On a theorem of Kreimer

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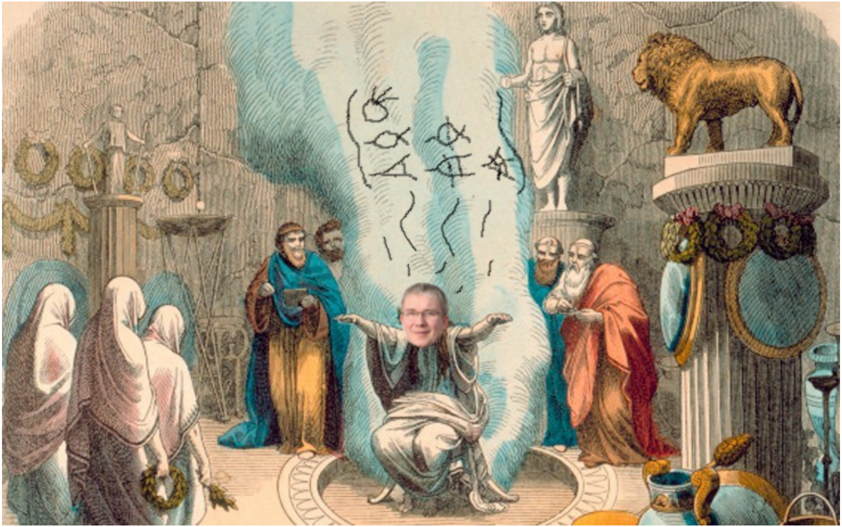
The Kreimer gang

On a *vision* of Kreimer:

Cutkosky rules in Outer space

arXiv:1512.01705 (Bloch, Kreimer), 1607.04861 (Kreimer) & 2008.09540 (Kreimer, MB)

A visionary at work

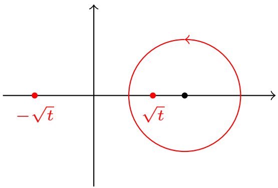


Baby example

Consider

$$f(t) = \int_\gamma \frac{dx}{x^2 - t}$$

where γ is a small circle around $1 \in \mathbb{C}$.



f is holomorphic at $t = 1$, can be analytically continued to $\mathbb{C} \setminus \{0\}$.

Feynman rules

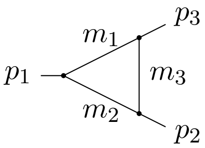
Momentum-space Feynman rules associate to a graph G on N edges, l loops and s legs the integral

$$I_G := \int_{(\mathbb{M}^d)^l} dk \prod_{i=1}^{N} \frac{1}{D_i}$$

where

- $D_i := q_i^2 - m_i^2 + i\epsilon$, the q_i being linear combinations of *loop momenta* $k_1, \ldots, k_l$ and the *external momenta* $p_1, \ldots, p_s$ (determined by momentum conservation),
- $\mathbb{M}^d$ is d-dimensional Minkowski space.

Example

Let $d = 4$ and $G =$ 

. Then the function

$I_G = I_G(p_1, p_2, p_3, m_1, m_2, m_3)$ is given by

$$\int_{\mathbb{M}^4} \frac{d^4k}{\left(k^2 - m_1^2 + i\epsilon\right)\left((k+p_1)^2 - m_2^2 + i\epsilon\right)\left((k+p_1+p_2)^2 - m_3^2 + i\epsilon\right)}.$$

Analytic structure of Feynman integrals

Problem

Understand the analytic structure of I_G.

- Pham (and many others) give a nice mathematical account of this problem, *almost* covering the case of Feynman integrals.
- Landau ('60) and Mühlbauer (this week!) formulate a necessary condition for such singularities to occur:

$$\forall i \in \{1, \ldots, N\} : x_i D_i = 0,$$
$$\forall j \in \{1, \ldots, l\} : \sum_{i \in E_{\mathrm{loop}(j)}} x_i q_i = 0.$$

A solution (in p-space) where all $D_i = 0$ is called a *leading* singularity, all others are referred to as *reduced* singularities of G, or I_G.

