New techniques for worldline integration

Christian Schubert

INSTITUTO DE FÍSICA
Y MATEMÁTICAS
U M S N H

Algebraic Structures in Perturbative Quantum Field Theory
IHES, November 19, 2020

Worldline path integral representation of scalar QED

R.P. Feynman, PR 80, (1950) 440

Green's function for the operator $-(\partial + ieA)^2 + m^2$

$$D^{xx'}[A] \equiv \langle x | \frac{1}{-(\partial + ieA)^2 + m^2} | x' \rangle$$

We work with euclidean conventions, defined by starting in Minkowski space with $(- + ++)$ and performing a Wick rotation

$$\begin{aligned} E = k^0 \quad &\rightarrow \quad ik_4 \\ t = x^0 \quad &\rightarrow \quad ix_4 \end{aligned}$$

We will also set $\hbar = c = 1$.

Worldline representation of the propagator

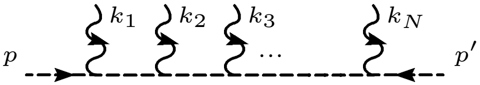
We exponentiate the denominator using a Schwinger proper-time parameter T:

$$D^{xx'}[A] = \langle x| \int_0^\infty dT \exp\Big[-T(-(\partial + ieA)^2 + m^2)\Big] |x'\rangle$$
$$= \int_0^\infty dT \; \mathrm{e}^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A(x(\tau))\right)}$$

Expanding the field in N plane waves,

$$A^\mu(x(\tau)) = \sum_{i=1}^N \varepsilon_i^\mu \, \mathrm{e}^{ik_i\cdot x(\tau)}$$

and Fourier transforming the endpoints we get the "photon-dressed propagator",



Worldline representation of the effective action

Similarly for the one-loop effective action:

$$\begin{aligned}\Gamma[A] &= -\mathrm{Tr}\ln\Big[-(\partial+ieA)^2+m^2\Big]\\ &= \int_0^\infty \frac{dT}{T}\,\mathrm{Tr}\exp\Big[-T(-(\partial+ieA)^2+m^2)\Big]\\ &= \int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int_{x(0)=x(T)}\mathcal{D}x(\tau)\,e^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2+ie\dot{x}\cdot A(x(\tau))\right)}\end{aligned}$$

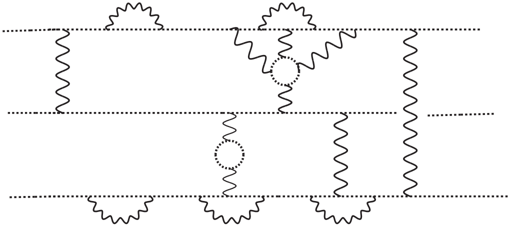
Expanding the field in N plane waves,

$$A^\mu(x(\tau))=\sum_{i=1}^N \varepsilon_i^\mu\,\mathrm{e}^{ik_i\cdot x(\tau)}$$

one gets the one-loop N - photon amplitudes.

Higher order QED processes

Arbitrary QED processes can be constructed from these building blocks by sewing:



From scalar to spinor QED

R.P. Feynman, PR 84 (1951) 108 (Spinor QED)

Add global factor of $-\frac{1}{2}$ and *spin factor* $\mathrm{Spin}[x,A]$,

$$\mathrm{Spin}[x,A] = \mathrm{tr}_\Gamma \mathcal{P} \exp\left[\frac{i}{4} e\,[\gamma^\mu,\gamma^\nu] \int_0^T d\tau F_{\mu\nu}(x(\tau))\right]$$

Modern way: Replace spin factor by a Grassmann path integral (E.S. Fradkin, NPB 76 (1966) 588)

$$\mathrm{Spin}[x,A] \to \int \mathcal{D}\psi(\tau)\,\exp\left[-\int_0^T d\tau\left(\frac{1}{2}\psi\cdot\dot\psi - ie\psi^\mu F_{\mu\nu}\psi^\nu\right)\right]$$

$$\begin{aligned}\psi(\tau_1)\psi(\tau_2) &= -\psi(\tau_2)\psi(\tau_1)\\ \psi(T) &= -\psi(0)\end{aligned}$$

Advantages of the Grassmann approach:

1. Removal of the path ordering.
2. Uncovers a supersymmetry between the orbital and spin degrees of freedom of the electron.

