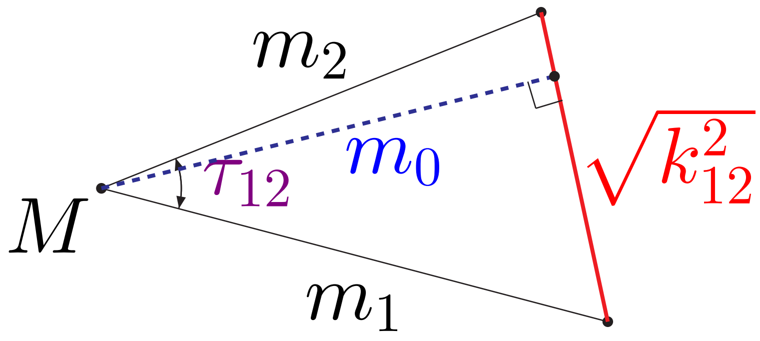
$$\left[\sum\sum\limits_{j<l} \alpha_j \alpha_l k_{jl}^2 - \left(\sum \alpha_i\right)\left(\sum \alpha_i m_i^2\right)\right] \Rightarrow -\left[\sum \alpha_i^2 m_i^2 + 2 \sum\sum\limits_{j<l} \alpha_j \alpha_l m_j m_l c_{jl}\right],$$

$$c_{jl} \equiv \frac{m_j^2 + m_l^2 - k_{jl}^2}{2 m_j m_l}, \qquad c_{jl} = \cos\tau_{jl} = \begin{cases} 1, & k_{jl}^2 = (m_j - m_l)^2 \quad \text{pseudothreshold} \\ -1, & k_{jl}^2 = (m_j + m_l)^2 \quad \text{threshold} \end{cases}$$

Direct geometrical interpretation: when $-1 \le c_{jl} \le 1$ (i.e., angles τ_{jl} are real)

Two-point function: the basic triangle

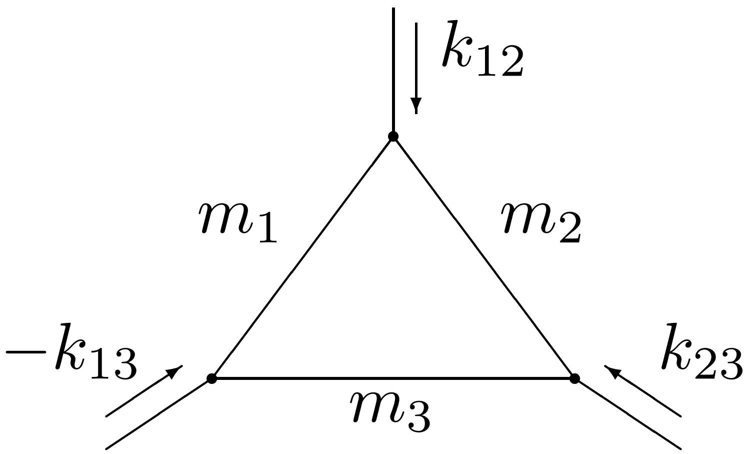


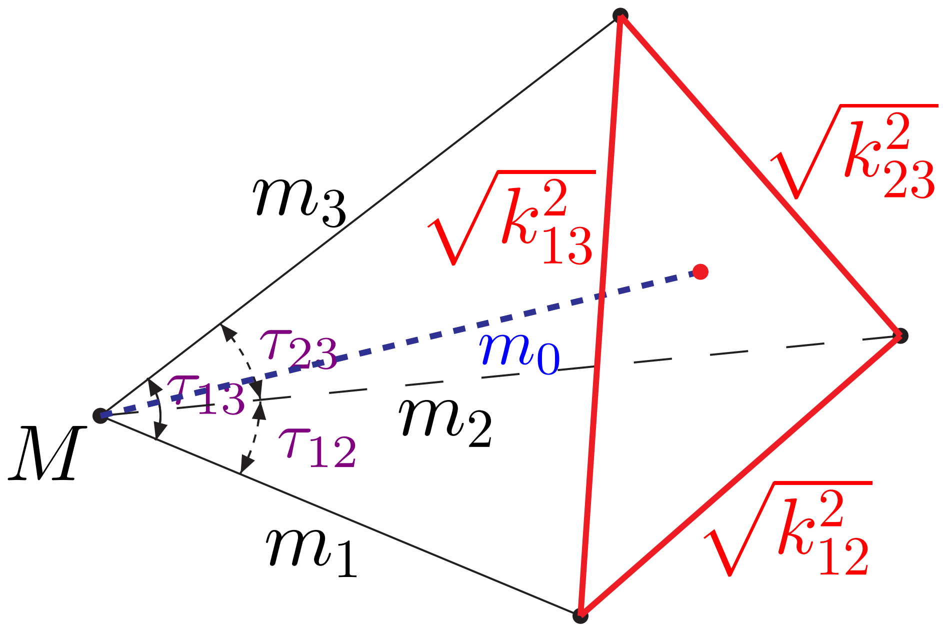


$$\cos\tau_{12} = c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1 m_2}$$

$$c_{12} = \cos\tau_{12} = \begin{cases} 1, & k_{12}^2 = (m_1 - m_2)^2 \quad \text{pseudothreshold} \quad (\tau_{12} = 0) \\ -1, & k_{12}^2 = (m_1 + m_2)^2 \quad \text{threshold} \quad (\tau_{12} = \pi) \end{cases}$$

Three-point function: the basic tetrahedron

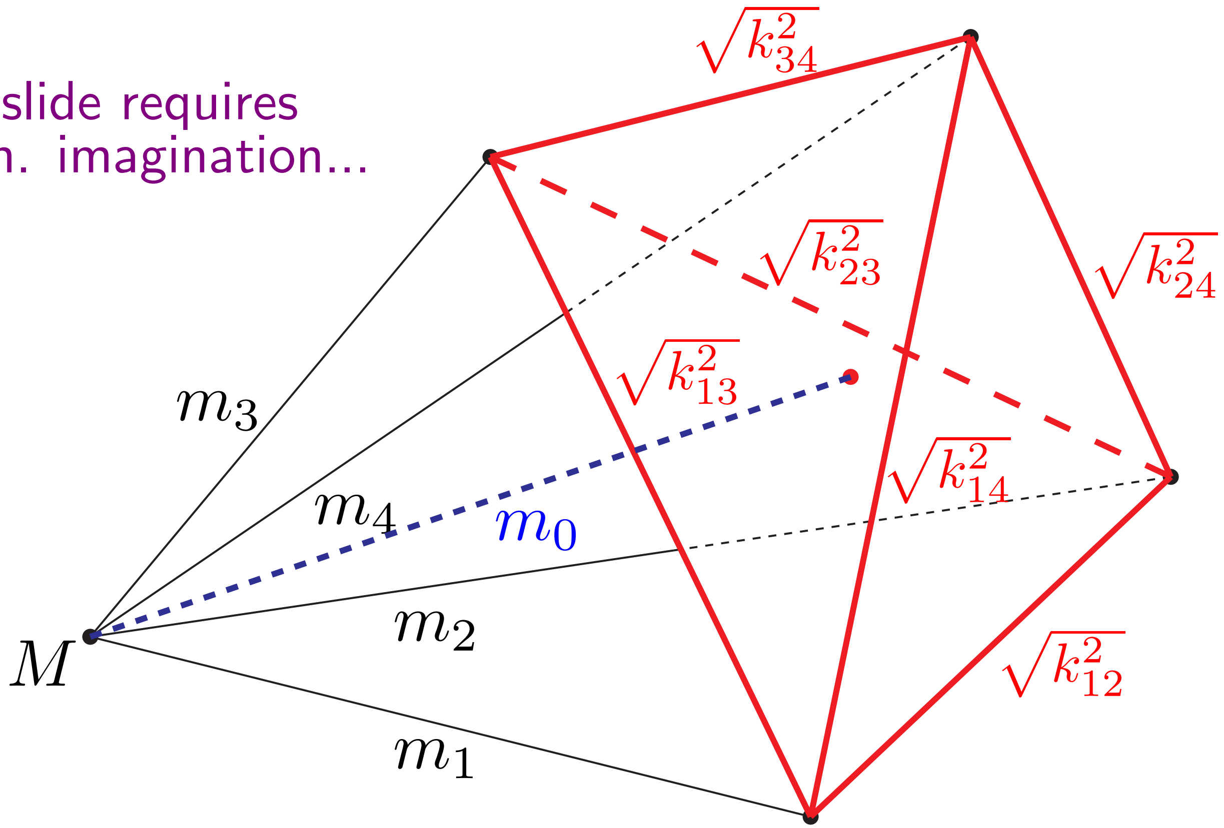


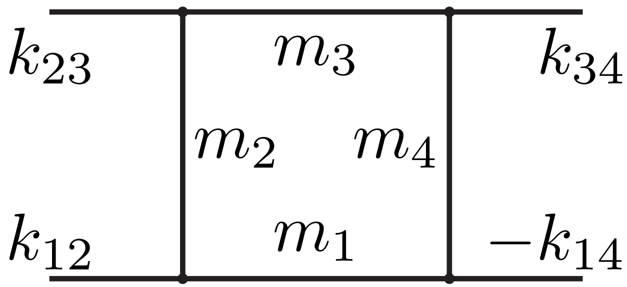


$$\cos\tau_{jl} = c_{jl} = \frac{m_j^2 + m_l^2 - k_{jl}^2}{2m_j m_l}$$

Four-point function: the basic simplex for $N = 4$

This slide requires 4-dim. imagination...





$k_{12}^2, k_{23}^2, k_{34}^2, k_{14}^2$ – external momenta squared

k_{13}^2, k_{24}^2 – Mandelstam variables s and t

$$D^{(4)} = \det \|c_{jl}\| \ , \qquad \Lambda^{(4)} = \det \|(k_{j4} \cdot k_{l4})\| \ ,$$

$$V^{(4)} = \frac{m_1 m_2 m_3 m_4}{4!} \sqrt{D^{(4)}}, \qquad \overline{V}_0^{(3)} = \frac{1}{3!} \sqrt{\Lambda^{(4)}}, \qquad m_0 = m_1 m_2 m_3 m_4 \sqrt{\frac{D^{(4)}}{\Lambda^{(4)}}}$$

From Feynman parameters to the geometrical approach

Using linear and *quadratic* substitutions of α variables, we arrive at

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\Gamma\left(N-\frac{n}{2}\right)\left(\Pi f_i\right)\int\limits_0^\infty\ldots\int\limits_0^\infty\frac{\left(\prod \mathsf{d}\alpha_i\right)\cdot\delta\left(\alpha^T\|C\|\alpha-1\right)}{\left(\sum\alpha_i f_i\right)^{n-N}}$$

Modified matrix: $C_{jl} = \left(\sqrt{F_j^{(N)}}c_{jl}\sqrt{F_l^{(N)}}\right)$, with $F_i^{(N)} = \frac{\partial}{\partial m_i^2}\left(m_i^2 D^{(N)}\right)$

obeying $\sum\limits_{l=1}^N c_{jl}F_l^{(N)}\frac{1}{m_l} = D^{(N)}\frac{1}{m_j} \Rightarrow \sum\limits_{l=1}^N C_{jl}\frac{\sqrt{F_l^{(N)}}}{m_l} = D^{(N)}\frac{\sqrt{F_j^{(N)}}}{m_j} \Rightarrow$

Eigenvector: $f_i = \frac{\sqrt{F_i^{(N)}}}{m_i}$, Eigenvalue: $D^{(N)}=\det\|c_{jl}\|$ (Gram determinant)

In what follows we will also need $\Lambda^{(N)}=\det\|(k_{jN}\cdot k_{lN})\|$ and $m_0 = \left(\prod\limits_{i=1}^N m_i\right)\sqrt{\frac{D^{(N)}}{\Lambda^{(N)}}}$.

Feynman parameters: diagonalization of the quadratic form

Let us “rotate” the variables $\alpha_i \to \beta_i$ so that $\alpha^T \|C\| \alpha = \sum_{i=1}^N \lambda_i \beta_i^2$.
One of the β’s (say β_N) is directed along f_i, so that $\lambda_N = D^{(N)}$ and $\sum \alpha_i f_i \propto \beta_N$.

