Tasmanian adventures

David Broadhurst, Open University, 20 November 2020,
at *Algebraic Structures in Perturbative Quantum Field Theory*,
a celebration of Dirk Kreimer's work, hosted by IHES, Bures-sur-Yvette

I report on two adventures with Dirk Kreimer in Tasmania, 25 years ago. One of these, concerning knots, is not even wrong. The other, concerning a conjectural 4-term relation, is either wrong or right. I suggest that younger colleagues have powerful tools that might be brought to bear on this 4-term conjecture.

1. Some of Dirk's achievements prior to Tasmania
2. Knots ands numbers
3. 4-term relations

1 Some of Dirk's achievements prior to Tasmania

We have heard much at this conference of the influence that Dirk has had since his discovery of the Hopf algebra of renormalization, first revealed to John Gracey and me in August 1997 at a private meeting in St Andrews, Scotland.

In this talk I call attention to some of his work before that memorable occasion.

I begin with some recollections of his graduate studies, from which I learnt a lot.

1. The problem of γ_5 in weak interactions: innovation and staying calm.
2. Two-loop propagators, form factors and double boxes, with general kinematics: deft analysis and good programming.
3. Systematic understanding of generalized hypergeometric functions.
4. Regularization, by Hadamard's finite part.

2 Knots and numbers

1. Dirk decorated the braids of positive knots and obtained Feynman diagrams with trivalent vertices. He shrank enough edges to obtain subdivergence-free counterterms of forms that I was able to evaluate. [Show Dirk's figures.]

2. We associated families of positive knots with combinations of multiple zeta values (MZVs). Our dictionary between knots and numbers exploited the pushdown of MZVs to alternating sums of lesser depth. This led to our conjecture for the number $D_{n,k}$ of primitive MZVs of weight n and depth k:

$$1-\frac{x^3y}{1-x^2}+\frac{x^{12}y^2(1-y^2)}{(1-x^4)(1-x^6)}=\prod_{n\geq 3}\prod_{k\geq 1}(1-x^ny^k)^{D_{n,k}}.$$

3. Our results included all the primitive contributions to the beta-function of ϕ^4 theory at 6 loops. We were keen to accomplish as much as possible at 7 loops.

4. Going from 7-loop ϕ^4 counterterms to knots was much more difficult: 4-valent vertices can be opened in 3 ways. Dirk had to consider a multiplicity of momentum routings, with link diagrams that he skeined to get positive knots, not all of which had been found to give MZVs. [Recall bushfire and lawn.]

2.1 MZVs associated with positive knots via counterterms

The $(2,2k+1)$ torus knot with braid word σ_1^{2k+1} is associated with ζ_{2k+1}.

The $(3,4)$ torus knot, with braid word $(\sigma_1\sigma_2)^4$, called 8_{19} in the knot tables, is associated with $N_{5,3}$, where

$$N_{a,b} \equiv \zeta(\overline{a},b) - \zeta(\overline{b},a)$$

with bars denoting signs alternating sums: $\zeta(\overline{5},3) = \sum_{m>n>0}(-1)^m/(m^5n^3)$. Then $N_{5,3}$ gives a combination of $\zeta_{5,3}$, ζ_8 and $\zeta_5\zeta_3$ at 6 loops

The $(3,5)$ torus knot, $10_{124} = (\sigma_1\sigma_2)^5$, is associated with $N_{7,3}$ at 7 loops, or beyond.

$N_{2k+5,3}$ occurs at $(k+6)$ loop, or beyond. At 8 loops, or beyond, we encounter $N_{9,3}$ and $N_{7,5} - \pi^{12}/(2^5 10!)$ in counterterms. The latter gives MZVs of depth 4.

At 9 loops, or beyond, we encounter weight-14 depth-4 MZVs We found these three knot-numbers at 14 crossings:

$$N_{11,3}, \quad N_{9,5} + \tfrac{5\pi^{14}}{7032946176}, \quad \zeta_{5,3,3,3} + \zeta_{3,5,3,3} - \zeta_{5,3,3}\zeta_3 + \tfrac{247851168\pi^{14}}{4331237155245}\,.$$

At 7 loops, a depth-3 combination $N_{3,5,3} \equiv \zeta_{3,5,3} - \zeta_3\zeta_{5,3}$ occurs, associated with the 4-braid 11-crossing positive knot $\sigma_1\sigma_2^3\sigma_3^2\sigma_1^2\sigma_2^2\sigma_3$.