String-inspired treatment of the worldline path integral

Polyakov 1987, Bern and Kosower 1991, Strassler 1992:
Perturbative approach to the evaluation of worldline path integrals using worldline Green's functions $G(\tau_1,\tau_2)$, $G_F(\tau_1,\tau_2)$

$$
\begin{aligned}
\langle x^\mu(\tau_1)x^\nu(\tau_2)\rangle &= -G(\tau_1,\tau_2)\,\delta^{\mu\nu} \\
G(\tau_1,\tau_2) &= |\tau_1-\tau_2| - \frac{\Big(\tau_1-\tau_2\Big)^2}{T}
\end{aligned}
$$

$$
\begin{aligned}
\langle \psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle &= G_F(\tau_1,\tau_2)\,\delta^{\mu\nu} \\
G_F(\tau_1,\tau_2) &= \mathrm{sign}(\tau_1-\tau_2)
\end{aligned}
$$

QED photon amplitudes

Scalar QED, one loop:

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2 T} \int \mathcal{D}x \exp\left[-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ieA_\mu \dot{x}^\mu\right)\right]$$

$$\exp\left[-\int_0^T d\tau\; ieA_\mu \dot{x}^\mu\right] = \sum_{N=0}^{\infty} \frac{(-ie)^N}{N!} \prod_{i=0}^{N} \int_0^T d\tau_i \left[\dot{x}^\mu(\tau_i) A_\mu(x(\tau_i))\right]$$

Expand the field in N plane waves,

$$A^\mu(x(\tau)) = \sum_{i=1}^{N} \varepsilon_i^\mu\, \mathrm{e}^{ik_i \cdot x(\tau)}$$

and pick out the term containing every ε_i once.

$$\Gamma[\{k_i,\varepsilon_i\}] = (-ie)^N \int \frac{dT}{T}\, \mathrm{e}^{-m^2 T} \int \mathcal{D}x\, V^A_{\rm scal}[k_1,\varepsilon_1] \ldots V^A_{\rm scal}[k_N,\varepsilon_N] \mathrm{e}^{-\int_0^T d\tau \frac{\dot{x}^2}{4}}$$

$V^A_{\rm scal}$ denotes the same photon vertex operator as is used in string perturbation theory,

$$V^A_{\rm scal}[k,\varepsilon] = \int_0^T d\tau\, \varepsilon \cdot \dot{x}(\tau)\, \mathrm{e}^{ikx(\tau)}$$

The zero mode $x_0 = \frac{1}{T}\int_0^T d\tau x(\tau)$ factors out and produces the momentum conservation factor $(2\pi)^D \delta(\sum k_i)$.

Formally exponentiate $\varepsilon \cdot \dot{x}(\tau)\, \mathrm{e}^{ikx(\tau)} = \mathrm{e}^{ikx+\varepsilon\cdot\dot{x}(\tau)}\,\big|_{\mathrm{lin}(\varepsilon)}$
"Completing the square" $\Rightarrow$ Bern-Kosower master formula

$$\Gamma[\{k_i,\varepsilon_i\}] = (-ie)^N \int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i$$

$$\times \exp\Big\{\sum_{i,j=1}^N \big[\frac{1}{2} G_{Bij} k_i \cdot k_j + i\dot{G}_{Bij} k_i \cdot \varepsilon_j + \frac{1}{2}\ddot{G}_{Bij}\varepsilon_i \cdot \varepsilon_j\big]\Big\}\,\Big|_{\mathrm{lin}(\varepsilon_1,\dots,\varepsilon_N)}$$

$$G(\tau_1,\tau_2) = |\tau_1-\tau_2| - \frac{(\tau_1-\tau_2)^2}{T}$$

$$\dot{G}(\tau_1,\tau_2) = \mathrm{sign}(\tau_1-\tau_2) - 2\frac{(\tau_1-\tau_2)}{T}$$

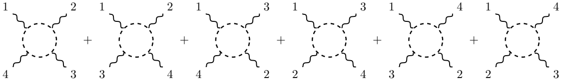
$$\ddot{G}(\tau_1,\tau_2) = 2\delta(\tau_1-\tau_2) - \frac{2}{T}$$

$(4\pi T)^{-\frac{D}{2}}$ = free path integral determinant.

