The Field Theory KLT Relations

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The field theory KLT relations are of the form

$$M_n = \sum_{a,b \in S_{n-2}} A_{YM}(1an)\, S(1a,1b)\, A_{YM}(1bn)$$

for M_n the gravity tree amplitude, $A_{YM}(1a,n)$ partial tree amplitudes; the sum is over permutations of $23 \cdots n-1$. The matrix is

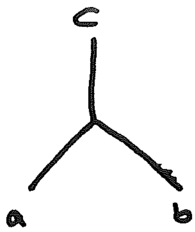
$$S(1a,1b) = \prod_{i=2}^{n-1} \Bigg(\sum_{\substack{j \\ j <_{1a} i \\ j >_{1b} i}} s_{ij} \Bigg)$$

2011 Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove

The aim of this talk is to derive this formula in an elementary, and algebraic way.

Partial Amplitudes

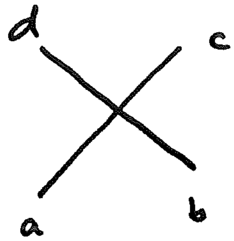
Consider Yang-Mills. The gluon vertices are:



$$f^{abc} = tr([\lambda^a, \lambda^b], \lambda^c)$$

for $\lambda^a \in su(N)$ a basis, with $tr(\lambda^a \lambda^b) = \delta^{ab}$,

and also:



$$(\dots) f^{abe} f^{ecd} + (\dots) f^{bce} f^{ead} + (\dots) f^{ace} f^{ebd}$$

$$= tr([[\lambda^a, \lambda^b], \lambda^c], \lambda^d) \qquad = tr([[\lambda^b, \lambda^c], \lambda^a], \lambda^d)$$

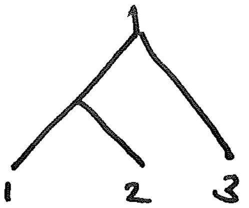
Given these rules, it can be seen that the full tree amplitude may be written

$$\boxed{A_{tree} = \sum_{\substack{\Gamma \\ \text{cubic trees}}} A_\Gamma \, C_\Gamma ,}$$

where, for each cubic tree, we choose a corresponding <u>Lie monomial</u> Γ, and the colour factor is given by bracketing the $su(N)$ elements according to Γ'.

e.g. $C_{[[12]3]} = tr([[\lambda^1, \lambda^2], \lambda^3], \lambda^4)$.

$[[12]3]$ corresponds to the tree



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Using the matrix representation ("fundamental"), $su(N) \subset Mat_{N\times N}$, then the Lie bracket is the commutator and so, e.g.,

$$C_{[12]} = tr([\lambda^1, \lambda^2], \lambda^3)$$
$$= tr(\lambda^1\lambda^2\lambda^3) - tr(\lambda^2\lambda^1\lambda^3).$$

This leads to the conventional representation of the tree amplitude:

$$A_{tree} = \sum_{a \in S_{n-1}} A(an)\, tr(\lambda^{a_1}\lambda^{a_2}\cdots\lambda^{n})$$

where $A(an)$ is called the <u>partial amplitude</u> and it is given by

$$A(an) = \sum_{\substack{\Gamma' \\ \text{cubic trees}}} (a, \Gamma')\, A_{\Gamma'},$$

where

$$(a, T') = \pm 1, 0$$

is the coefficient of the word a in the Lie monomial T'.

e.g.

$$(312, [[12]3]) = -1$$
$$(132, [[12]3]) = 0 .$$

Remark. The above remarks hold for ~~[illegible]~~ any gauge theory whose Lagrangian has only 'single trace' terms.

Lie Polynomials

Let $L(A)$ be the multilinear part of the free Lie algebra on A: i.e. $L(A)$ is spanned by Lie monomials with no repeated letters.

It is a subspace of $W(A)$: the linear span of words on the set A with no repeated letters.

Then

$$L(A) \subset W(A)$$

is specified by Ree's thm as

$$\Gamma \in L(A) \quad \text{iff} \quad (\Gamma, a \sqcup\!\sqcup b) = 0$$
for all $a, b \neq$ empty.

where the inner product

$$(\ ,\ ): \ W(A) \times W(A) \to k$$

is

$$(a, b) = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}.$$

Write $Sh(A) \subset W(A)$ for the subspace spanned by nontrivial shuffles, $a \sqcup\!\sqcup b$, $a, b \neq$ empty.

<u>Ree</u>: $L(A) = Sh(A)^{\perp}$, or, dually, $L(A)^{\vee} = W(A)/Sh(A)$.

<u>Example</u>: The partial amplitudes of a gauge theory,

$$A(a) = \sum (a, \Gamma) A_{\Gamma},$$

satisfy

$$A(a \sqcup\!\sqcup b) = 0, \qquad a, b \neq \text{empty}.$$

Dynkin-Specht-Wever: The map

$$\ell : W(A) \longrightarrow L(A)$$

that sends

$$\ell : 12\cdots n \longmapsto [[[1,2],3]\dots,n]$$

is <u>surjective</u>.