At higher loops, we found these families of depth-3 combinations in counterterms:

$$\begin{aligned}
N_{2m+1,2n+1,2m+1} &= \zeta(2m+1,2n+1,2m+1) - \zeta(2m+1)\,\zeta(2m+1,2n+1) \\
&\quad + \sum_{k=1}^{m-1} \binom{2n+2k}{2k} \zeta_P(2n+2k+1, 2m-2k+1, 2m+1) \\
&\quad - \sum_{k=0}^{n-1} \binom{2m+2k}{2k} \zeta_P(2m+2k+1, 2n-2k+1, 2m+1)\,, \\
N_{2m,2n+1,2m} &= \zeta(2m,2n+1,2m) + \zeta(2m)\,\{\zeta(2m,2n+1) + \zeta(2m+2n+1)\} \\
&\quad + \sum_{k=1}^{m-1} \binom{2n+2k}{2k} \zeta_P(2n+2k+1, 2m-2k, 2m) \\
&\quad + \sum_{k=0}^{n-1} \binom{2m+2k}{2k+1} \zeta_P(2m+2k+1, 2n-2k, 2m)\,,
\end{aligned}$$

where $\zeta_P(a,b,c) = \zeta(a)\,\{2\,\zeta(b,c) + \zeta(b+c)\}$.

We identified braid words of 5 classes of positive knots $\mathcal{K}$ associated with MZVs, along with their HOMFLY polynomials, $X_{\mathcal{K}}(q,\lambda)$:

$\mathcal{K}$	$X_{\mathcal{K}}(q,\lambda)$
$\mathcal{T}_{2k+1}=\sigma_1^{2k+1}$	$T_{2k+1}=\lambda^k(1+q^2(1-\lambda)p_k)$
$\mathcal{R}_{k,m}=\sigma_1\sigma_2^{2k+1}\sigma_1\sigma_2^{2m+1}$	$R_{k,m}=T_{2k+2m+3}+q^3p_kp_m\Lambda_{k+m+1}$
$\mathcal{R}_{k,m,n}=\sigma_1\sigma_2^{2k}\sigma_1\sigma_2^{2m}\sigma_1\sigma_2^{2n+1}$	$R_{k,m,n}=R_{1,k+m+n-1}+q^6p_{k-1}p_{m-1}r_n\Lambda_{k+m+n+1}$
$\mathcal{S}_k=\sigma_1\sigma_2^3\sigma_3^2\sigma_1^2\sigma_2^{2k}\sigma_3$	$S_k=T_3^2T_{2k+3}+q^2p_kr_2(q^2(\lambda-2)+q-2)\Lambda_{k+3}$
$\mathcal{S}_{k,m,n}=\sigma_1\sigma_2^{2k+1}\sigma_3\sigma_1^{2m}\sigma_2^{2n+1}\sigma_3$	$S_{k,m,n}=T_{2k+2m+2n+3}+q^3(p_kp_m+p_mp_n+p_np_k$ $+(q^2(3-\lambda)-2q)p_kp_mp_n)\Lambda_{k+m+n+1}$

with $p_n=(1-q^{2n})/(1-q^2)$, $r_n=(1+q^{2n-1})/(1+q)$, $\Lambda_n=\lambda^n(1-\lambda)(1-\lambda q^2)$.

Noting that $\mathcal{S}_{1,1,1}=\mathcal{S}_1$ and $\mathcal{S}_{m,n,0}=\mathcal{R}_{m,n,0}=\mathcal{R}_{m,n}$, we obtained the third row of these enumerations by crossing number and weight:

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
positive n-crossing knots	1	0	1	0	1	1	1	3	2	7	9	17	47	?	?
primitive weight-n MZVs	1	0	1	0	1	1	1	1	2	2	3	3	4	5	7
identified n-crossing knots	1	0	1	0	1	1	1	1	2	2	3	3	4	5	5

with two associations missing for MZVs of weight 17, at 10 loops.