Advantages

1. Highly compact generating function for the N - photon amplitudes, valid off-shell.
2. No need to fix the ordering of the photons along the loop.
3. Simple pattern-matching rules for putting on spin or color (Bern-Kosower).
4. The quartic vertices contained in the $\delta(\tau_i - \tau_j)$ can be easily removed by integration by parts, which at the same time leads to the appearance of photon field strength tensors (Strassler).

N=4 (Scalar or Spinor QED)



After a large number of integrations by parts:

$$\hat{\Gamma} = \hat{\Gamma}^{(1)} + \hat{\Gamma}^{(2)} + \hat{\Gamma}^{(3)} + \hat{\Gamma}^{(4)} + \hat{\Gamma}^{(5)}$$

$$\hat{\Gamma}^{(1)} = \hat{\Gamma}^{(1)}_{(1234)} T^{(1)}_{(1234)} + \hat{\Gamma}^{(1)}_{(1243)} T^{(1)}_{(1243)} + \hat{\Gamma}^{(1)}_{(1324)} T^{(1)}_{(1324)}$$

$$\hat{\Gamma}^{(2)} = \hat{\Gamma}^{(2)}_{(12)(34)} T^{(2)}_{(12)(34)} + \hat{\Gamma}^{(2)}_{(13)(24)} T^{(2)}_{(13)(24)} + \hat{\Gamma}^{(2)}_{(14)(23)} T^{(2)}_{(14)(23)}$$

$$\hat{\Gamma}^{(3)} = \sum_{i=1,2,3} \hat{\Gamma}^{(3)}_{(123)i} T^{(3)r_4}_{(123)i} + \sum_{i=2,3,4} \hat{\Gamma}^{(3)}_{(234)i} T^{(3)r_1}_{(234)i} + \sum_{i=3,4,1} \hat{\Gamma}^{(3)}_{(341)i} T^{(3)r_2}_{(341)i} + \sum_{i=4,1,2} \hat{\Gamma}^{(3)}_{(412)i} T^{(3)r_3}_{(412)i}$$

$$\hat{\Gamma}^{(4)} = \sum_{i<j} \hat{\Gamma}^{(4)}_{(ij)ii} T^{(4)}_{(ij)ii} + \sum_{i<j} \hat{\Gamma}^{(4)}_{(ij)jj} T^{(4)}_{(ij)jj}$$

$$\hat{\Gamma}^{(5)} = \sum_{i<j} \hat{\Gamma}^{(5)}_{(ij)ij} T^{(5)}_{(ij)ij} + \sum_{i<j} \hat{\Gamma}^{(5)}_{(ij)ji} T^{(5)}_{(ij)ji}$$

Tensor basis for the off-shell four-photon amplitudes

The basis of five tensors $T^{(i)}$ is identical with the one found in 1971 by Costantini, De Tollis and Pistoni using the QED Ward identity:

$$
\begin{aligned}
T^{(1)}_{(1234)} &\equiv Z(1234)\,, \\
T^{(2)}_{(12)(34)} &\equiv Z(12)Z(34)\,, \\
T^{(3)r_4}_{(123)i} &\equiv Z(123)\,\frac{r_4\cdot f_4\cdot k_i}{r_4\cdot k_4} \quad (i=1,2,3)\,, \\
T^{(4)}_{(12)11} &\equiv Z(12)\,\frac{k_1\cdot f_3\cdot f_4\cdot k_1}{k_3\cdot k_4}\,, \\
T^{(5)}_{(12)12} &\equiv Z(12)\,\frac{k_1\cdot f_3\cdot f_4\cdot k_2}{k_3\cdot k_4}\,.
\end{aligned}
$$

$$
\begin{aligned}
f_i^{\mu\nu} &\equiv k_i^\mu\varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu \\
Z(ij) &\equiv \frac{1}{2}\mathrm{tr}\big(f_i f_j\big) = \varepsilon_i\cdot k_j\varepsilon_j\cdot k_i - \varepsilon_i\cdot\varepsilon_j k_i\cdot k_j \\
Z(i_1 i_2\ldots i_n) &\equiv \mathrm{tr}\Big(\prod_{j=1}^{n} f_{i_j}\Big) \quad (n\geq 3)
\end{aligned}
$$

