Wick products and combinatorial Hopf algebras ⊛

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Aim: understand relations between different families of Wick polynomials in non-commutative probability.

Setting: non-commutative shuffle Hopf algebra (a.k.a. dendriform algebra)

Rmk.: M. Anshelevich described these polynomials using generating function calculus (2004/9).

1) Classical case

A commutative unital associative K-algebra

φ linear unital form : $\varphi : A \to k$

$$T(A) = \bigoplus_{n>0} A^{\otimes n} \qquad \overline{T}(A) = k\mathbb{1} \oplus T(A)$$

$$\Delta^{ш} : \overline{T}(A) \to \overline{T}(A) \otimes \overline{T}(A)$$

$$\Delta^{ш}(a_1 \cdots a_n) = \sum_{S \subseteq [n]} a_S \otimes a_{[n] \setminus S}$$

$S = \{s_1 < \dots < s_l\}$

$$a_S := a_{s_1} \cdots a_{s_l}$$

Recall: $\mu \sqcup\!\sqcup \nu := (\mu \otimes \nu)\Delta^{\sqcup\!\sqcup}$

Extend φ to $\phi : \overline{T}(A) \to k$, $\phi(\mathbb{1}) = 1$

$$\phi(a_1 \cdots a_n) := \varphi(\underbrace{a_1 \bullet \cdots \bullet a_n}_{\in A})$$

We can find: $c : \overline{T}(A) \to k$, $c(\mathbb{1}) = 0$

$$\phi = \exp^{\sqcup\!\sqcup}(c)$$

Thm.:

$$\phi(a_1 \cdots a_n) = \exp^{\sqcup\!\sqcup}(c)(a_1 \cdots a_n) = \sum_{\pi \in P(n)} \prod_{\pi_i \in \pi} c(a_{\pi_i})$$

Classical moment-cumulant relations

$$m_n = \phi(a^n), \quad \kappa_n = c(a^n)$$

$$M(t) = 1 + \sum_{n>0} m_n \frac{t^n}{n!}, \quad K(t) = \sum_{n>0} \kappa_n \frac{t^n}{n!}$$

$$M(t) = \exp\left(K(t)\right)$$

$$m_1 = \kappa_1, \quad m_2 = \kappa_2 + \kappa_1^2, \quad m_3 = \kappa_3 + 2\kappa_2\kappa_1 + \kappa_1^3$$

$$m_4 = \kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4$$

$$m_n = \sum_{j=1}^{n} \binom{n-1}{j-1} \kappa_j m_{n-j}$$

Thm.:

$$W_T := (\mathrm{id} \otimes \phi^{-1})\,\Delta^{ш}, \quad \bar{T}(A) \to \bar{T}(A)$$

$$\phi \circ W = \varepsilon, \qquad \partial_a \circ W_T = W_T \circ \partial_a$$

$$W_T^{\circ -1} = (\mathrm{id} \otimes \phi)\,\Delta^{ш}, \qquad \varepsilon \circ W_T^{-1} = \phi$$

$$W_T(a_1) = a_1 - \phi(a_1)\mathbb{1}$$

$$W_T(a_1 a_2) = a_1 a_2 - \phi(a_1)a_2 - \phi(a_2)a_1 - (\phi(a_1a_2) - 2\phi(a_1)\phi(a_2))\mathbb{1}$$

$$\begin{aligned} W_T(a_1a_2a_3) = a_1a_2a_3 &- \phi(a_i)\,a_j a_k \\ &- a_i(\phi(a_j a_k) - 2\phi(a_j)\phi(a_k)) \\ &- (\phi(a_i a_j a_k) - \phi(a_1)\phi(a_2a_3) - \phi(a_2)\phi(a_1a_3) \\ &- \phi(a_3)\phi(a_1a_2) + 6\,\phi(a_1)\phi(a_2)\phi(a_3)) \end{aligned}$$

$$\mathrm{ev}(W_T(a^n)) = W_n(a)$$

$$\begin{cases} W_n'(a) = n\,W_{n-1}(a) \\ \varphi(W_n(a)) = 0 \end{cases}$$

$$:\exp:(ta) = \sum_{n\geq 0} W_n(a)\frac{t^n}{n!} = e^{ta-K(t)} = \frac{e^{ta}}{M(t)}$$

We can define on $\overline{T}(A)$ a new product

$$w_1 \bullet w_2 = W_T\left(\overset{\circ -1}{W_T}(w_1)\,\overset{\circ -1}{W_T}(w_2)\right)$$

$$W_T(a^n)\bullet W_T(a^m) = W_T(a^{n+m})$$

2) Non-commutative case

Def.: A non-commutative probability space (A, φ) consists of an associative algebra and a linear map $\varphi : A \to k$, $\varphi(1_A) = 1$.