Assume (for a moment) that all $\lambda_i > 0$ and rescale $\beta_i = \frac{\gamma_i}{\sqrt{\lambda_i}} \Rightarrow$

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N} \pi^{n/2}\, \Gamma\left(N - \tfrac{n}{2}\right) \frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}} \int\limits_{\Omega^{(N)}} \!\!\ldots\! \int \frac{\prod \mathrm{d}\gamma_i}{{\gamma_N}^{n-N}}\, \delta\left(\sum \gamma_i^2 - 1\right)$$

- Remarkably: the same N-dimensional solid angle $\Omega^{(N)}$ as in the *basic simplex*!
- Special case: $N = n$ ($N = 2$ in 2d, $N = 3$ in 3d, $N = 4$ in 4d, etc.)
  $\to$ no denominator, just the non-Euclidean (hyper)volume (content)!
- If some of λ_i are negative – *hyperbolic* surface (instead of *spherical*)
  $\leftrightarrow$ analytical continuation!

A.I.D. and R. Delbourgo, J. Math. Phys. **39** (1998) 4299;
A.I.D., Phys. Rev. **D61** (2000) 087701;
A.I.D. and M.Yu. Kalmykov, Nucl. Phys. B (PS) **89** (2000) 283; Nucl. Phys. **B605** (2001) 266;
A.I.D., AIHENP-99 Proceedings (hep-th/9908032); Nucl.Instr.Meth. **A559** (2006) 293;
A.I.D., J. Phys. (Conf. Ser.) 762 (2016) 012068; J. Phys. (Conf. Ser.) 1085 (2018) 052016.

Feynman parameters versus geometrical approach

Feynman parametric representation (with $c_{jl} \equiv \frac{m_j^2+m_l^2-k_{jl}^2}{2m_j m_l}$):

$$J^{(N)}(n;1,\ldots,1) = \mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\int_0^1\!\ldots\!\int_0^1 \frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\sum\alpha_i-1\right)}{\left[\sum\alpha_i^2 m_i^2+2\sum\limits_{j<l}\sum\alpha_j\alpha_l m_j m_l c_{jl}\right]^{N-n/2}}$$

• depends on the masses and k_{jl}^2 through m_i and c_{jl} in the quadratic form

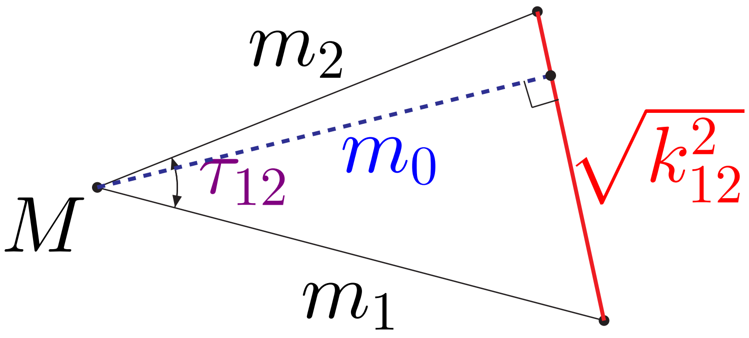
Geometric representation:

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}}\int_{\Omega^{(N)}}\!\!\ldots\int\frac{\prod\mathrm{d}\gamma_i}{\gamma_N^{n-N}}\,\delta\left(\sum\gamma_i^2-1\right)$$

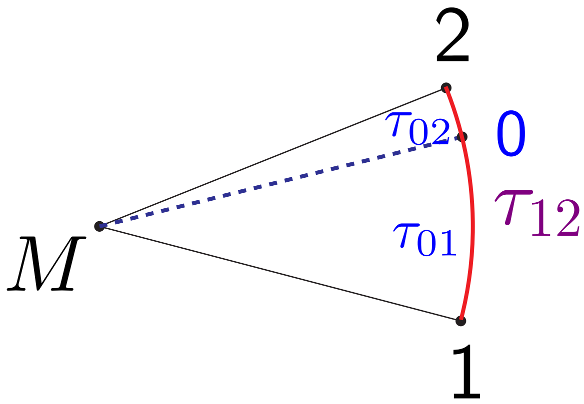
• except for the pre-factor, depends on the masses and k_{jl}^2 only through the integration limits defined by the N-dimensional solid angle $\Omega^{(N)}$

• *ready for splitting!*

Two-point function, geometrical approach



the basic triangle



the arc τ_{12}

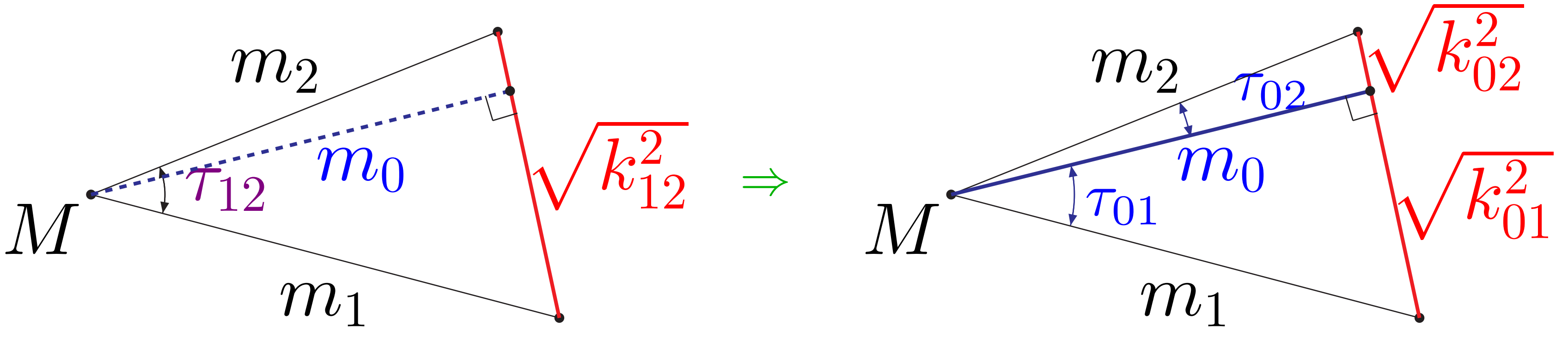
$$\cos\tau_{12} \equiv c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1 m_2}, \quad D^{(2)} = 1 - c_{12}^2 = \sin^2\tau_{12}, \quad \Lambda^{(2)} = k_{12}^2,$$

$$\cos\tau_{0i} = \frac{m_0}{m_i}, \quad \tau_{01} + \tau_{02} = \tau_{12}$$

Area of the triangle : $\frac{1}{2} m_1 m_2 \sqrt{D^{(2)}} = \frac{1}{2} m_1 m_2 \sin\tau_{12} = \frac{1}{2} m_0 \sqrt{\Lambda^{(2)}} = \frac{1}{2} m_0 \sqrt{k_{12}^2}$

$$\Rightarrow m_0 = m_1 m_2 \sqrt{\frac{D^{(2)}}{\Lambda^{(2)}}} = \frac{m_1 m_2 \sin\tau_{12}}{\sqrt{k_{12}^2}}$$

Two-point function, splitting the basic triangle



$$\underbrace{k_{01}^2 = m_1^2 - m_0^2} = \frac{(k_{12}^2 + m_1^2 - m_2^2)^2}{4k_{12}^2}, \qquad \underbrace{k_{02}^2 = m_2^2 - m_0^2} = \frac{(k_{12}^2 - m_1^2 + m_2^2)^2}{4k_{12}^2}$$

Basically, we are applying the Pythagorean theorem to Feynman diagrams:

$$J^{(2)}(n;1,1|k_{12}^2;m_1,m_2) = \frac{1}{2k_{12}^2}\Big\{(k_{12}^2 + m_1^2 - m_2^2)\, J^{(2)}(n;1,1|\underbrace{k_{01}^2;m_1,m_0})$$
$$+(k_{12}^2 - m_1^2 + m_2^2)\, J^{(2)}(n;1,1|\underbrace{k_{02}^2;m_2,m_0})\Big\}$$

For other examples of functional relations between integrals with different momenta and masses see, e.g., in O.V. Tarasov, Phys.Lett. **B670** (2008) 67

Two-point function: number of variables and the quadratic form

Number of dimensionless variables, before and after splitting:

in $J^{(2)}\left(n;1,1\middle|k_{12}^2;m_1,m_2\right)$: $3-1(\text{dimension})=2$

in $J^{(2)}\left(n;1,1\middle|k_{01}^2;m_1,m_0\right)$: $3-1(k_{01}^2=m_1^2-m_0^2)-1(\text{dimension})=1$

Quadratic form in Feynman parametric integral:

in $J^{(2)}\left(n;1,1\middle|k_{12}^2;m_1,m_2\right)$: $[\alpha_1\alpha_2 k_{12}^2-\alpha_1 m_1^2-\alpha_2 m_2^2]$

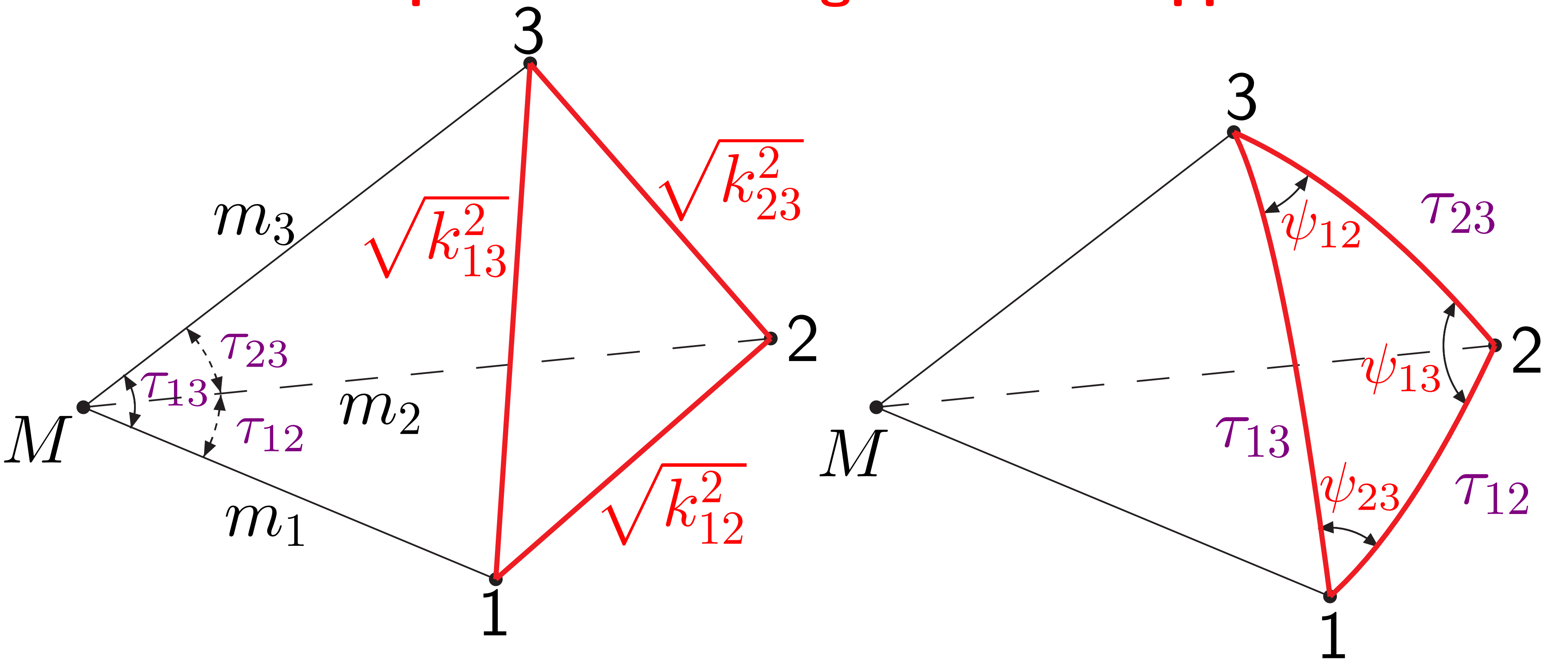
in $J^{(2)}\left(n;1,1\middle|k_{01}^2;m_1,m_0\right)$: $[\alpha_1\alpha_2 k_{01}^2-\alpha_1 m_1^2-\alpha_2 m_0^2]=-[\alpha_1^2 k_{01}^2+m_0^2]$

Result in arbitrary dimension:

$$J^{(2)}\left(n;1,1\middle|k_{01}^2;m_1,m_0\right) = \mathrm{i}\pi^{n/2}\Gamma(2-n/2)\int_0^1\int_0^1 \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\delta(\alpha_1+\alpha_2-1)}{[\alpha_1^2k_{01}^2+m_0^2]^{2-n/2}}$$

$$= \mathrm{i}\pi^{n/2}\frac{\Gamma(2-n/2)}{(m_0^2)^{2-n/2}}\;{}_2F_1\left(\left.\begin{matrix}\frac{1}{2},\ 2-\frac{n}{2}\\ \frac{3}{2}\end{matrix}\right|-\frac{k_{01}^2}{m_0^2}\right)$$