Optimized worldline parameter integrals for $N = 4$

$$\hat{\Gamma}^{(k)}_{\dots} = \int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} \, \mathrm{e}^{-m^2 T} \int_0^1 \prod_{i=1}^4 du_i \, \hat{\gamma}^{(k)}_{\dots}(\dot{G}_{ij}) \, \mathrm{e}^{T \sum_{i<j=1}^4 G_{ij} k_i \cdot k_j}$$

where, for spinor QED,

$$\begin{aligned}
\hat{\gamma}^{(1)}_{(1234)} &= \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41} - G_{F12}G_{F23}G_{F34}G_{F41} \\
\hat{\gamma}^{(2)}_{(12)(34)} &= \big(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\big)\big(\dot{G}_{34}\dot{G}_{43} - G_{F34}G_{F43}\big) \\
\hat{\gamma}^{(3)}_{(123)i} &= \big(\dot{G}_{12}\dot{G}_{23}\dot{G}_{31} - G_{F12}G_{F23}G_{F31}\big)\dot{G}_{4i} \\
\hat{\gamma}^{(4)}_{(12)11} &= \big(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\big)\dot{G}_{13}\dot{G}_{41} \\
\hat{\gamma}^{(5)}_{(12)12} &= \big(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\big)\dot{G}_{13}\dot{G}_{42}
\end{aligned}$$

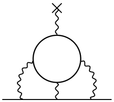
(plus permutations thereof). The coefficient functions for scalar QED are obtained from these simply by deleting all the G_{Fij}.

Work in progress

N. Ahmadiniaz, C. Lopez-Arcos, C. Lopez-Lopez and C. Schubert, in preparation:

1. First calculation of the four-photon amplitudes fully off-shell.
2. With general kinematics, as well as with one or two photons taken in the low-energy limit.
3. Tensor reduction is done on the amplitude as a whole, without fixing an ordering.

Should become useful for high-order calculations in QED with four-photon sub-diagrams,

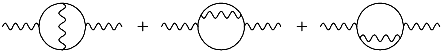


as well as for photonic processes in external fields.

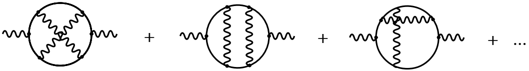
On to multiloop

Dealing with the amplitude as a whole becomes important when one uses the one-loop amplitudes to construct higher-loop amplitudes by sewing:

From the four-photon amplitude we can construct the two-loop quenched photon propagator,



From the one-loop six-photon amplitude we get the three-loop quenched propagator



etcetera

This type of sums of diagrams is known to suffer from extensive cancellations...

The sad story of the quenched QED beta function

The quenched QED beta function β was a big topic at the Multiloop Workshop at Aspen 1995 (D. Kreimer, D. Broadhurst, K. Chetyrkin, ...). The coefficients known at the time (up to four loops in spinor QED, up to three loops in scalar QED) were all rational:

$$\beta_3^{\mathrm{spin}} = -2$$

after a complete cancellation of ζ_3s between diagrams, and

$$\beta_4^{\mathrm{spin}} = -46$$

after a complete cancellation of ζ_3s and ζ_5s. And in scalar QED,

$$\beta_3^{\mathrm{scal}} = \frac{29}{2}$$

Thus everybody believed at the time that the quenched coeffcients would stay rational to all loop orders (and David, Dirk and Bob Delbourgo even had a knot-theoretical explanation.....) but then at five loops ζ_3 refused to cancel out in β_5^{spin} (Baikov et al. JHEP 1207 (2012) 017). The Lord sometimes *can* be malicious. Still, although incomplete the cancellations are remarkable and ought to find an explanation!

The fundamental problem of worldline integration

Returning to the one-loop level, using that G_{Fij}s can always be eliminated by

$$G_{Fij}G_{Fjk}G_{Fki} = -(\dot{G}_{ij} + \dot{G}_{jk} + \dot{G}_{ki})$$

the most general integral that one will ever have to compute in the worldline approach to QED (or any abelian theory) is of the form

$$\int_0^1 du_1 du_2 \cdots du_N \, \mathrm{Pol}(\dot{G}_{ij}) \, \mathrm{e}^{\sum_{i<j=1}^{N} \lambda_{ij}^2 G_{ij}}$$

with arbitrary N and polynomial $\mathrm{Pol}(\dot{G}_{ij})$, where

$$G_{ij} = |u_i - u_j| - (u_i - u_j)^2, \, \dot{G}_{ij} = \mathrm{sgn}(u_i - u_j) - 2(u_i - u_j)$$

Without decomposing the integrand into ordered sectors!