Analytic structure of Feynman integrals

If we knew the types of these singularities and the discontinuities along the associated branch cuts, we could, *in principle*, construct the function I_G from this data (Hilbert transform).

Conjecture (Cutkosky '60)

The discontinuity of I_G with respect to the Landau singularity associated to $D_1 = \ldots = D_k = 0$ is given by

$$\mathrm{Disc}(I_G) = \int_{(\mathbb{M}^d)^l} \prod_{i=1}^{k} \delta^+(D_i) \prod_{j=k+1}^{N} \frac{1}{D_j}$$

where $\delta^+(q^2 - m^2) := \theta(q_0)\delta\big(q_0 - \sqrt{\vec{q}^2 + m^2}\big)\frac{1}{2\sqrt{\vec{q}^2+m^2}}$.

Analytic structure of Feynman integrals

An unfinished proof can be found in arXiv:1512.01705 (Bloch, Kreimer). Alternative approach: "Regroup" Feynman integrals.

Theorem (Kreimer, MB)

The (unrenormalized) Feynman integral I_G can be written as a sum of cut integrals associated to spanning trees of G,

$$I_G = \sum_{T \in \mathcal{T}(G)} I_{G,T}$$

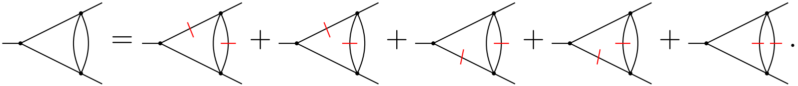
with

$$I_{G,T} := \int \prod_{e \in T} \frac{1}{D_e} \prod_{e' \notin T} \delta^+(D_{e'})$$

the original integral where all edges not in T have been cut.

Example

1. Diagrammatically,



2. In terms of integrals,

$$
\begin{aligned}
I = {} & \pi i \int_{\mathbb{R}^3} d^3\vec{k}\, \frac{1}{k_0} \frac{1}{(k-p)^2 - m_2^2 + i\epsilon}\bigg|_{k_0=\sqrt{\vec{k}^2+m_1^2-i\epsilon}} \\
& + \frac{1}{k_0 - p_0} \frac{1}{k^2 - m_1^2 + i\epsilon}\bigg|_{k_0=p_0+\sqrt{(\vec{k}-\vec{p})^2+m_2^2-i\epsilon}}.
\end{aligned}
$$

An alternative point of view

Let's go to Outer space!

Parametric Feynman rules

Using the Schwinger trick we can rewrite a Feynman integral as

$$I_G := \int_{\Delta_G} \omega_G$$

where $\Delta_G := \{[x_1 : \ldots : x_N] \in \mathbb{P}(\mathbb{R}^N) \mid x_i \geq 0\} \cong \Delta^{N-1}$ and

$$\omega_G := \psi_G^{-\frac{d}{2}} \Theta_G^{N-h_1(G)\frac{d}{2}} \sum_{i=1}^{N} (-1)^i x_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_N$$

with ψ_G and Θ_G two *graph polynomials*, Θ_G depending on the masses $m_1, \ldots, m_N$ and momenta $p_1, \ldots, p_s$.

Parametric Feynman rules

Crucial identities

1. $\delta_{x_e}[\omega_G] = \omega_{G/e}$ if e is not a self-loop,
2. $\mathrm{Res}_{\{x_e=0|e\in E_\gamma\}}[\omega_G] = \omega_\gamma \otimes \omega_{G/\gamma}$ if γ is *divergent*.

(and similar for Δ_G and its blow-up/compactification)

The first one allows to relate (reduced) Landau singularities (of the first type) of different Feynman integrals, the second one is fundamental for *renormalization*.

A moduli space of colored graphs

Suppose we are given only a finite set C of masses to "color" our Feynman graphs with, possibly with further restrictions on the coloring maps $c : E_G \to C$.