Three-point function: geometrical approach

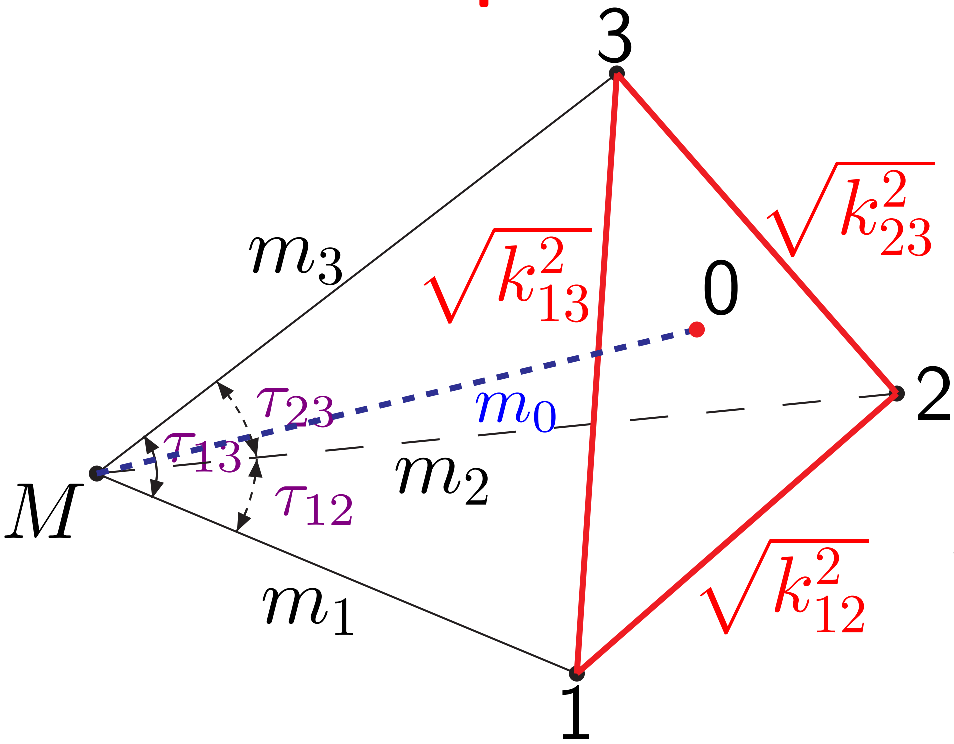


Special case $n = 3 \Rightarrow$ the area of spherical triangle ("spherical excess"):

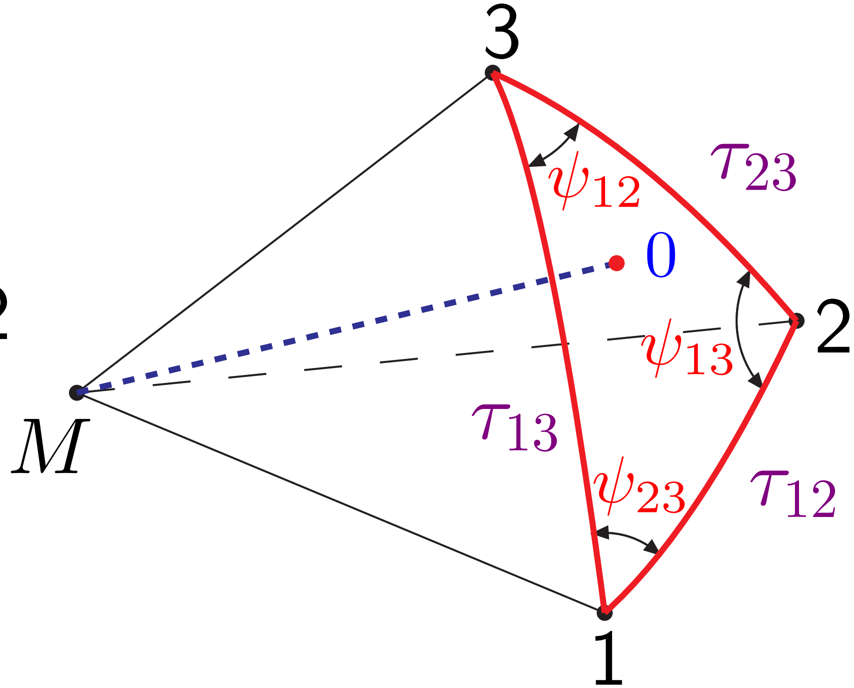
$$\Omega^{(3;3)} = \psi_{12} + \psi_{23} + \psi_{31} - \pi \; .$$