Some examples of worldline integrals

Chain integrals:

$$\int_0^1 du_2 \dots du_n \dot{G}_{12}\dot{G}_{23}\dots\dot{G}_{n(n+1)} = -\frac{2^n}{n!}B_n(|u_1-u_{n+1}|)\mathrm{sign}^n(u_1-u_{n+1})$$

$$\int_0^1 du_2 \dots du_n G_{F12}G_{F23}\dots G_{Fn(n+1)} = \frac{2^{n-1}}{(n-1)!}E_{n-1}(|u_1-u_{n+1}|)\mathrm{sign}^n(u_1-u_{n+1})$$

($B_n(x)$, $E_n(x)$ Bernoulli and Euler polynomials.)

3-point integral:

$$\int_0^1 du\, \dot{G}(u,u_1)\dot{G}(u,u_2)\dot{G}(u,u_3) = -\frac{1}{6}(\dot{G}_{12}-\dot{G}_{23})(\dot{G}_{23}-\dot{G}_{31})(\dot{G}_{31}-\dot{G}_{12})$$

General n-point integral of a polynomial in $\dot{G}$:

$$\int_0^1 du\, \dot{G}(u,u_1)^{k_1}\dot{G}(u,u_2)^{k_2}\cdots\dot{G}(u,u_n)^{k_n} = \frac{1}{2n}\sum_{i=1}^n\prod_{j\neq i}\sum_{l_j=0}^{k_j}\binom{k_j}{l_j}\dot{G}_{ij}^{k_j-l_j}\sum_{l_i=0}^{k_i}\binom{k_i}{l_i}$$

$$\times\frac{(-1)^{\sum_{j=1}^n l_j}}{(1+\sum_{j=1}^n l_j)n^{\sum_{j=1}^n l_j}}\left\{\Big(\sum_{j\neq i}\dot{G}_{ij}+1\Big)^{1+\sum_{j=1}^n l_j}-(-1)^{k_i-l_i}\Big(\sum_{j\neq i}\dot{G}_{ij}-1\Big)^{1+\sum_{j=1}^n l_j}\right\}$$

This formula settles all polynomial integrals by recursion

.

Bernoulli numbers and polynomials

Worldline integration naturally relates to the theory of Bernoulli numbers and polynomials. The path integral was performed in the Hilbert space H'_P of periodic functions orthogonal to the constant functions (because of the elimination of the zero mode). In this space the ordinary nth derivative ∂_P is invertible, and the integral kernel of the inverse is given essentially by the nth Bernoulli polynomial $B_n(x)$:

$$\begin{aligned} < u_1 \mid \partial_P^{-n} \mid u_{n+1} > &= -\frac{1}{n!} B_n(|u_1 - u_{n+1}|) \mathrm{sgn}^n(u_1 - u_{n+1}) \quad (n \geq 1) \\ \langle u_i | \partial^0 | u_j \rangle &= \delta(u_i - u_j) - 1 \end{aligned}$$

Related to the well-known Fourier series

$$B_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{k \neq 0} \frac{\mathrm{e}^{2\pi i k x}}{k^n}$$

but here it is usually assumed that $0 < x < 1$.

All + helicity N - photon amplitudes in the low-energy limit

Low-energy limit of the "all +" amplitudes in scalar or spinor QED, at one and two loops:

$$\Gamma^{(1,2)(EH)}_{\rm spin}[\varepsilon_1^+;\dots\varepsilon_K^+;\varepsilon_{K+1}^-;\dots;\varepsilon_N^-] = -\frac{\alpha\pi m^4}{8\pi^2}\Big(\frac{2e}{m^2}\Big)^N c^{(1,2)}_{\rm spin}\Big(\frac{K}{2},\frac{N-K}{2}\Big)\chi_K^+\chi_{N-K}^-$$

$$\Gamma^{(1,2)(EH)}_{\rm scal}[\varepsilon_1^+;\dots\varepsilon_K^+;\varepsilon_{K+1}^-;\dots;\varepsilon_N^-] = \frac{\alpha\pi m^4}{16\pi^2}\Big(\frac{2e}{m^2}\Big)^N c^{(2)}_{\rm scal}\Big(\frac{K}{2},\frac{N-K}{2}\Big)\chi_K^+\chi_{N-K}^-$$

$$c^{(1)}_{\rm spin}(n,0) = \frac{(-1)^{n+1}B_{2n}}{2n(2n-2)} = c^{(1)}_{\rm scal}(n,0)$$