Definition

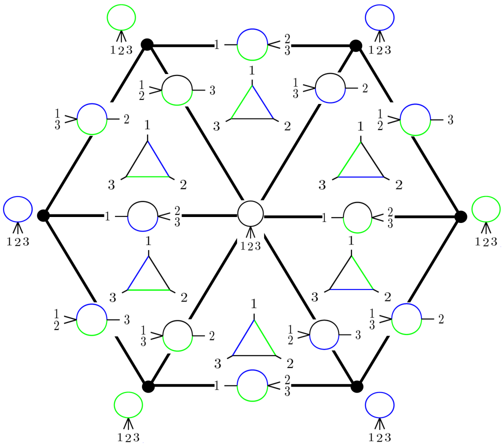
The *moduli space of (metric) colored graphs with* l *loops and* s *legs* is defined as

$$\mathcal{MG}^C_{l,s} := \Big(\bigcup_{G \in \mathbb{G}^C_{l,s}} \dot{\Delta}_G \Big)_{/\sim},$$

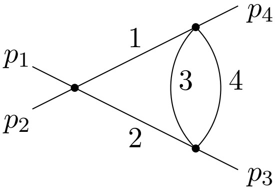
where $\mathbb{G}^C_{l,s}$ is the set of all 1PI Feynman diagrams with all vertices at least three-valent, internal edges colored by C, and $\sim$ is induced by edge collapses (and graph isomorphisms).

Example

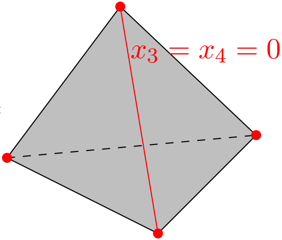
$\mathcal{MG}_{1,3}^{\{1,2,3\},\mathrm{inj}}$ looks like

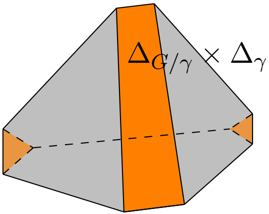


Example

$G =$ 

represents a “cell” σ_G in the moduli space $\mathcal{MG}_{2,4}^{\{1,2,3,4\},\mathsf{inj}}$:

$\sigma_G =$ 

$\widetilde{\sigma}_G =$ 

Feynman integrals on $\mathcal{MG}_{l,s}^{C}$

Recall $I_G = \int_{\sigma_G} \omega_G$.

We see

- the integration domain σ_G is a cell in $\mathcal{MG}_{l,s}^{C}$,
- the integrand ω_G is a (compactly supported) distribution density on $\mathcal{MG}_{l,s}^{C}$.

This allows to

- formulate amplitudes as "semi-discrete" volumes of $\mathcal{MG}_{l,s}^{C}$,
- study renormalization on a Borel-Serre compactification $\widetilde{\mathcal{MG}}_{l,s}^{C}$ of $\mathcal{MG}_{l,s}^{C}$.

The topology of $\mathcal{MG}^C_{l,s}$

Theorem (Mühlbauer, MB)

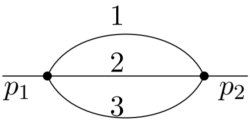
1. $H_{s-1}(\mathcal{MG}^{\{1,\dots,s\},\textit{inj}}_{1,s};\mathbb{Z}) \cong \mathbb{Z}^{\frac{(s-1)!}{2}}$.
2. *The Betti numbers* $h_{s-1}(\mathcal{MG}^{\{1,\dots,m\}}_{1,s})$ *grow polynomially of degree* s *with the number of colors* m*.*
3. *For all* $i < s-1$ *we have* $h_i(\mathcal{MG}^{\{1,\dots,m\}}_{1,s}) = h_i(\mathcal{MG}_{1,s})$*.*

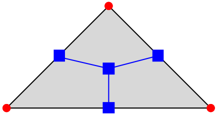
There are many interesting maps between these moduli spaces changing the number of colors, legs or loop numbers. Can these be used to study $H_*(\mathcal{MG}_{l,s};\mathbb{Q}) \cong H_*(\Gamma_{l,s};\mathbb{Q})$? (here $\Gamma_{l,0} = \mathrm{Out}(F_l)$, $\Gamma_{l,1} = \mathrm{Aut}(F_l)$, ...)