Compare with: B. G. Nickel, J. Math. Phys. **19** (1978) 542

Three-point function: geometrical approach

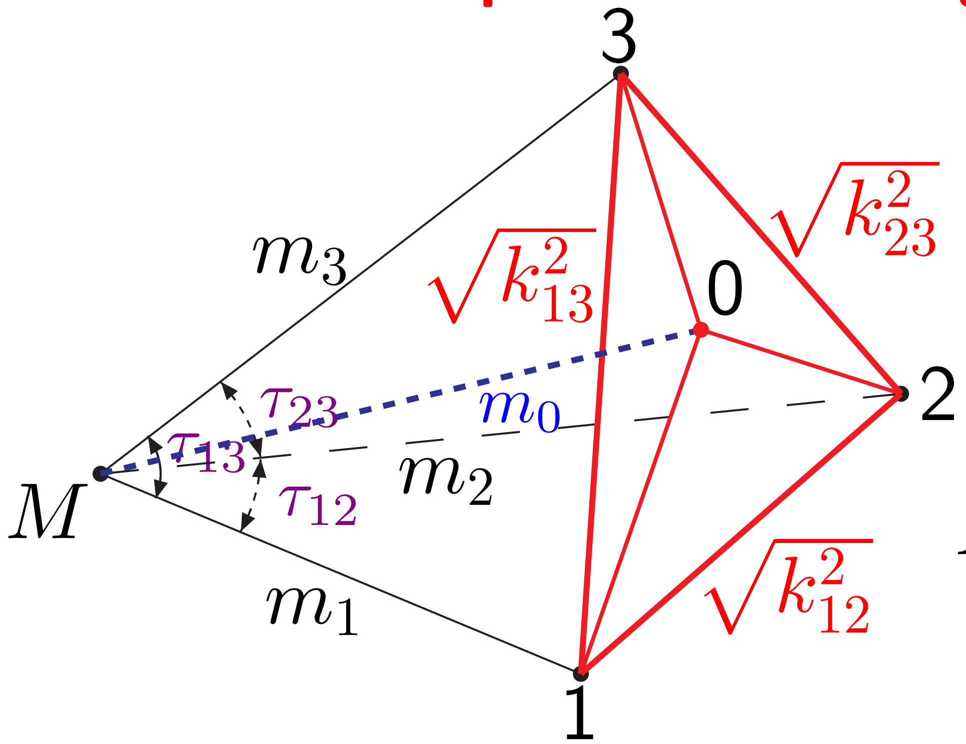


the basic tetrahedron

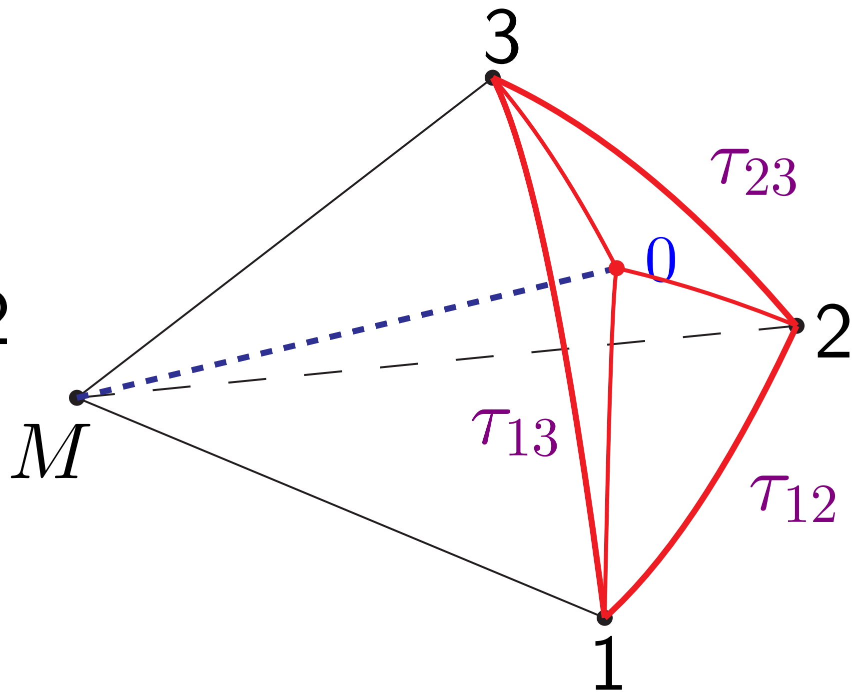


the solid angle

Three-point function: geometrical approach

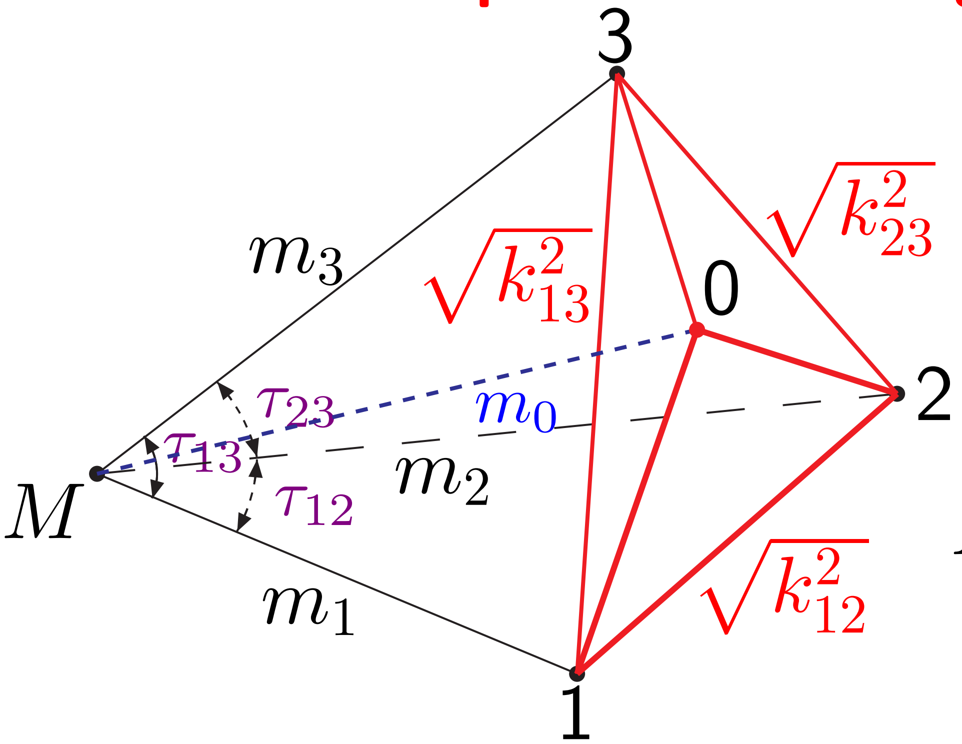


the basic tetrahedron

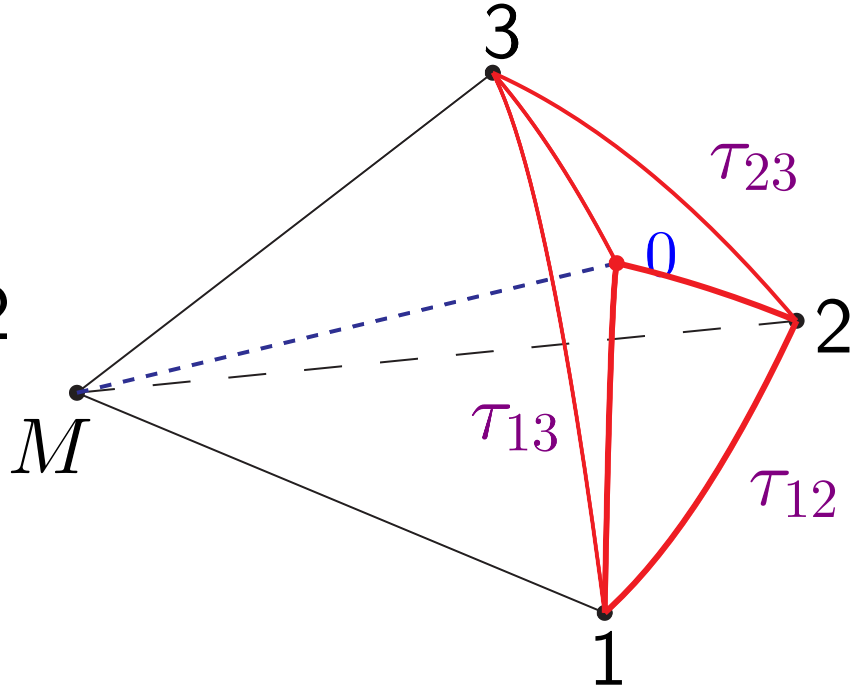


the solid angle

Three-point function: geometrical approach

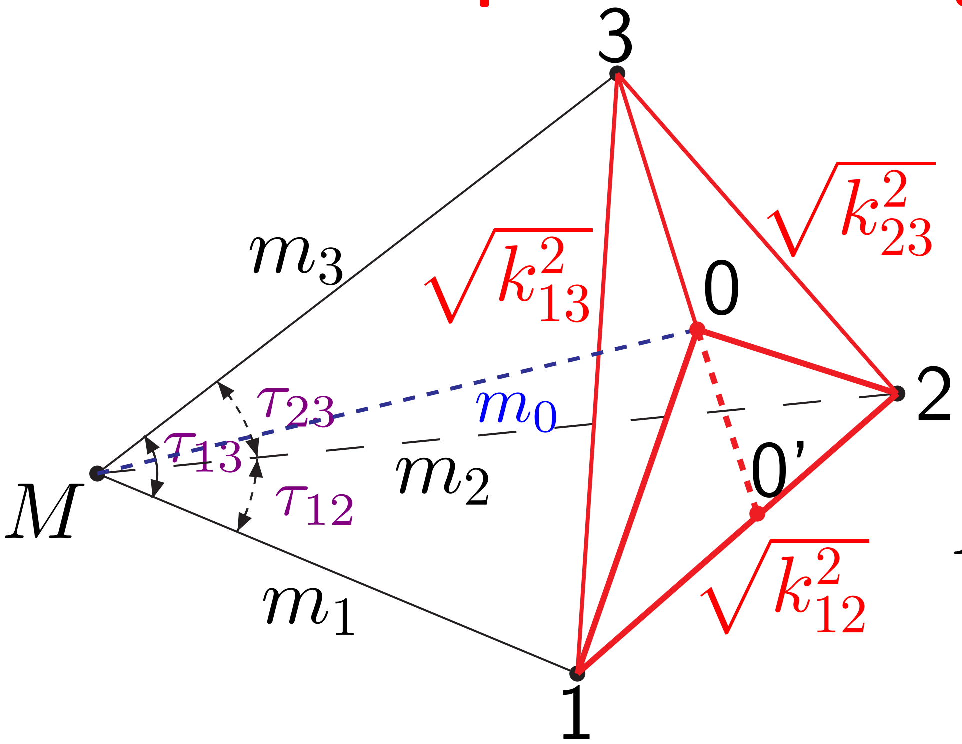


the basic tetrahedron

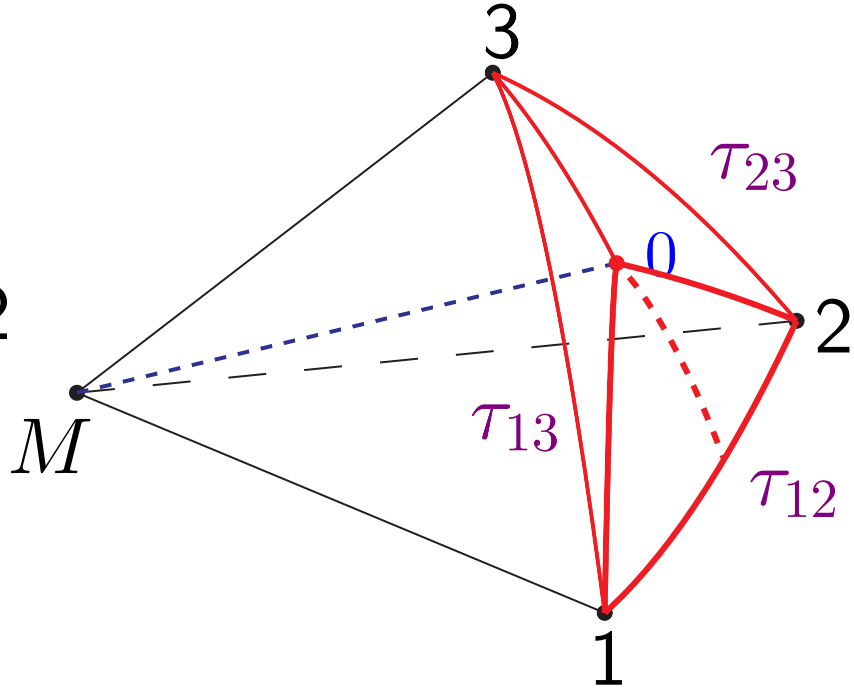


the solid angle

Three-point function: geometrical approach



the basic tetrahedron



the solid angle

Three-point function: number of variables and the quadratic form

Number of dimensionless variables, before and after splitting:

in $J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$: $6-1(\text{dimension})=5$

in $J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)$: $6-2(\text{relations})-1(\text{dimension})=3$

in $J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$: $6-3(\text{relations})-1(\text{dimension})=2$

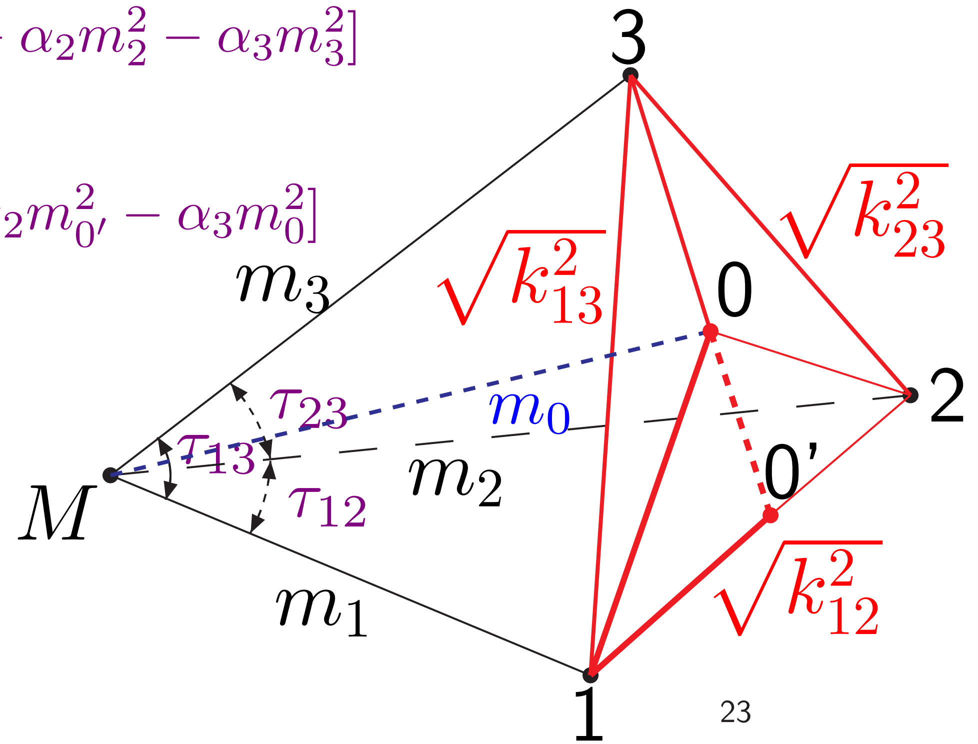
Quadratic form in Feynman parametric integral:

in $J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$:

$$[\alpha_1\alpha_2k_{12}^2+\alpha_1\alpha_3k_{13}^2+\alpha_2\alpha_3k_{23}^2-\alpha_1m_1^2-\alpha_2m_2^2-\alpha_3m_3^2]$$

in $J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$:

$$[\alpha_1\alpha_2k_{10'}^2+\alpha_1\alpha_3k_{01}^2+\alpha_2\alpha_3k_{00'}^2-\alpha_1m_1^2-\alpha_2m_{0'}^2-\alpha_3m_0^2]$$
$$=-[\alpha_1^2k_{10'}^2+(\alpha_1+\alpha_2)^2k_{00'}^2+m_0^2]$$



Three-point function: result in arbitrary dimension

$$J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$$

$$= -\mathrm{i}\pi^{n/2}\Gamma(3-n/2)\int_0^1\int_0^1\int_0^1 \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\delta(\alpha_1+\alpha_2+\alpha_3-1)}{[\alpha_1^2k_{10'}^2+(\alpha_1+\alpha_2)^2k_{00'}^2+m_0^2]^{3-n/2}}$$

$$= -\frac{\mathrm{i}\pi^{n/2}\Gamma(2-n/2)}{2(m_0^2)^{2-n/2}k_{00'}^2}\left\{\sqrt{\frac{k_{00'}^2}{k_{10'}^2}}\arctan\sqrt{\frac{k_{10'}^2}{k_{00'}^2}} - \left(\frac{m_0^2}{m_{0'}^2}\right)^{2-n/2} F_1\left(\tfrac{1}{2},1,2-\tfrac{n}{2};\tfrac{3}{2}\middle|-\frac{k_{10'}^2}{k_{00'}^2},-\frac{k_{10'}^2}{m_{0'}^2}\right)\right\}$$

where F_1 is Appell hypergeometric function of two variables,

$$F_1\left(a,b_1,b_2;c\middle|x,y\right) = \sum_{j_1,j_2}\frac{(a)_{j_1+j_2}\,(b_1)_{j_1}\,(b_2)_{j_2}}{(c)_{j_1+j_2}}\,\frac{x^{j_1}\,y^{j_2}}{j_1!\,j_2!}$$

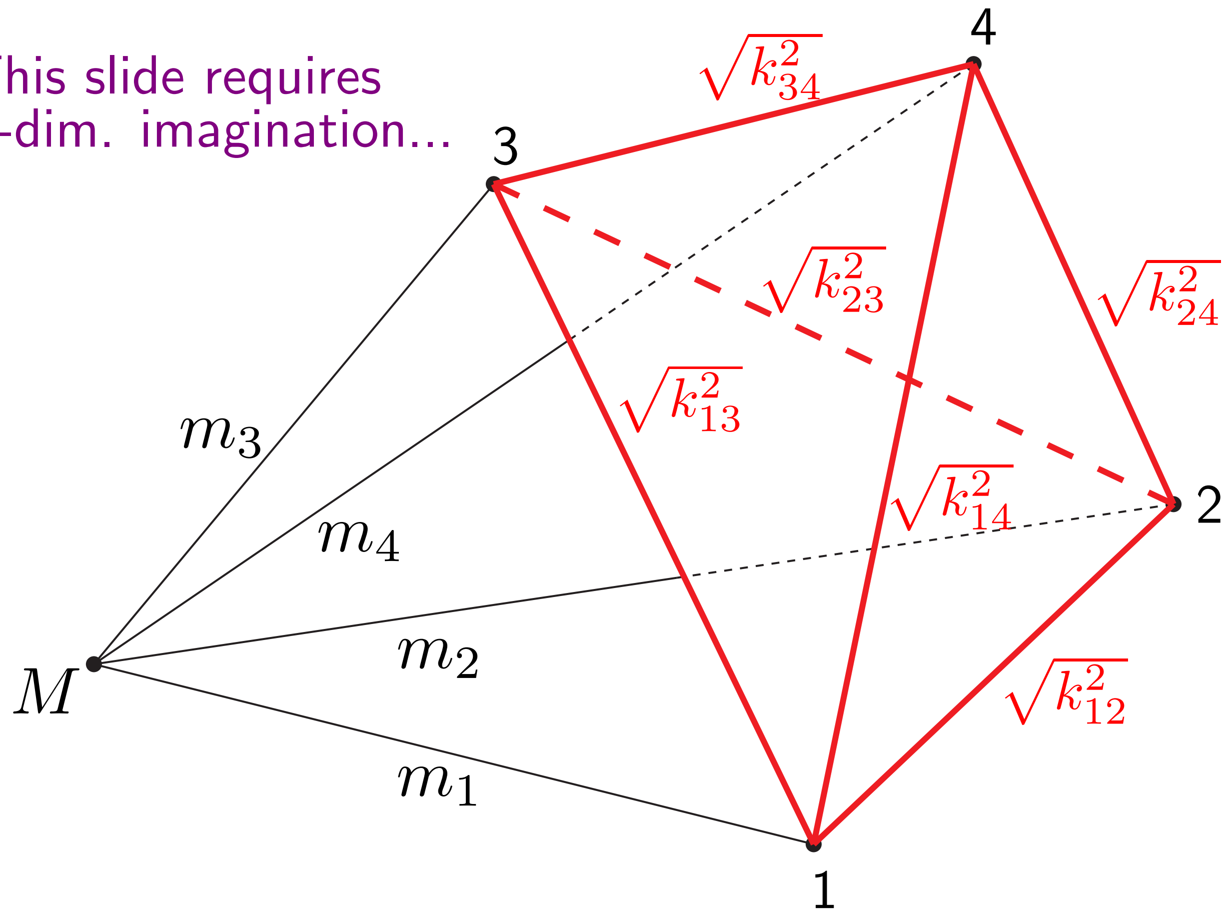
A.I.D., hep-th/9908032; Nucl.Instr.Meth. A559 (2006) 293