$$c^{(2)}_{\rm spin}\Big(n,0\Big) = \frac{1}{(2\pi)^2}\Big\{\frac{2n-3}{2n-2}B_{2n-2} + 3\sum_{k=1}^{n-1}\frac{B_{2k}}{2k}\frac{B_{2n-2k}}{(2n-2k)}\Big\}$$

$$c^{(2)}_{\rm scal}\Big(n,0\Big) = \frac{1}{(2\pi)^2}\Big\{\frac{2n-3}{2n-2}B_{2n-2} + \frac{3}{2}\sum_{k=1}^{n-1}\frac{B_{2k}}{2k}\frac{B_{2n-2k}}{(2n-2k)}\Big\}$$

where $n = N/2$.

The Miki and Faber-Pandharipande-Zagier identities

Calculating the two-loop self-dual Euler-Heisenberg Lagrangian in two ways (with G.V. Dunne in 2002)

$$c_n^{(2)} = \frac{1}{(2\pi)^2}\left\{\frac{2n-3}{2n-2}\,B_{2n-2} + \frac{3}{2}\sum_{k=1}^{n-1}\frac{B_{2k}}{2k}\,\frac{B_{2n-2k}}{(2n-2k)}\right\}$$

$$c_n^{(2)} = \frac{1}{(2\pi)^2}\left\{\frac{2n-3}{2n-2}\,B_{2n-2} + 3\left[\psi(2n+1) - \frac{2}{2n-1} + \gamma - 1\right]\frac{B_{2n}}{2n} + 3\sum_{k=1}^{n-1}\binom{2n-2}{2k-2}\frac{B_{2k}}{2k}\,\frac{B_{2n-2k}}{2n-2k}\right\}$$

Equivalence? Ask Richard Stanley ....

Theorem: (Miki 1977): For integer $n \geq 2$,

$$\sum_{k=1}^{n-1}\frac{B_{2k}B_{2n-2k}}{(2k)(2n-2k)} = \sum_{k=1}^{n-1}\frac{B_{2k}B_{2n-2k}}{(2k)(2n-2k)}\binom{2n}{2k} + \frac{B_{2n}}{n}H_{2n}$$

Here H_i denotes the ith harmonic number,

$$H_i \equiv \sum_{j=1}^{i}\frac{1}{j} = \psi(i+1) + \gamma$$

$\psi(x) = \Gamma'(x)/\Gamma(x)$, γ is Euler's constant.

Similarly, from a *string theory* calculation:

<u>*Theorem:*</u> (Faber and Pandharipande 1998, proof by Zagier) For integer $n \geq 2$,

$$\sum_{k=1}^{n-1} \frac{\bar{B}_{2k}\bar{B}_{2n-2k}}{(2k)(2n-2k)} = \frac{1}{n}\sum_{k=1}^{n} \frac{B_{2k}\bar{B}_{2n-2k}}{(2k)} \binom{2n}{2k} + \frac{\bar{B}_{2n}}{n} H_{2n-1}$$

$$\text{where} \quad \bar{B}_n \equiv \left(\frac{1-2^{n-1}}{2^{n-1}}\right) B_n$$

Corresponds to a certain identity between *Hodge integrals*.

Generalizations

C.V. Dunne and C.S., CNTP 7 (2013) 225: Used the worldline formalism as a guiding principle to

1. Unify the proofs of the Miki and FPZ identities.
2. Generalized them at the quadratic level.
3. Generalized them to the cubic level.
4. Outlined the construction of even higher-order identities.

General worldline integral in ϕ^3 theory

Worldline representation of the one-loop N-point amplitude for scalar ϕ^3-theory in D dimensions (Polyakov 1987)

$$
\begin{aligned}
I_N(p_1,\dots,p_N) &= \frac{1}{2}(4\pi)^{-D/2}(2\pi)^D\,\delta\Big(\sum_{i=1}^N p_i\Big)\,\hat{I}_N(p_1,\dots,p_N)\,,\\
\hat{I}_N(p_1,\dots,p_N) &= \int_0^\infty \frac{dT}{T}\,T^{N-D/2}e^{-m^2T}\int_0^1 du_1\dots du_N \exp\left[T\sum_{i<j=1}^N G_{ij}p_{ij}\right]
\end{aligned}
$$

$G_{ij}=|u_i-u_j|-(u_i-u_j)^2$, $p_{ij}\equiv p_i\cdot p_j$. Even for a fixed ordering, off-shell these integrals are still not easy......