Back to Feynman integrals

Our theorem "$I_G = \sum_{T \in \mathcal{T}(G)} I_{G,T}$", as well as other instances of loop-tree duality, (could) have a nice reformulation:

The space $\mathcal{MG}_{l,s}^{C}$ is of dimension $3l - 4 + s$. It deformation retracts onto a subspace of dimension $2l - 3 + s$ that has a natural decomposition into cubes (cf. sector decomposition). Each cell σ_G retracts onto a union of cubes, indexed by pairs (G, T), $T \in \mathcal{T}(G)$.



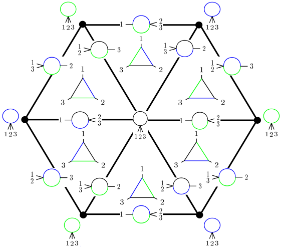


Loop-tree duality appears as the result of fiber integration!

Singularities of Feynman amplitudes

The natural cell decomposition of $\mathcal{MG}_{l,s}^{C}$ encodes relations between "neighboring" Feynman integrals.

Let us consider a theory with only cubic interactions (all graphs three-regular). Then the l-loop amplitude is the integral of a $3l-4+s$-form on $\mathcal{MG}_{l,s}^{C}$ (or a $2l-3+s$-form on the associated cube complex).
That is, we consider only contributions from the top-dimensional cells of $\mathcal{MG}_{l,s}^{C}$.



Singularities of Feynman amplitudes

However, the lower dimensional cells still carry information:

We can map the boundary strata of a cell σ_G to poles of the integrand of I_G (or the associated Landau varieties),

$$\rho : G \supset \gamma \longmapsto L_{G/\gamma}$$

where

$$L_{G/\gamma} := \bigcup_{I \subset E_G \setminus E_\gamma} \big\{\text{Sol. of Landau's eq. for } \{D_i = 0\}\big\}.$$

This is a map of posets between the set of subgraphs of G, ordered by reverse inclusion, and the singularities of I_G.
Extending ρ to $\mathbb{G}^C_{l,s}$ allows to encode if and where two Feynman integrals I_G and $I_{G'}$ have singularities in common.

A graph complex

Definition

Let $GC^C_{l,s} := \mathbb{Z}_2\langle G \mid G \in \mathbb{G}^C_{l,s}\rangle$, graded by #edges -1, and equip it with a differential d defined by

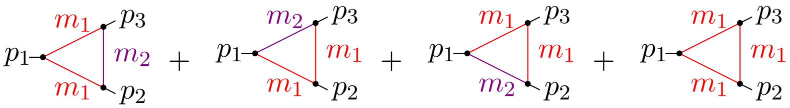
$$d(G,c) := \sum_{e \in E_G} (G/e, c_e)$$

where $c_e := c\big|_{E_G \setminus \{e\}}$. If e is a tadpole, set $G/e = 0$.

Experimental observation: The top rank homology classes seem to partition the set of graphs that contribute to the l loop and s legs amplitude into "nice" subamplitudes.

Example

The element



represent a class in $H_2(GC_{1,3}^{\{1,2\}}) \cong \mathbb{Z}_2^2$. The function $I_{G_1+\ldots+G_4}$ has (reduced) singularities along

$$\{p_i^2 = 4m_1^2, p_i^2 = 0, p_i^2 = (m_1 \pm m_2)^2 \mid i = 1, 2, 3\}.$$

The other class and its set of singularities is given by the same expressions, with m_1 and m_2 interchanged.
So, the full amplitude splits into $\mathcal{A}_{l,s}(p) = I_1(p) + I_2(p)$ with both summands (and their singularities) related by a S_2-symmetry.

Theorem (Kreimer, MB)

The "partition property" holds for all $H_{s-1}(GC_{1,s}^{\{1,\dots,s\},inj})$, $s \geq 3$.

Conjecture (Kreimer, MB)

The "partition property" holds for all $H_{s-1}(GC_{1,s}^{\{1,\dots,m\}})$, $s \geq 3$ and $m \geq 2$.

For $l > 1$ we know nothing about the homology of this complex, might have to switch to the cubical world

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- of course, in total agreement with Dirk's prophecy!

Thank you for your attention
and, once again, happy birthday, Dirk!