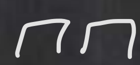
Thm.: (Speicher, '97)

free $\quad \varphi(a_1 \cdot \ldots \cdot a_n) = \sum_{\pi \in NC(n)} \prod_{\pi_i \in \pi} r(a_{\pi_i})$

Thm.: (Speicher & Woroudi, '97)

boolean $\quad \varphi(a_1 \cdot \ldots \cdot a_n) = \sum_{\pi \in Int(n)} \prod_{\pi_i \in \pi} b(a_{\pi_i})$

$$\varphi(a^4) = r(a^4) + 4\,r(a^3)r(a) + 2\,r(a^2)r(a^2) + 6\,r(a^2)\,r(a)^2 + r(a)^4$$

 $\notin NC(4)$,  ,  $\in NC(4)$

Rmk.: Hasebe & Saigo, 2011

monotone $\quad \varphi(a_1\cdot\ldots\cdot a_n) = \sum_{\pi\in NC(n)} \frac{1}{t(\pi)!} \prod_{\pi_i\in\pi} r(a_{\pi_i})$

Generating series: $w=\{w_1,\dots,w_n\}$, $a_1,\dots,a_n\in A$

$$R(w) = \sum_{\underline{n}} r(a_{\underline{n}})\,w_{\underline{n}}$$

$$M(w) = 1 + \sum_{\underline{n}} \varphi(a_{\underline{n}})\,w_{\underline{n}}$$

$$\eta(w) = \sum_{\underline{n}} b(a_{\underline{n}})\,w_{\underline{n}}$$

free: $M(w) = 1 + R(z)$

$$z_i := w_i M(w)$$

boolean: $M(w) = 1 + \eta(w) M(w)$

<u>Rmk.</u>: Cvitanović, '80

$$Z[J] \underline{=} 1 + W[J Z[J]]$$

Relation between generating functionals for full and connected planar Green's functions.

$H = \overline{T}(T(A))$ with second tensor prod. denoted by $|$

$$\Delta(a_1 \cdots a_n) = \sum_{S \subseteq [n]} a_S \otimes a_{J_1^S} | \cdots | a_{J_k^S}$$

$a_1 a_2 \dots a_6$, $S = \{2, 4, 5\}$:

$$a_S = a_2 a_4 a_5$$

$$a_{J_1^S} | a_{J_2^S} | a_{J_3^S} = a_1 | a_3 | a_6$$

$$a_1 a_2 a_3 a_4 a_5 a_6$$

Splitting: $\Delta = \Delta_< + \Delta_>$

$$\Delta_<(a_1 \cdots a_n) := \sum_{1 \in S \subseteq [n]} \underline{a_S} \otimes a_{J_1^S} | \cdots | a_{J_k^S}$$

$$\Delta_>(a_1 \cdots a_n) := \sum_{1 \notin S \subseteq [n]} a_S \otimes a_{J_1^S} | \cdots | a_{J_k^S}$$

On the dual side:

$$\mu \prec \nu := (\mu \otimes \nu)\Delta_{\prec} \qquad \mu \succ \nu := (\mu \otimes \nu)\Delta_{\succ}$$

$$\mu * \nu = (\mu \otimes \nu)\Delta = \mu \succ \nu + \mu \prec \nu$$

non-com. shuffle alg. shuffle relations

$$\begin{cases} \alpha \succ (\beta \succ \gamma) = (\alpha * \beta) \succ \gamma \\ \alpha \succ (\beta \prec \gamma) = (\alpha \succ \beta) \prec \gamma \\ (\alpha \prec \beta) \prec \gamma = \alpha \prec (\beta * \gamma) \end{cases}$$

Rmk.: $\Delta^{ш} = \Delta^{ш}_{\prec} + \Delta^{ш}_{\succ}$, $\Delta^{ш}_{\succ} = \tau \circ \Delta^{ш}_{\prec}$

$\mu \prec \nu = \nu \succ \mu$ $\qquad (\alpha \prec \beta) \prec \gamma = \alpha \prec (\beta ш \gamma)$

com. shuffle algebra

$\phi : \overline{T}(A) \to k$ extended to H: $\Phi : \overline{T}(T(A)) \to k$
character

Thm.: (EF & Patras, 2014)

α, β, γ infinitesimal characters

$$
\begin{aligned}
\Phi &= \varepsilon + \alpha \prec \Phi = \mathcal{E}_{\prec}(\alpha) \\
&= \varepsilon + \Phi \succ \beta = \mathcal{E}_{\succ}(\beta) \\
&= \exp^{*}(\gamma)
\end{aligned}
$$

Rmk.:

$$\Phi = \exp^{*}(\Omega(\alpha)) = \exp^{*}(-\Omega(-\beta)) = \exp^{*}(\gamma)$$

$$\Phi^{-1} = \mathcal{E}_{\succ}(-\alpha) = \exp^{*}(-\gamma) = \mathcal{E}_{\prec}(-\beta)$$

Thm.: (EF & Patras, 2014/17)

$$\Phi(\underbrace{a_1 \cdots a_n}_{w}) = \alpha(w) + \alpha \prec \alpha(w) + \cdots + \alpha \prec (\cdots \prec \alpha)(w)$$

$$\vdots$$

$$= \sum_{\pi \in NC(n)} \prod_{\pi_i \in \pi} \alpha(a_{\pi_i})$$

$$= \beta(w) + \beta \succ \beta(w) + \cdots + (\beta \succ \cdots) \succ \beta(w)$$

$$= \sum_{\pi \in Int(n)} \prod_{\pi_i \in \pi} \beta(a_{\pi_i})$$

Rmk.:

$\Phi = \exp^*(\gamma)$ gives monotone moment–cumulant relations

Def.:

$$W : \overline{T}(T(A)) \longrightarrow \overline{T}(T(A))$$

$$W = (\mathrm{id} \otimes \Phi^{-1})\Delta \qquad \text{free Wick map}$$

or $\mathrm{id} = (W \otimes \Phi)\Delta$, $\{W(a_1 \cdots a_n) \mid a_i \in A\}$ free Wick polynom.