2.2 What really happens at 7 loops and beyond?

There were two positive positive knots with 10 crossings which we could not associate to MZVs. There were three ϕ^4 counterterms at 7 loops that we could not identify. We concluded that MZVs would not suffice for ϕ^4 counterterms.

Positive knots, and hence the transcendentals associated with them by field theory, are richer in structure than MZVs. [arXiv:hep-th/9609128]

Maxim Konstsevich did not heed this and made a strong conjecture for Symanzik polynomials over $\mathbf{F}_q$, disproven by Prakash Belkale and Patrick Brosnan.

Later, I found that two unidentified counterterms are reducible to MZVs, namely

$$\begin{aligned} P_{7,8} &= \frac{22383}{20}\zeta_{11} + \frac{4572}{5}\left(\zeta_{3,5,3} - \zeta_3\zeta_{5,3}\right) - 700\zeta_3^2\zeta_5 \\ &\quad + 1792\zeta_3\left(\frac{9}{320}\left(12\zeta_{5,3} - 29\zeta_8\right) + \frac{45}{64}\zeta_5\zeta_3\right), \\ P_{7,9} &= \frac{92943}{160}\zeta_{11} + \frac{3381}{20}\left(\zeta_{3,5,3} - \zeta_3\zeta_{5,3}\right) - \frac{1155}{4}\zeta_3^2\zeta_5 \\ &\quad + 896\zeta_3\left(\frac{9}{320}\left(12\zeta_{5,3} - 29\zeta_8\right) + \frac{45}{64}\zeta_5\zeta_3\right). \end{aligned}$$

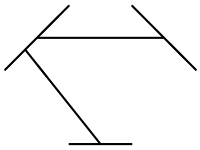
These two reductions to MZVs, at 7 loops, are somewhat surprising. Francis Brown had predicted that alternating sums would suffice. Erik Panzer and Oliver Schnetz proved this, obtaining ornate combinations of alternating sums. Then the MZV datamine, developed by Johannes Blümlein, Jos Vermaseren and me, proves the two reductions to MZVs that I had found empirically.

The remaining 7-loop counterterm, now called $P_{7,11}$ in the census of Schnetz, was predicted by Brown to reduce to polylogarithms of the sixth root of unity and was proven to do so by Panzer, in an amazing feat of analysis.

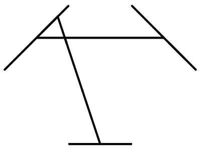
At 8 loops, polylogarithms fail to deliver all of the counterterms. There is a period whose obstruction to polylogarithmic reduction comes from a singular K3 surface associated to a cusp form with modular weight 3, coming from the symmetric square of an elliptic curve with conductor 49, as shown by Brown and Schnetz.

My subjective summary: Dirk's intuition that MZVs would not suffice at 7 loops and beyond was borne out by later analysis.

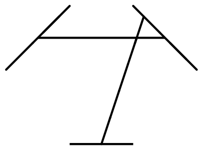
3 4-terms relations



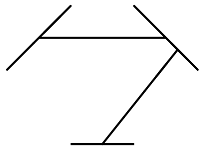
G_1



G_2



G_3



G_4

These four subgraphs generate every four-term relation. In each case, three arcs of a circle are indicated, with a chord connecting the upper pair. These arcs form part of a hamiltonian circuit that passes through every vertex of each diagram. The connections of vertices on other parts of the hamiltonian circuit need not yet concern us. From the bottom arc, connections are made, in turn, to the four parts of the hamiltonian circuit that are adjacent to the chord.

We assume that the four terms:

(i) are free of subdivergences;

(ii) differ only by the subgraphs shown;

(iii) have trivial vertices, involving no vectorial (or higher tensorial) structure;

(iv) involve no propagator with spin $s > \frac{1}{2}$;

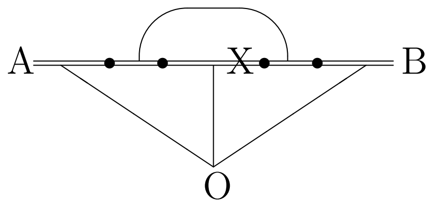
(v) modify one of the dimensionless couplings of a renormalizable theory.