Basis of $L(A)$: The set of monomials $\ell(a)$, for all words $a \in W(A)$ that begin with their <u>smallest</u> letter, is a basis of $L(A)$.

Dually, the $a + Sh(A) \in L(A)^\vee$, for words a that begin with their smallest letter, is a basis.

The two bases are dual, so any $a + Sh(A) \in L(A)^\vee$ has an expansion

$$a = \sum (a, \ell(ib))\, ib \qquad \text{in } L(A)^\vee$$

where i is the smallest letter of a.

One checks that

$$\ell(ib) = \sum_{c,d} (c \sqcup\!\sqcup d, b)\, \check{c}\, i\, d,$$

where $\check{c} = (-1)^{|c|}\, \overline{c}$.

So it follows that, if $a = bic$,

$$a = i(\tilde{b} \sqcup\!\sqcup c) \quad \text{in} \quad L(A)^\vee.$$

__Example__: This gives the "Kleiss-Kuijf" relation

$$A(bic) = A(i(\tilde{b} \sqcup\!\sqcup c))$$

on partial amplitudes.

Mandelstams & the KLT relation

Introduce Mandelstam variables s_I for subsets $I \subset N$, satisfying:

$$s_I = \sum_{\{i,j\} \subset I} s_{ij}.$$

Write M for the ring of rational functions of these variables.

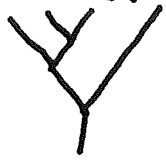
For a Lie monomial $\pm \Gamma \in L(A)$, there is a monomial

$$s_\Gamma = \prod_{I \in \Gamma} s_I$$

where the product is over the subsets I that arise from pairs of brackets in Γ.

e.g.

$$\Gamma = [[1[23]]4] \longleftrightarrow \begin{array}{l} \{23\} \\ \{123\} \\ \{1234\} \end{array}$$



so that $\quad S_\Gamma = S_{23}\, S_{123}\, S_{1234}$.

n.b. The pairs of brackets in Γ correspond to edges of the tree associated to Γ.

The KLT relation is a statement about the following object:

$$T := \sum_{\substack{\Gamma \\ \text{binary trees}}} \frac{\Gamma \otimes \Gamma}{S_\Gamma} \in L(A) \otimes L(A) \otimes M.$$

(T is a 'prototype' of the gravity tree amplitude.)

T defines a map,

$$T : L(A)^{\vee} \otimes M \longrightarrow L(A) \otimes M$$

$$: \quad a \longmapsto T(a) = \sum \frac{(a,\Gamma)\Gamma}{S_\Gamma}.$$

<u>Claim</u>: The map T is invertible, with inverse given by a 'KLT map.'

To define the KLT map, introduce a bracket

$$\{\ ,\ \} : L(A)^\vee \times L(A)^\vee \longrightarrow L(A)^\vee$$

defined by

$$\{i,j\} = S_{ij}\ ij, \quad \text{for letters } i,j,$$

and

$$\{iaj, b\} = i\{aj,b\} - j\{ia,b\}$$
$$\{a, ibj\} = \{a,ib\}j - \{a,bj\}i.$$

e.g. $$\{a,i\} = \sum_{a=bc} \Big(\sum_{j\in c} S_{ij}\Big)\, bic,$$

though it takes some work to get this from the definition.

$$\boxed{\underline{\text{Lemma}}: \quad T(\{a,b\}) = [T(a), T(b)].}$$

Nesting this gives, e.g., that

$$T(\{\{1,2\},\{3,4\}\}) = [[1,2],[3,4]]$$

and so on.

Given a Lie monomial Γ written as a nested bracketing, write

$$\{\Gamma\} \in L(A)^\vee$$

for the expression obtained by replacing every $[\ ,\ ]$ by $\{\ ,\ \}$. Then the Lemma implies that

$$T(\{\Gamma\}) = \Gamma.$$

Define ~~a~~ ~~an~~ the <u>KLT map</u> to be

$$S : L(A) \longrightarrow L(A)^\vee$$

$$\Gamma \longmapsto \{\Gamma\}$$

Theorem: S is well defined as stated because $\{\ ,\ \}$ is a Lie bracket.

Moreover, S and T are inverses.

The matrix elements of the map S are the entries of the <u>KLT matrix</u>.

To be explicit,

$$S(\ell(ia)) = \ell\{ia\}$$

$$= \sum_{b \in S_{n-2}} (\ell\{ia\}, \ell(ib))\, ib$$

these coefficients are conventionally written as

$$S(ia, ib)$$.

In fact, a basis expansion gives

$$T = \sum \frac{T' \otimes T'}{S_{T'}} = \sum_{a} T(a) \otimes \ell(a)$$

over a basis

but

$$\ell(a) = T(S(\ell\{a\}))$$

$$= \sum_{b} S(a,b)\, T(b),$$

over a basis

so

$$\boxed{T = \sum_{a,b} S(a,b)\, T(a) \otimes T(b)}$$

which is the KLT relation.