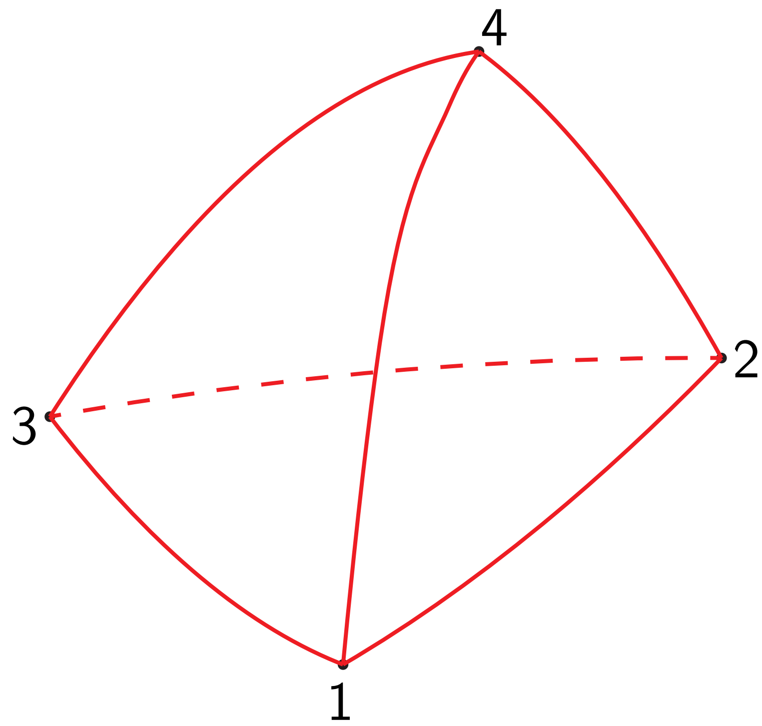
See also: O.V. Tarasov, Nucl. Phys. B (PS) **89** (2000) 237

J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

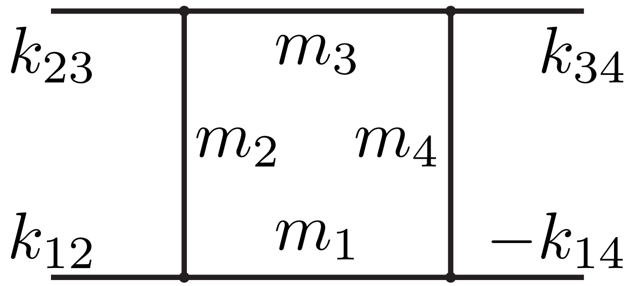
Four-point function: basic simplex and non-Euclidean tetrahedron

This slide requires
4-dim. imagination...





$k_{12}^2, k_{23}^2, k_{34}^2, k_{14}^2$ – external momenta squared
k_{13}^2, k_{24}^2 – Mandelstam variables s and t



Four-point function: basic simplex and non-Euclidean tetrahedron

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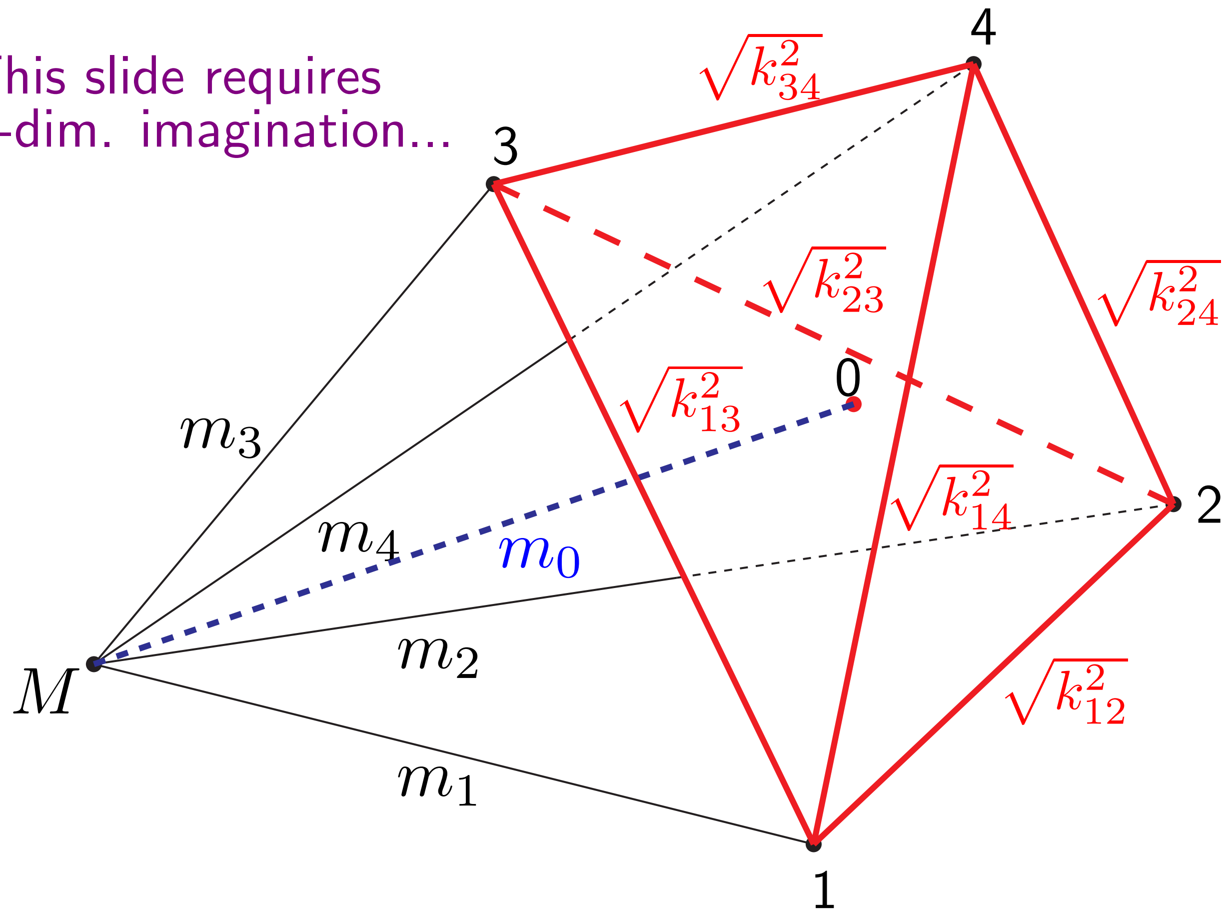
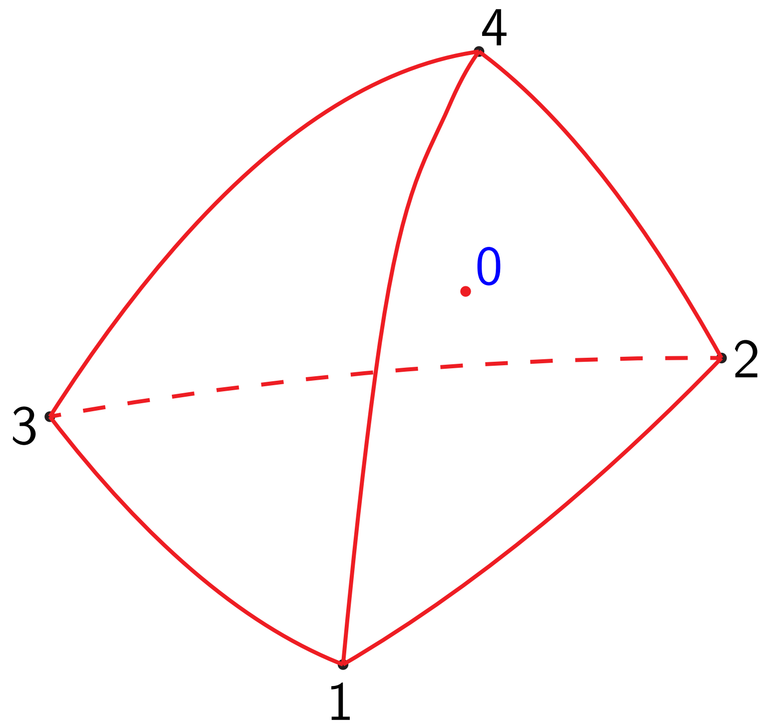
4-dim. imagination...