1. I_2 contains ${}_2F_1$
2. I_3 contains ${}_2F_1,\,F_1$
3. I_4 contains ${}_2F_1,\,F_1,\,Lauricella-Saran$

A. Davydychev, T. Riemann, O. Tarasov....

The scalar effective action

Corresponding formula for the one-loop effective action in ϕ^3 theory:

$$\mathcal{L}[\phi](x) = \frac{1}{2}\sum_{N=0}^{\infty}\frac{(-1)^N}{N!}\int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}(4\pi T)^{-D/2}\int_0^T d\tau_1\int_0^T d\tau_2\cdots\int_0^T d\tau_N$$
$$\times\exp\Bigl[-\frac{1}{2}\sum_{i,j=1}^{N}G(\tau_i,\tau_j)\partial^{(i)}\cdot\partial^{(j)}\Bigr]W^{(1)}(x)\cdots W^{(N)}(x)\,.$$

Arbitrary self-interaction $U(\phi)$, $W = U''(\phi)$. Heat kernel expansion:

$$\mathcal{L}[\phi] = \frac{1}{2}\int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}(4\pi T)^{-D/2}\sum_{n=0}^{\infty}T^n\,O_n W]$$

Basic one-loop integral

Expanding all exponentials, one would have to compute

$$I_N(n_{12}, n_{13} \ldots, n_{(N-1)N}) \equiv \int_0^1 du_1 du_2 \cdots du_N \prod_{i<j=1}^{N} G_{ij}^{n_{ij}}$$

Individually trivial, but not easy to obtain a closed-form expression....

Expansion in inverse derivatives/Bernoulli polynomials

Abbreviate $\int_{12\dots n} \equiv \int_0^1 du_1 \cdots \int_0^1 du_n$ Expand each exponential using the following identity,

$$e^{p_{ab}G_{ab}} = 1 + 2\sum_{n=1}^{\infty} p_{ab}^{n-1/2} H_{2n-1}\left(\frac{\sqrt{p_{ab}}}{2}\right) \overline{\langle u_a|\partial^{-2n}|u_b\rangle}$$

Here the $H_n(x)$ are Hermite polynomials,

$$\overline{\langle a|\partial^{-2n}|b\rangle} \equiv \langle u_a|\partial^{-2n}|u_b\rangle - \langle u_a|\partial^{-2n}|u_a\rangle$$

By integration, this also gives

$$\frac{\sqrt{\pi}}{2x}\mathrm{erf}(x)\,\mathrm{e}^{x^2} = 1 + \sum_{n=1}^{\infty} 2^{2n}\hat{B}_{2n}x^{2n-1}H_{2n-1}(x)$$

$(\hat{B}_n \equiv \frac{B_n}{n!})$. Curiously, we have not been able to find this series representation of the error function in the mathematical literature.

$N = 3$

In the three-point case one can use this to write

$$\begin{aligned} e^{p_{12}G_{12}+p_{13}G_{13}+p_{23}G_{23}} &= \left\{1+2\sum_{i=1}^{\infty} p_{12}^{i-\frac{1}{2}} H_{2i-1}\left(\frac{\sqrt{p_{12}}}{2}\right)\left[\langle u_1|\partial^{-2i}|u_2\rangle + \hat{B}_{2i}\right]\right\} \\ &\times \left\{1+2\sum_{j=1}^{\infty} p_{13}^{j-\frac{1}{2}} H_{2j-1}\left(\frac{\sqrt{p_{13}}}{2}\right)\left[\langle u_1|\partial^{-2j}|u_3\rangle + \hat{B}_{2j}\right]\right\} \\ &\times \left\{1+2\sum_{k=1}^{\infty} p_{23}^{k-\frac{1}{2}} H_{2k-1}\left(\frac{\sqrt{p_{23}}}{2}\right)\left[\langle u_2|\partial^{-2k}|u_3\rangle + \hat{B}_{2k}\right]\right\} \end{aligned}$$

Since $\int_0^1 du_{i,j}\langle u_i|\partial^{-2i}|u_j\rangle = 0$, the three $\langle u_i|\partial^{-2n}|u_j\rangle$ must go together, and then by $\int_0^1 du|u\rangle\langle u| = \mathbb{1}$,

$$\int_{123}\langle u_1|\partial^{-2i}|u_2\rangle\langle u_2|\partial^{-2k}|u_3\rangle\langle u_3|\partial^{-2j}|u_1\rangle = \mathrm{Tr}(\partial^{-2(i+j+k)}) = -\hat{B}_{2(i+j+k)}$$