Thm.: 1) W is multiplicative

$$W(w_1 \mid w_2) = W(w_1) \mid W(w_2)$$

2) The free Wick polynomials are centered

$$\Phi \circ W = \varepsilon$$

Define: $a \in A$, $\partial_a := (\xi_a \otimes \mathrm{id})\Delta$, $\xi_a(a) = 1$ and zero else

$$\partial_a(aw) = w, \quad \partial_a(w_1 \underset{\cdot}{a} w_2) = w_1 \, w_2$$

$$\partial_a(a w_1 a) = w_1 a + a w_1$$

<u>Thm.</u>: $\quad \partial_a \circ W = W \circ \partial_a$

$$W(a_1) = a_1 - \varphi(a_1)\mathbb{1}$$

$$W(a_1 a_2) = a_1 a_2 - a_1 \varphi(a_2) - a_2 \varphi(a_1) - (\varphi(a_1 \cdot a_2) - 2\varphi(a_1)\varphi(a_2))\mathbb{1}$$

$$\begin{aligned} W(a_1 a_2 a_3) = {} & a_1 a_2 a_3 - a_1 a_2 \varphi(a_3) - a_1 a_3 \varphi(a_2) - a_2 a_3 \varphi(a_3) \\ & - a_3(\varphi(a_1 \cdot a_2) - 2\varphi(a_1)\varphi(a_2)) \\ & - a_1(\varphi(a_2 \cdot a_3) - 2\varphi(a_2)\varphi(a_3)) \\ & + a_2 \varphi(a_1)\varphi(a_3) \\ & - (\varphi(a_1 \cdot a_2 \cdot a_3) - 2\varphi(a_1)\varphi(a_2 \cdot a_3) - 2\varphi(a_3)\varphi(a_1 \cdot a_2) \\ & \quad - \varphi(a_2)\varphi(a_1 a_3) + 5\varphi(a_1)\varphi(a_2)\varphi(a_3)) \end{aligned}$$

$$\Phi^{-1} = \mathcal{E}_{>}(-\alpha)$$

$$W = id \otimes \mathcal{E}_{>}(-\alpha)$$

Computing

$$W(a_1 \cdots a_n) = \sum_{S \subseteq [n]} a_S \, \mathcal{E}_{>}(-\alpha)(a_{J_1^S}) \cdots \mathcal{E}_{>}(-\alpha)(a_{J_k^S})$$

we recover:

Thm.: (Anshelevich, 2004)

$$W(a_1 \cdots a_n) = \sum_{S \subseteq [n]} a_S \sum_{\substack{\pi \in \mathrm{Int}([n] \setminus S) \\ \pi \cup S \in NC(n)}} (-1)^{|\pi|} \prod_{\pi_i \in \pi} \alpha(a_{\pi_i})$$

Boolean side of the picture

$$W = id \otimes \Phi^{-1} = id \otimes \mathcal{E}_<(-\beta)$$

Thm.: $$W = e + (id - e - id < \beta) < \Phi^{-1}$$

$e = \eta \circ \varepsilon$

Proof uses the recursion $W = e + (id - e) < \Phi^{-1} - W > \alpha$
and the relation between cumulants: $\alpha = \Phi > \beta < \Phi^{-1}$

Def.: Boolean Wick map

$$W' := id - id < \beta$$

<u>Thm.</u>: 1) Boolean Wick polynomials are centered

$$\Phi \circ W' = \Phi - \Phi > \beta = \varepsilon$$

2) Free and boolean Wick polynomials are related

$$W' = e + (W - e) < \Phi$$

<u>Rmk.</u>: 1)

$$W = id * \Phi^{-1}$$

$$W' = W < \Phi$$

right action of G on $\underline{\mathrm{End}}(\overline{T}(A))$ via $*$ and $<$

$$W^c = W' < \Psi^{-1}$$

conditionally free Wick polynomials

2) W is invertible (for composition)

new product on $\overline{T}(A)$ $\quad x = a_1 \cdots a_n$, $y = a_{n+1} \cdots a_{n+m}$

$$x \bullet y := W\left(W^{-1}(x)\, W^{-1}(y)\right)$$

$$= \sum_{S \subseteq [n+m]} a_S\, \Phi(a_{J_1^S}) \cdots \Phi(a_{J_k^S})$$

$$W(a)^{\bullet n} = \underline{W}(a^n)$$

Thank you!