Four-point function: basic simplex and non-Euclidean tetrahedron

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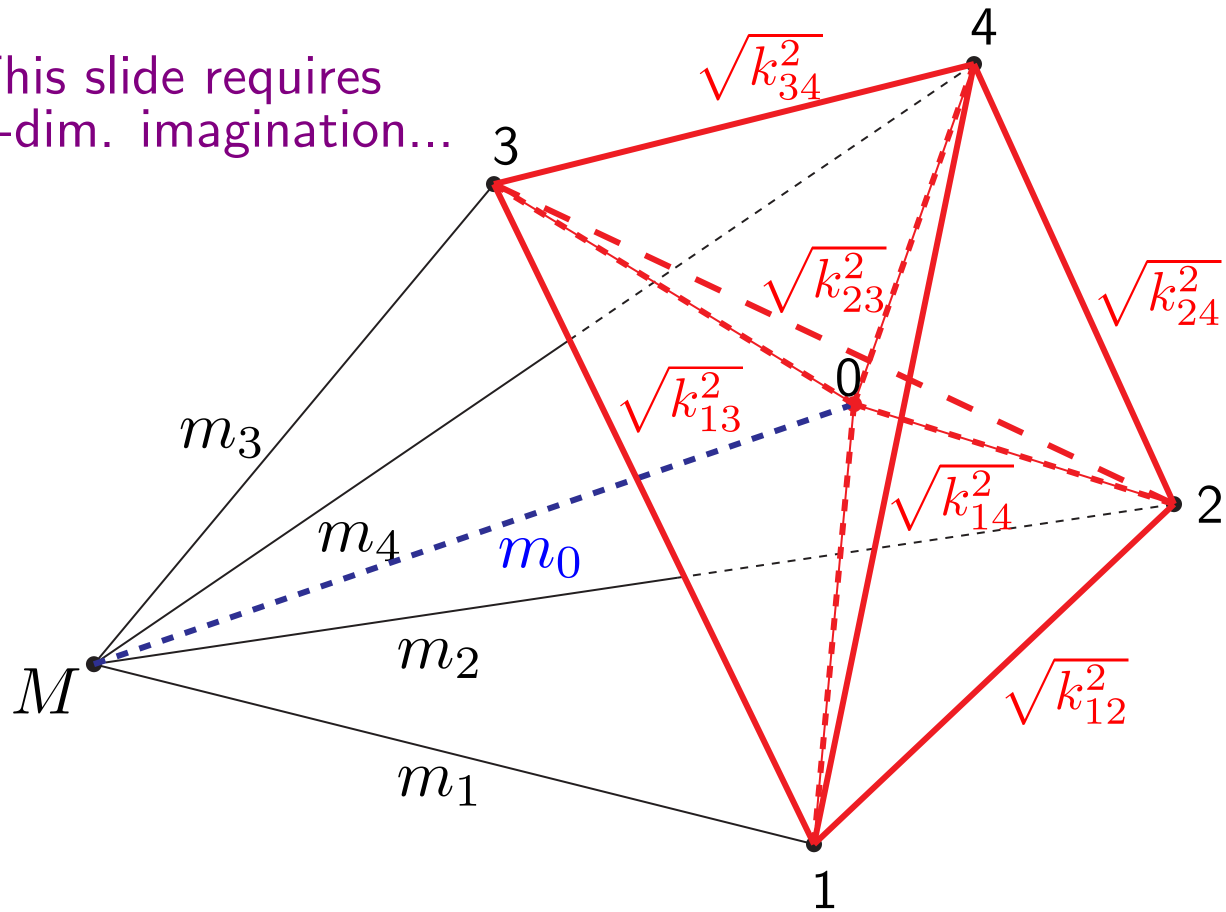
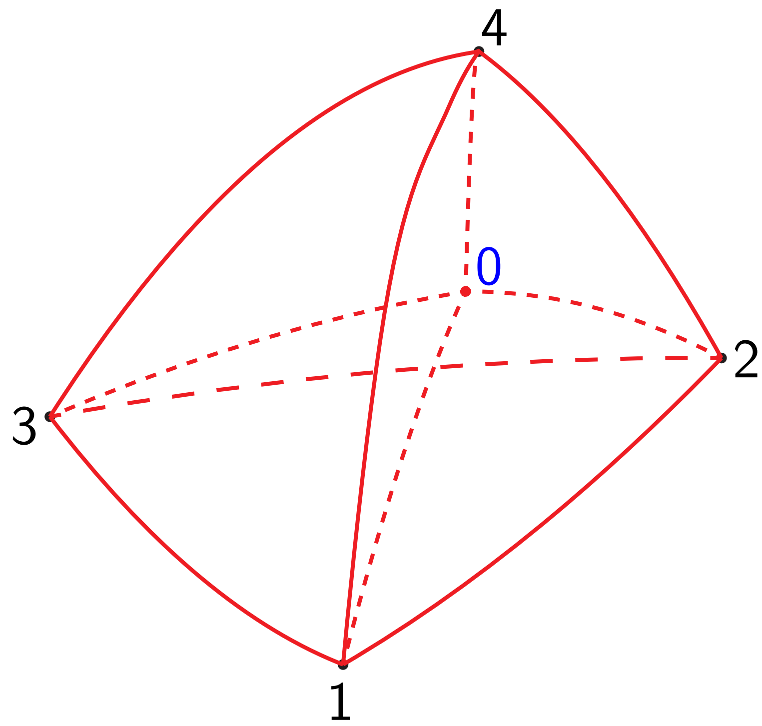
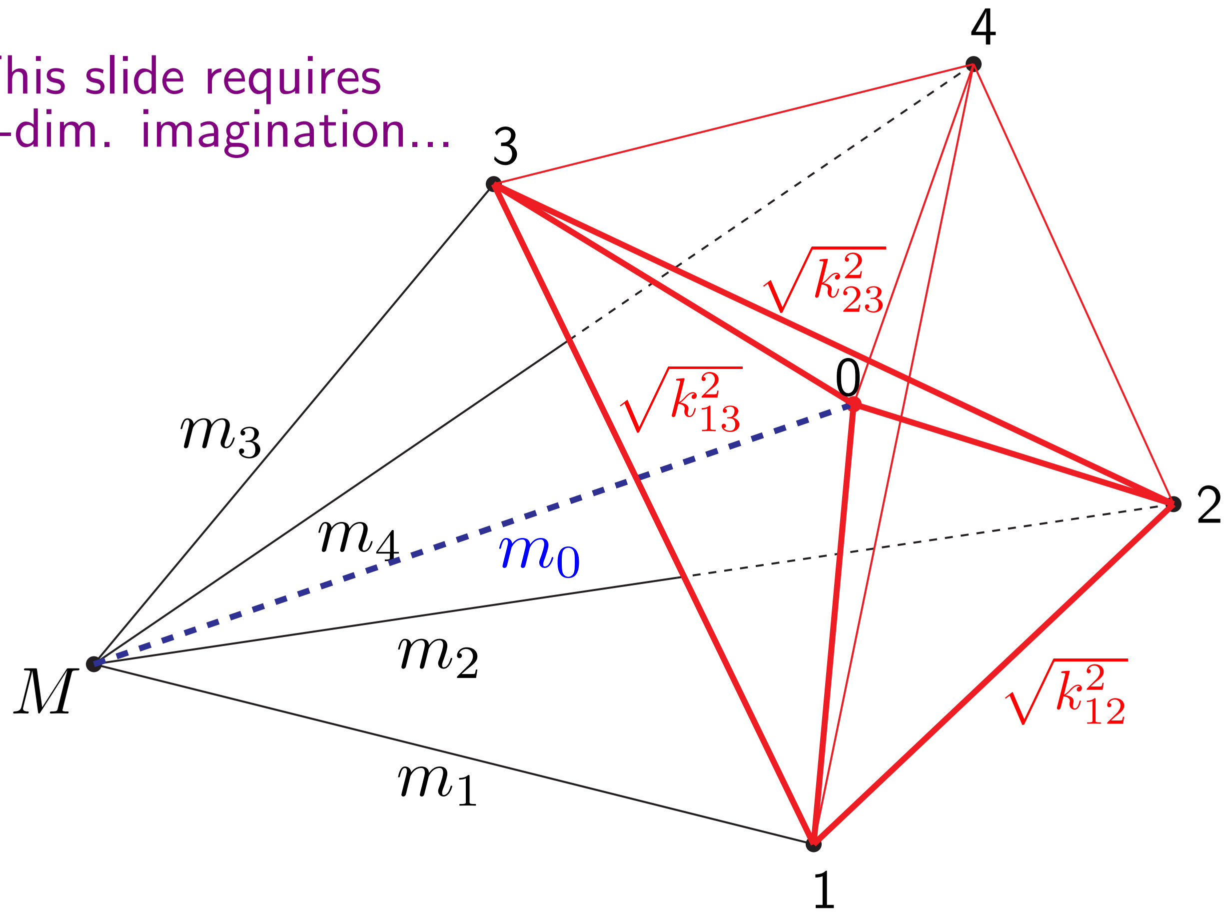
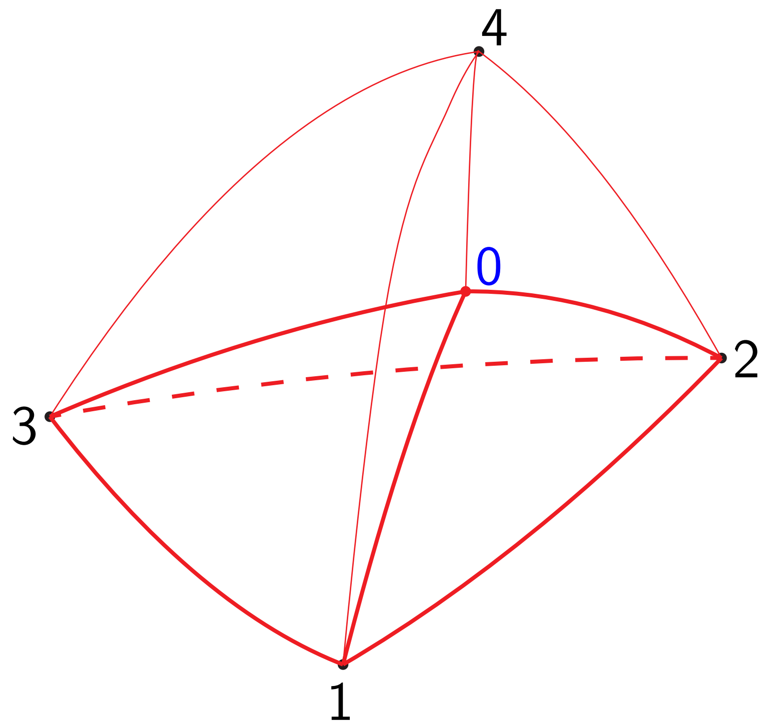




Four-point function: basic simplex and non-Euclidean tetrahedron

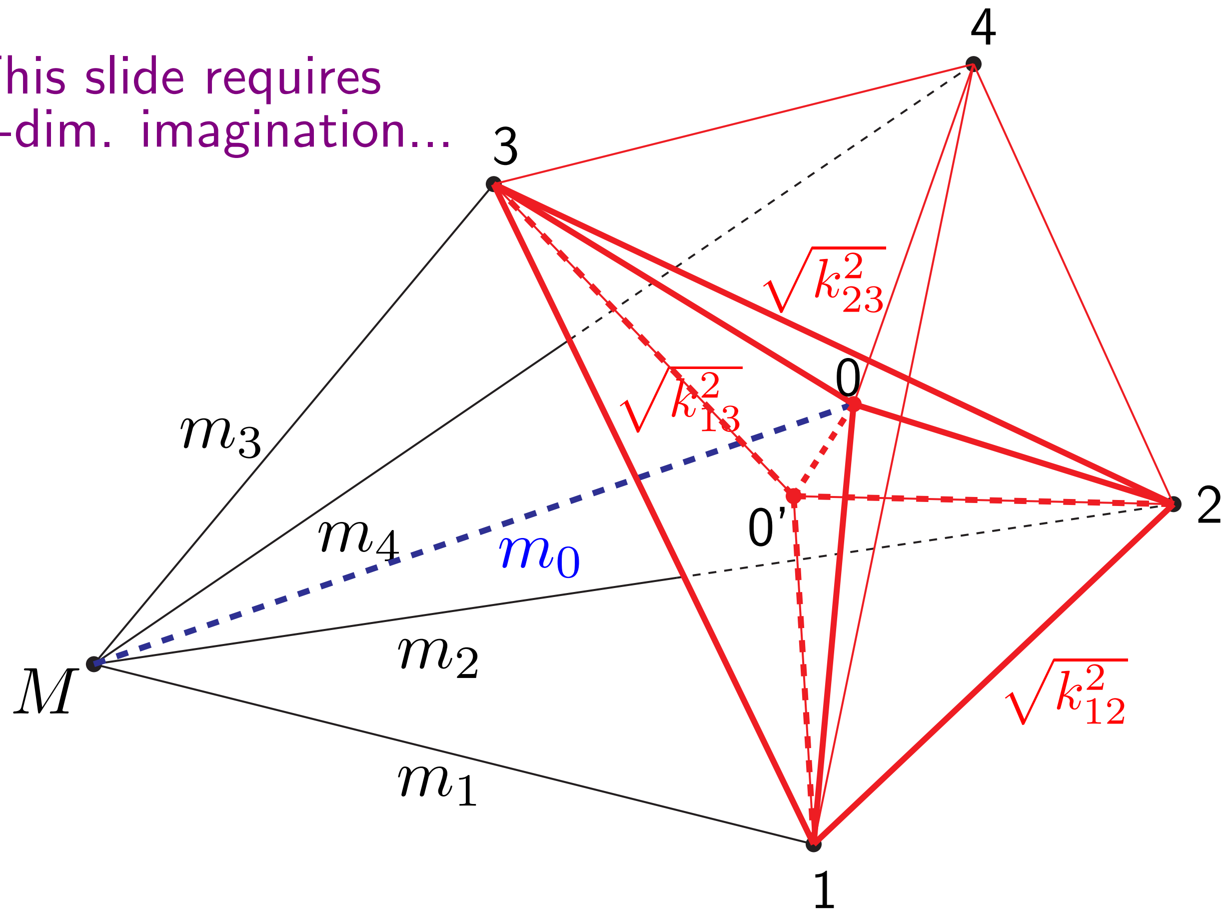
This slide requires
4-dim. imagination...