$N = 3$ coefficients

In this way we get a closed form-expression for the $N = 3$ coefficients,

$$I_3(a, b, c) \equiv \int_{123} G_{12}^a G_{13}^b G_{23}^c = a!b!c! \sum_{i=\lfloor 1+a/2 \rfloor}^{a} \sum_{j=\lfloor 1+b/2 \rfloor}^{b} \sum_{k=\lfloor 1+c/2 \rfloor}^{c} h_i^a h_j^b h_k^c \left(\hat{B}_{2i} \hat{B}_{2j} \hat{B}_{2k} - \hat{B}_{2(i+j+k)} \right)$$

Here we have assumed that a, b, c are all different from zero, and the coefficients h_i^a are found from the rearrangement

$$2 \sum_{i=1}^{\infty} \lambda^{i-\frac{1}{2}} H_{2i-1} \left(\frac{\sqrt{\lambda}}{2} \right) \hat{B}_{2i} = \sum_{a=1}^{\infty} \lambda^a \sum_{i=\lfloor 1+a/2 \rfloor}^{a} h_i^a \hat{B}_{2i}$$

From the explicit formula for the Hermite polynomials

$$H_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m \frac{(2x)^{n-2m}}{m!(n-2m)!}$$

$$h_i^a = (-1)^{a+1} \frac{2(2i-1)!}{(2i-a-1)!(2a-2i+1)!}$$

$N = 4$

Starting from $N = 4$, we encounter the "cubic worldline vertex"

$$V_3^{ijk} \equiv \int_0^1 du \langle u|\partial^{-i}|u_1\rangle\langle u|\partial^{-j}|u_2\rangle\langle u|\partial^{-k}|u_3\rangle$$

which can be eliminated by integration by parts:

$$\begin{aligned}
V_3^{ijk} &\equiv \int_0^1 du \langle u|\partial^{-i}|u_1\rangle\langle u|\partial^{-j}|u_2\rangle\langle u|\partial^{-k}|u_3\rangle \\
&= \sum_{a=i}^{i+j-1} (-1)^a \binom{a-1}{i-1} \int_0^1 du \langle u|\partial^{0}|u_1\rangle\langle u|\partial^{-(i+j-a)}|u_2\rangle\langle u|\partial^{-(k+a)}|u_3\rangle \\
&\quad + \{i \leftrightarrow j, u_1 \leftrightarrow u_2\} \\
&= \sum_{a=i}^{i+j-1} (-1)^a \binom{a-1}{i-1} \Big[\langle u_1|\partial^{-(i+j-a)}|u_2\rangle\langle u_1|\partial^{-(k+a)}|u_3\rangle \\
&\quad -(-1)^{i+j-a} \int_0^1 du \langle u_2|\partial^{-(i+j-a)}|u\rangle\langle u|\partial^{-(k+a)}|u_3\rangle\Big] + \{i \leftrightarrow j, u_1 \leftrightarrow u_2\} \\
&= \sum_{a=i}^{i+j-1} (-1)^a \binom{a-1}{i-1} \Big[\langle u_1|\partial^{-(i+j-a)}|u_2\rangle\langle u_1|\partial^{-(k+a)}|u_3\rangle - (-1)^{i+j-a}\langle u_2|\partial^{-(i+j+k)}|u_3\rangle\Big] \\
&\quad + \sum_{a=j}^{i+j-1} (-1)^a \binom{a-1}{j-1} \Big[\langle u_2|\partial^{-(i+j-a)}|u_1\rangle\langle u_2|\partial^{-(k+a)}|u_3\rangle - (-1)^{i+j-a}\langle u_1|\partial^{-(i+j+k)}|u_3\rangle\Big]
\end{aligned}$$

Outlook

1. Will get closed formulas for the coefficients $I_N(n_{12},\ldots)$ for moderate N.
2. How to make contact with the description in terms of hypergeometric functions?
3. Might be useful for asymptotic purposes.. for large n,

$$B_{2n} \quad \sim \quad (-1)^{n+1} 2 \frac{(2n)!}{(2\pi)^{2n}}$$

4. Generalize to gauge theory....
5. Get back to multiloop....