The necessity of this set of provisos is not established. In a paper by Dirk Kreimer it is, however, claimed to be *sufficient* to derive the four-term relation

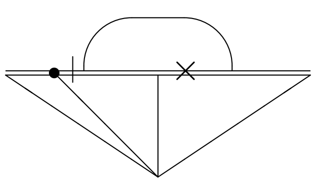
$$\langle G_1 - G_2 + G_3 - G_4 \rangle = 0$$

where $\langle G_k \rangle$ is the corresponding counterterm, i.e. the coefficient of overall logarithmic divergence of the k-th of the four diagrams, numbered in cyclic order, These counterterms may be calculated by nullifying external momenta and internal masses, and cutting the diagram wheresoever one pleases, since infrared problems are excluded by the provisos.

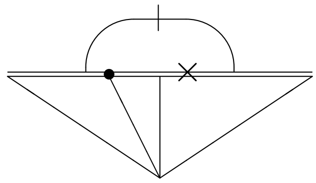
To generate the four-loop test, consider the figure below, whose four blobs indicate the connections that will be made to the origin O. The horizontal double line represents the propagation of a Dirac fermion field, ψ, with a Yukawa coupling, $\overline{\psi}\phi\psi$, to a scalar boson field, ϕ. At X there is a Yukawa coupling to an external boson, which prevents subdivergences. The asymmetry which it introduces also guarantees non-triviality of the four-term relation. Now we connect the origin O to each of the four blobs, in turn, so that O becomes a ϕ^4 vertex. Masses are then set to zero, and the external momenta at A, B and X are nullified, to give the 4 terms.



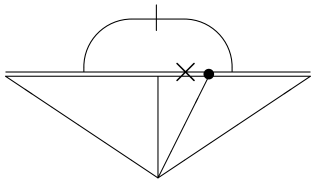
The four terms, after nullification, cut at convenient places:



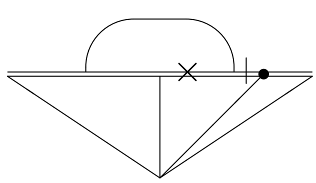
$\langle G_1 \rangle$



$\langle G_2 \rangle$



$\langle G_3 \rangle$



$\langle G_4 \rangle$

Explicit expressions for the four counterterms may be compactly written using

$$d\mu_n = \frac{(p_0^2)^{1+n\varepsilon}}{(p_n - p_0)^2} \prod_{k=1}^{n} \frac{d^D p_k}{\pi^{D/2}} \frac{G(1+\varepsilon)}{[G(1)]^2} \frac{1}{p_k^2} \frac{1}{(p_{k-1} - p_k)^2}$$

as a n-loop integration measure in $D \equiv 4 - 2\varepsilon$ euclidean dimensions, with p_0 as the cut momentum, and $G(\alpha) \equiv \Gamma(D/2 - \alpha)/\Gamma(\alpha)$.

The four terms of are given by

$$
\begin{aligned}
\langle G_1 \rangle &= \tfrac{1}{4} \lim_{\varepsilon \to 0} \mathrm{Tr} \int d\mu_2 \frac{1}{\not{p}_1} \not{p}_{02} \\
\langle G_2 \rangle &= \tfrac{1}{4} \lim_{\varepsilon \to 0} \mathrm{Tr} \int d\mu_3 \frac{1}{\not{p}_{10}} \not{p}_1 \not{p}_2 \frac{1}{\not{p}_{30}} \\
\langle G_3 \rangle &= \tfrac{1}{4} \lim_{\varepsilon \to 0} \mathrm{Tr} \int d\mu_3 \frac{1}{\not{p}_{10}} \not{p}_1 \not{p}_3 \frac{1}{\not{p}_{30}} \\
\langle G_4 \rangle &= \tfrac{1}{4} \lim_{\varepsilon \to 0} \mathrm{Tr} \int d\mu_2 \frac{1}{\not{p}_1} \not{p}_{12}
\end{aligned}
$$

with $p_{ij} \equiv p_i - p_j$.