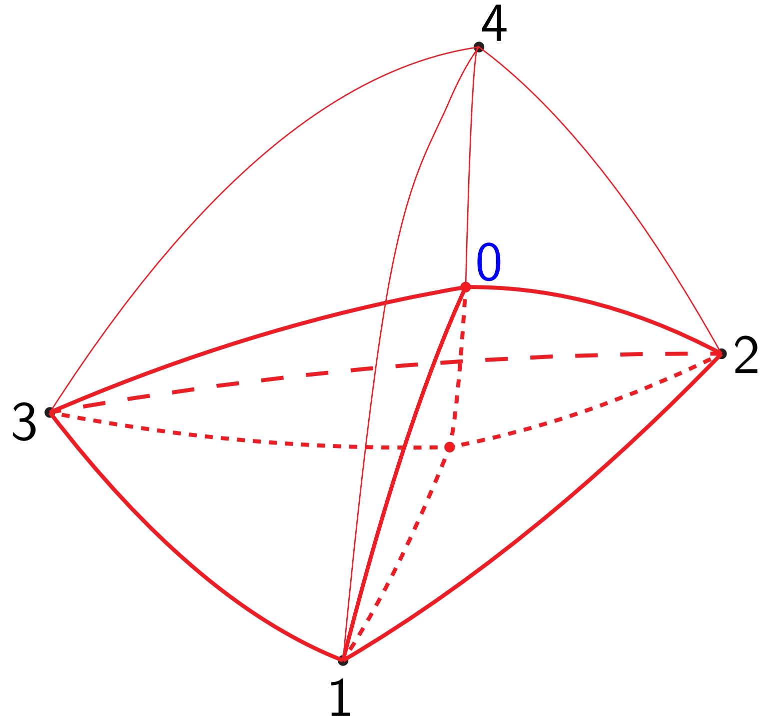




Four-point function: basic simplex and non-Euclidean tetrahedron

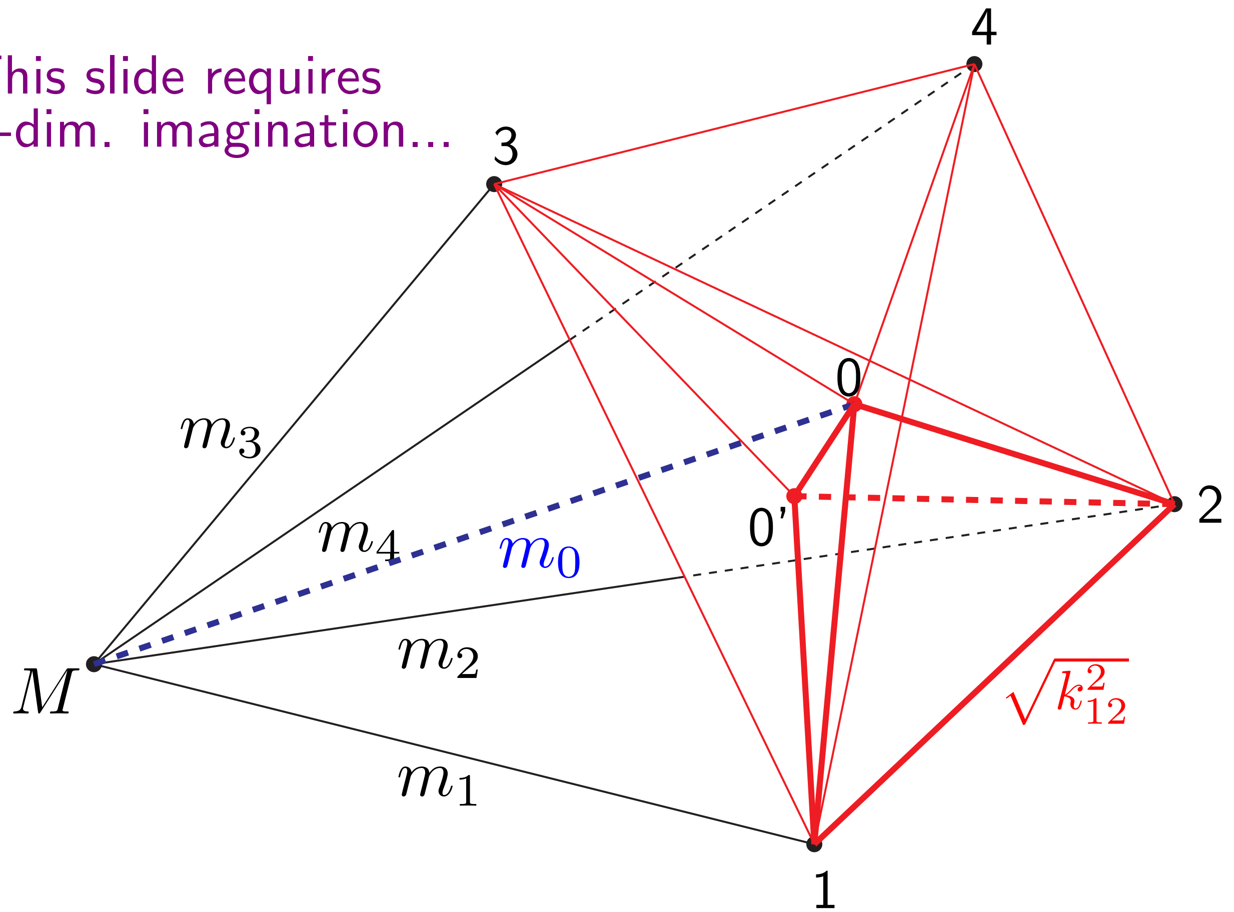
This slide requires
4-dim. imagination...

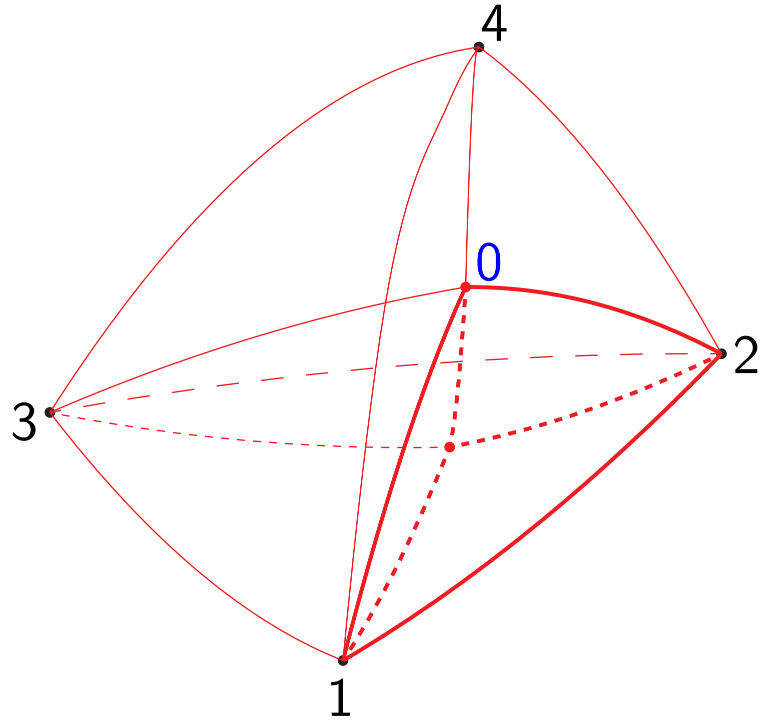




Four-point function: basic simplex and non-Euclidean tetrahedron

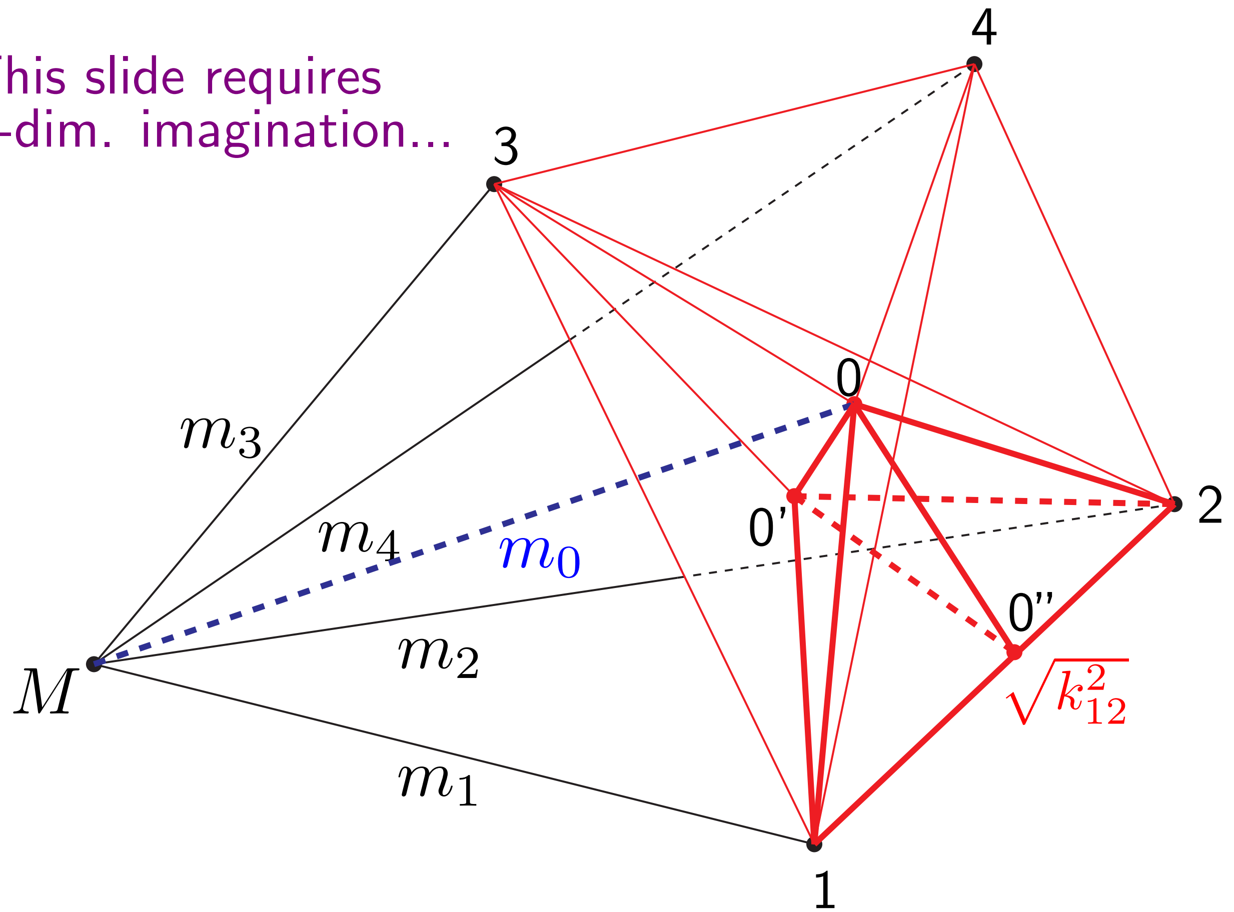
This slide requires 4-dim. imagination...

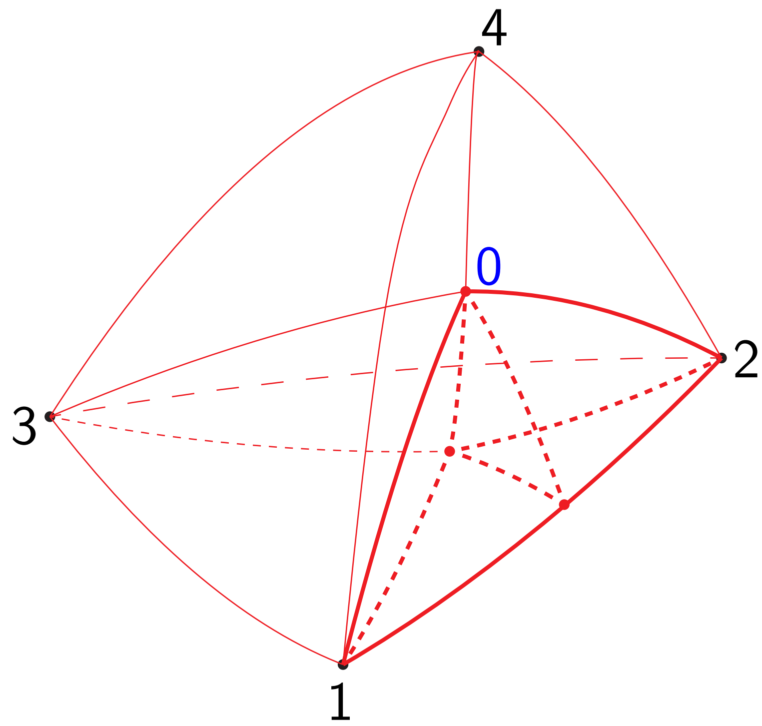




Four-point function: basic simplex and non-Euclidean tetrahedron

This slide requires
4-dim. imagination...