To proceed, we use the following properties of the measures:

$$
\begin{aligned}
\int d\mu_1 &= -\frac{1}{\varepsilon} \\
\lim_{\varepsilon \to 0} \int d\mu_n &= \binom{2n}{n} \zeta_{2n-1} \\
\int d\mu_2 \frac{p_0 \cdot p_1}{p_1^2} &= \frac{1+2\varepsilon}{2} \int d\mu_2 \\
\int d\mu_2 \frac{p_1 \cdot p_2}{p_1^2} &= \frac{1+\varepsilon}{2} \int d\mu_2
\end{aligned}
$$

These results lead to immediate evaluation of $\langle G_1 \rangle = 0$ and $\langle G_4 \rangle = 3\zeta_3$.

To complete the *experimentum crucis*, we use integration by parts on the central triangles of $\langle G_2 \rangle$ and $\langle G_3 \rangle$ Each term so generated lacks a fermion propagator. Subintegration then reduces the integrals to combinations of terms of two-loop form, each with a propagator raised to a non-integer power. This method is intrinsically D-dimensional; at $\varepsilon = 0$ separate contributions diverge. Performing the subintegrations and relabelling momenta, we obtain

$$\int d\mu_3 \frac{1}{\not{p}_{10}} \not{p}_1 \not{p}_k \frac{1}{\not{p}_{30}} = \int d\mu_2 \frac{1}{\not{p}_{10}} H_k \frac{1}{\not{p}_{20}}$$

for $k = 2, 3$, with

$$\begin{aligned}(D-3)H_2 &= \frac{\not{p}_1(\not{p}_1 + \not{p}_2)(E_{10} - E_{12})}{\varepsilon} + (\not{p}_0\not{p}_1 + \not{p}_1\not{p}_2)E_{10} \\ (D-4)H_3 &= \frac{2\not{p}_1\not{p}_2(E_{10} - E_{12})}{\varepsilon} + 2\not{p}_0\not{p}_2 E_{10}\end{aligned}$$

and $E_{ij} \equiv (p_0^2/p_{ij}^2)^\varepsilon$. Evaluation of $\langle G_3 \rangle$, from H_3, thus requires one to expand two-loop integrals to $\mathrm{O}(\varepsilon^2)$. However, we found that this did not generate ζ_5.

The final step was accomplished by developing ε-expansions of Saalschützian ${}_3F_2$ series, as in my work with Gracey and Kreimer on critical exponents. This gives $\langle G_2\rangle = 3\zeta_3$ and $\langle G_3\rangle = 6\zeta_3$ and verifies the four-term relation in its sole non-trivial appearance below five loops:

$$\langle G_1 - G_2 + G_3 - G_4\rangle = 0 - 3\zeta_3 + 6\zeta_3 - 3\zeta_3 = 0.$$

3.1 Vector couplings and vector propagators

If we replace the Yukawa coupling by γ_μ, there is no 4-term relation:

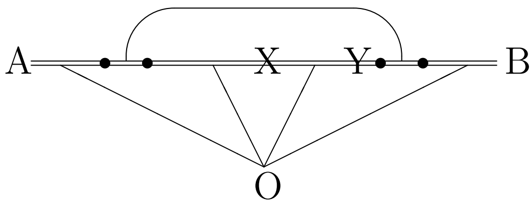
$$\begin{aligned}
\langle\tilde{G}_1\rangle &= \tfrac{1}{16}\lim_{\varepsilon\to0}\mathrm{Tr}\,\gamma^\mu\int d\mu_2\frac{1}{\not{p}_1}\not{p}_{12}\gamma_\mu\frac{1}{\not{p}_{12}}\not{p}_{02} = \frac{3\zeta_3-1}{2}\\
\langle\tilde{G}_2\rangle &= \tfrac{1}{16}\lim_{\varepsilon\to0}\mathrm{Tr}\,\gamma^\mu\int d\mu_3\frac{1}{\not{p}_{10}}\not{p}_1\not{p}_2\not{p}_3\gamma_\mu\frac{1}{\not{p}_3}\frac{1}{\not{p}_{30}} = \frac{3\zeta_3+1}{2}\\
\langle\tilde{G}_3\rangle &= \tfrac{1}{16}\lim_{\varepsilon\to0}\mathrm{Tr}\,\gamma^\mu\int d\mu_3\frac{1}{\not{p}_{10}}\not{p}_1\not{p}_2\gamma_\mu\frac{1}{\not{p}_2}\not{p}_3\frac{1}{\not{p}_{30}} = -3\zeta_3\\
\langle\tilde{G}_4\rangle &= \tfrac{1}{16}\lim_{\varepsilon\to0}\mathrm{Tr}\,\gamma^\mu\int d\mu_2\frac{1}{\not{p}_1}\not{p}_{12}\not{p}_{02}\gamma_\mu\frac{1}{\not{p}_{02}} = -\tfrac{3}{2}\zeta_3
\end{aligned}$$

Similarly, there is no four-term relation when the chord is a vector boson.

3.2 Indications of richer structure at five loops

There is a class of five-loop subdivergence-free counterterms that may be obtained from integration by parts: those whose momentum flow is that of the wheel with five spokes. Consider a putative 4-term relation, generated as follows.

To generate four terms, at five loops, connect O to each blob, in turn.



Each term is a radiative correction to a $\overline{\psi}\phi^2\psi$ coupling, induced by Yukawa couplings and a non-renormalizable ϕ^5 interaction. Thanks to ideas from John Gracey, we evaluated the terms using integration by parts for five-spoke wheels, via recurrence relations on 15 exponents of Lorentz scalars.

We found that the counterterms

$$
\begin{aligned}
\langle\overline{G}_1\rangle &= \tfrac{1}{4}\lim_{\varepsilon\to 0}\mathrm{Tr}\int d\mu_3 \not{p}_1\frac{1}{\not{p}_{30}} = -2\zeta_3\\
\langle\overline{G}_2\rangle &= \tfrac{1}{4}\lim_{\varepsilon\to 0}\mathrm{Tr}\int d\mu_4 \frac{1}{\not{p}_{10}}\not{p}_1\not{p}_2\frac{1}{\not{p}_{40}} = 4\zeta_3\\
\langle\overline{G}_3\rangle &= \tfrac{1}{4}\lim_{\varepsilon\to 0}\mathrm{Tr}\int d\mu_4 \frac{1}{\not{p}_{10}}\not{p}_1\not{p}_4\frac{1}{\not{p}_{40}} = 20\zeta_5\\
\langle\overline{G}_4\rangle &= \tfrac{1}{4}\lim_{\varepsilon\to 0}\mathrm{Tr}\int d\mu_3 \frac{1}{\not{p}_{10}}\not{p}_1 = 10\zeta_5
\end{aligned}
$$

fail to satisfy a four-term relation. This failure was the origin of proviso that four-term relations are associated with *renormalizable* field theories.

Remarkably, a four-term relation *is* obtained, if one moves the external vertex Y, on the p_4 line of $\langle\overline{G}_2\rangle$, to the p_3 line where X resides, giving

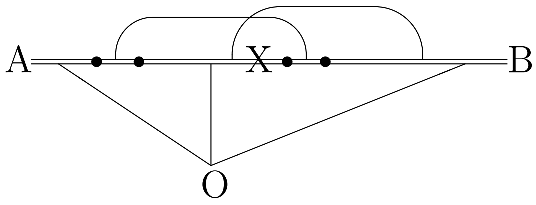
$$\langle\overline{G}_2^{\,*}\rangle = \tfrac{1}{4}\lim_{\varepsilon\to 0}\mathrm{Tr}\int d\mu_4 \frac{1}{\not{p}_{10}}\not{p}_1\not{p}_2\frac{1}{\not{p}_3}\not{p}_4\frac{1}{\not{p}_{40}} = 10\zeta_5 - 2\zeta_3$$

and hence non-trivial five-loop cancellation

$$\langle\overline{G}_1 - \overline{G}_2^{\,*} + \overline{G}_3 - \overline{G}_4\rangle = -2\zeta_3 - (10\zeta_5 - 2\zeta_3) + 20\zeta_5 - 10\zeta_5 = 0.$$

3.3 Exercise for modern quantum computation

Question for Dirk: Is each of the 4 terms indicated below free of subdivergences?



Questions for Erik, Michi, Oliver, et alia: Does Dirk's 4-term relation hold at 5 loops in Yukawa plus ϕ^4 theory? How about ϕ^3 theory in 6 dimensions?

Summary

Dirk is a skillful analyst, an inspiring combinatoricist and a deeply influential algebraic thinker. He combines all of this with a quiet self-confidence and a concern for colleagues that has enriched my life and many others. Thank you, kind friend.