Four-point function: number of dimensionless variables

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{12}^2,k_{23}^2,k_{34}^2,k_{14}^2,k_{13}^2,k_{24}^2\};\{m_1,m_2,m_3,m_4\}\right)$:

$10-1$(dimension) $=9$

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{12}^2,k_{23}^2,k_{03}^2,k_{01}^2,k_{13}^2,k_{02}^2\};\{m_1,m_2,m_3,m_0\}\right)$
(after splitting the tetrahedron 1234 into four tetrahedra):

$10-3$(relations) -1(dimension) $=6$

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{12}^2,k_{20'}^2,k_{00'}^2,k_{01}^2,k_{10'}^2,k_{02}^2\};\{m_1,m_2,m_{0'},m_0\}\right)$
(after splitting the tetrahedron 0123 into three tetrahedra):

$10-5$(relations) -1(dimension) $=4$

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{10''}^2,k_{0'0''}^2,k_{00'}^2,k_{01}^2,k_{10'}^2,k_{00''}^2\};\{m_1,m_{0''},m_{0'},m_0\}\right)$
(after splitting each of the resulting tetrahedra into two):

$10-6$(relations) -1(dimension) $=3$

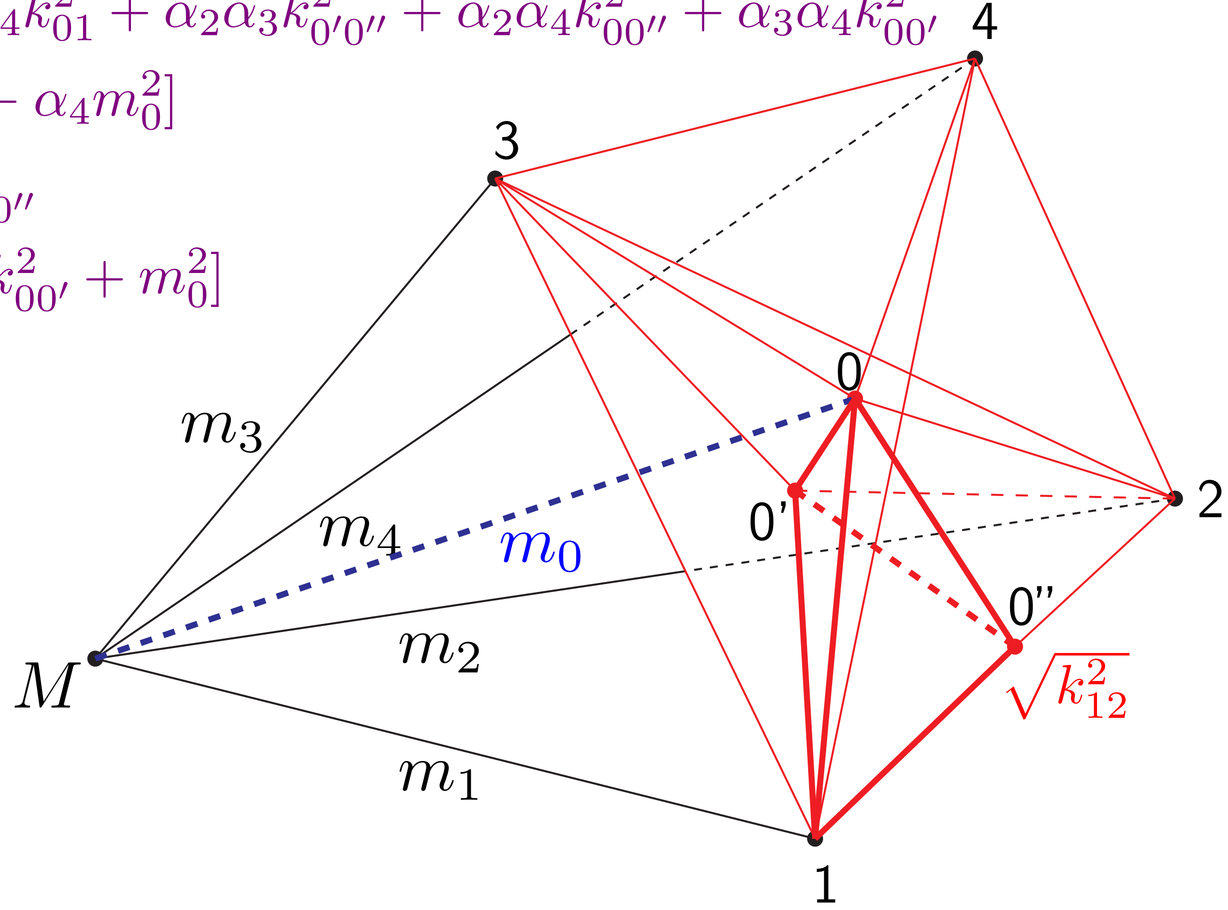
Four-point function: quadratic form in Feynman parametric integral

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{12}^2,k_{23}^2,k_{34}^2,k_{14}^2,k_{13}^2,k_{24}^2\};\{m_1,m_2,m_3,m_4\}\right)$:

$$[\alpha_1\alpha_2k_{12}^2+\alpha_1\alpha_3k_{13}^2+\alpha_1\alpha_4k_{14}^2+\alpha_2\alpha_3k_{23}^2+\alpha_2\alpha_4k_{24}^2+\alpha_3\alpha_4k_{34}^2$$
$$-\alpha_1m_1^2-\alpha_2m_2^2-\alpha_3m_3^2-\alpha_4m_4^2]$$

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{10''}^2,k_{0'0''}^2,k_{00'}^2,k_{01}^2,k_{10'}^2,k_{00''}^2\};\{m_1,m_{0''},m_{0'},m_0\}\right)$:

$$[\alpha_1\alpha_2k_{10''}^2+\alpha_1\alpha_3k_{10'}^2+\alpha_1\alpha_4k_{01}^2+\alpha_2\alpha_3k_{0'0''}^2+\alpha_2\alpha_4k_{00''}^2+\alpha_3\alpha_4k_{00'}^2$$
$$-\alpha_1m_1^2-\alpha_2m_{0''}^2-\alpha_3m_{0'}^2-\alpha_4m_0^2]$$
$$=-[\alpha_1^2k_{10''}^2+(\alpha_1+\alpha_2)^2k_{0'0''}^2$$
$$+(\alpha_1+\alpha_2+\alpha_3)^2k_{00'}^2+m_0^2]$$



Four-point function: result in arbitrary dimension

$$J^{(4)}\left(n;1,1,1,1\middle|\{k^2_{10''},k^2_{0'0''},k^2_{00'},k^2_{01},k^2_{10'},k^2_{00''}\};\{m_1,m_{0''},m_{0'},m_0\}\right)$$

$$= \mathrm{i}\pi^{n/2}\Gamma(4-n/2)\int\limits_0^1\int\limits_0^1\int\limits_0^1\int\limits_0^1 \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\mathrm{d}\alpha_4\,\delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1)}{[\alpha_1^2 k^2_{10''}+(\alpha_1+\alpha_2)^2 k^2_{0'0''}+(\alpha_1+\alpha_2+\alpha_3)^2 k^2_{00'}+m_0^2]^{4-n/2}}$$

$$= \frac{\mathrm{i}\pi^{n/2}\Gamma(3-n/2)}{2k^2_{0'0''}(m_0^2)^{3-n/2}}\left\{\sqrt{\frac{k^2_{0'0''}}{k^2_{10''}}}\;\arctan\sqrt{\frac{k^2_{10''}}{k^2_{0'0''}}}\;\;{}_2F_1\left(\begin{matrix}\frac12, 3-\frac{n}{2}\\ \frac32\end{matrix}\middle| -\frac{k^2_{00'}}{m_0^2}\right)\right.$$

$$\left. -\left(\frac{m_0^2}{m_{0'}^2}\right)^{2-n/2} F_N\left(1,1,3-\tfrac{n}{2},\tfrac12,\tfrac{n-3}{2},\tfrac12;\tfrac32,\tfrac32,\tfrac32\middle| -\frac{k^2_{10''}}{k^2_{0'0''}}, -\frac{k^2_{00'}}{m_0^2}, -\frac{k^2_{10'}}{m_{0'}^2}\right)\right\}$$

where F_N is one of the Lauricella-Saran functions,

$$F_N(a_1,a_2,a_3,b_1,b_2,b_1;c_1,c_2,c_2|x,y,z) = \sum_{j_1,j_2,j_3} \frac{(a_1)_{j_1}(a_2)_{j_2}(a_3)_{j_3}(b_1)_{j_1+j_3}(b_2)_{j_2}}{(c_1)_{j_1}(c_2)_{j_2+j_3}}\,\frac{x^{j_1}y^{j_2}z^{j_3}}{j_1!j_2!j_3!}$$

A.I.D., J. Phys. (Conf. Ser.) 1085 (2018) 052016 (arXiv:1711.07351)

See also: J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303 (F_S can be transformed into F_N)

Reduced number of variables and simplified quadratic forms

	total # of dimensionless variables	# of splitting pieces	reduced # of variables
$N = 2$	$3 - 1 = 2$	2	1
$N = 3$	$6 - 1 = 5$	6	2
$N = 4$	$10 - 1 = 9$	24	3
arbitrary N	$\frac{1}{2}(N-1)(N+2)$	$N!$	$N - 1$

$J^{(2)}\left(n; 1, 1 \middle| k_{01}^2; m_1, m_0\right)$:

$$\Rightarrow -[\alpha_1^2 k_{01}^2 + m_0^2]$$

$J^{(3)}\left(n; 1, 1, 1 \middle| k_{00'}^2, k_{01}^2, k_{10'}^2; m_1, m_{0'}, m_0\right)$:

$$\Rightarrow -[\alpha_1^2 k_{10'}^2 + (\alpha_1 + \alpha_2)^2 k_{00'}^2 + m_0^2]$$

$J^{(4)}\left(n; 1, 1, 1, 1 \middle| \{k_{10''}^2, k_{0'0''}^2, k_{00'}^2, k_{01}^2, k_{10'}^2, k_{00''}^2\}; \{m_1, m_{0''}, m_{0'}, m_0\}\right)$:

$$\Rightarrow -[\alpha_1^2 k_{10''}^2 + (\alpha_1 + \alpha_2)^2 k_{0'0''}^2 + (\alpha_1 + \alpha_2 + \alpha_3)^2 k_{00'}^2 + m_0^2]$$

$\Rightarrow$ for $N > 4$ we should also expect squares of sums of partial sums of α's

Summary

- A geometrical way to calculate dimensionally-regulated Feynman diagrams is reviewed.

- All variables (k_{jl}^2 and m_i) acquire direct geometrical meaning; the integration region corresponds to an N-dimensional solid angle ($\Omega^{(N)}$); thresholds (and pseudothresholds) can be associated with situations when some hypervolumes vanish; the dependence on the momenta and masses is moved from the integrand (as in Feynman parametric representation) into the integration limits.

- In the one-loop N-point case, results can be related to certain volume integrals in non-Euclidean geometry. For example, the result for the four-point function can be associated with the content of a spherical or hyperbolic tetrahedron in three-dimensional spherical or hyperbolic space (Lobachevsky, Schläfli, ...).

- Analytical continuation of the results to other regions of kinematical variables (momenta and masses of the particles) is discussed. In a number of cases, analytic results can be presented in terms of the (generalized) polylogarithms and associated functions. In more complicated cases, multiple polylogarithms may appear.

Summary (continued)

- Geometrical splitting provides straightforward way to reduce general integrals to those with lesser number of independent variables and predict the set and the number of these variables in the resulting integrals; it also allows to derive functional relations between integrals with different momenta and masses.

- Resulting integrals (after splitting) can be calculated either within geometrical approach (by integrating over non-Euclidean simplices), or by going back to the Feynman parametric representation, which becomes greatly simplified due to right-triangle connections between the invariants.

- Explicit results for general N-point integrals in arbitrary dimension can be presented in terms of hypergeometric functions of $(N-1)$ variables, in particular:
  - for the 2-point diagram we get the hypergeometric function ${}_2F_1$;
  - for the 3-point diagram we get Appell hypergeometric function F_1;
  - for the 4-point diagram we get the Lauricella-Saran function F_N (which can be transformed into F_S).

Geometrical ideas in recent research (a very incomplete list!)

A. Gorsky and A. Zhiboedov, J. Phys. **A42** (2009) 355214

S. Bloch and D. Kreimer, Commun. Num. Theor. Phys. **4** (2010) 703

O. Schnetz, arXiv:1010:5334 (2010)

L. Mason and D. Skinner, J. Phys. **A44** (2009) 135401

D. Nandan, M.F. Paulos, M. Spradlin and A. Volovich, JHEP **05** (2013) 105

O.V. Tarasov, JHEP **02** (2019) 173

M. Libine, J. Functional Analysis **278** (2020) 108388

B. Kol, S. Mazumdar, JHEP **03** (2020) 156

J.L. Bourjaily, E. Gardi, A.J. McLeod and C. Vergu, JHEP **08** (2020) 029

F. Loebbert, J. Miczajka, D. Müller and H. Múnkler, arXiv:2010.08552 (2020)

etc.