Generalized Gross-Neveu universality class with non-abelian symmetry

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Overview

Introduction

One area of interest in quantum field theory concerns the description of phase transitions which can be in particle physics or in material sciences

In general, dynamics of electron transport in materials are described by spin models on discrete lattices

The phase transitions that such materials undergo can be described by continuum quantum field theories with the same symmetry properties

One example is in the area of graphene which is a sheet of carbon atoms

Electrons are located at the corners of a honeycomb or hexagonal lattice and can be studied with Monte Carlo or lattice methods

When the sheet is stretched it can undergo a quantum phase change from a conductor to a Mott-insulating phase for instance

The properties of the phase transition are described by Gross-Neveu-Yukawa interactions with or without QED

Equally the Mott transition could be a mimic of spontaneous symmetry breaking in the Standard Model

There is also a connection with continuum field theories whose Wilson-Fisher fixed points describe various phase transitions in graphene

The focus here will be on the Gross-Neveu model but with variations in the interaction

Specifically the case where the core interaction will have a non-abelian symmetry structure

Recently such symmetries have arisen in a new spin model context where there is a fractionalized Gross-Neveu model with a novel spectrum

Renormalization and phase transitions

In the continuum quantum field theory underlying a second order phase transition the renormalization group equation plays a key role

Phase transitions correspond to fixed points in the renormalization group flow being defined by

$$\beta(g^*) ~=~ 0$$

where g^* is the critical coupling and similarly for multi-coupling theories

The trivial solution at $g^* = 0$ is known as the Gaussian fixed point

One important non-trivial example is the Wilson-Fisher fixed point which is present in the d-dimensional theory

If

$$\beta(g) ~=~ -~\epsilon g ~+~ a g^2 ~+~ b g^3 ~+~ O(g^4)$$

where $d = D_c - 2\epsilon$ and D_c is the critical dimension of the field theory then the fixed point is at

$$g^* ~=~ -~\frac{\epsilon}{a} ~+~ O(\epsilon^2)$$

For $SU_f(N)$ Gross-Neveu

$$g^* = - \frac{\epsilon}{2(N-1)} + O(\epsilon^2)$$

which illustrates the dependence on the symmetry group parameters

At either type of fixed point the evaluation of the renormalization group functions leads to observables termed critical exponents

For example $\eta = \gamma_\psi(g^*)$ and $\omega = \beta'(g^*)$ and these will depend on d and any parameters such as colour group Casimirs

They define the properties of the universal quantum field theory which describes the Wilson-Fisher fixed point in *all* spacetime dimensions

They can be computed up to an approximation using methods such as large N, functional renormalization group, d-dimensional conformal field theory and matched Padé approximants based on ϵ expansion

Main condensed matter interest concerns the three dimensional theory

The aim is to compute the large N critical exponents of this more general non-abelian Gross-Neveu model at three orders in d-dimensions

A secondary motivation is to gain insight into the potential group Casimir structure of the renormalization group invariant exponents in the universal theory

This ought to give a flavour of what to expect for similar large N exponents in a non-abelian gauge theory

Basic Lagrangian

The Gross-Neveu (GN) model is a renormalizable quantum field theory in two dimensions based on a 4-fermi interaction

The original Ising Gross-Neveu Lagrangian is

$$L^{\text{GN}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i - m\bar{\psi}^i\psi^i + \frac{g^2}{2}\left(\bar{\psi}^i\psi^i\right)^2$$

where $1 \leq i \leq N$ for the $SU_f(N)$ flavour symmetry or it can be rewritten using an auxiliary field

$$L^{\text{GN}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i - m\bar{\psi}^i\psi^i + g\sigma\bar{\psi}^i\psi^i - \frac{1}{2}\sigma^2$$

In the underlying critical universal theory the σ field becomes dynamical and moreover corresponds to the bound state of two fermions that is known to be part of the spectrum in the true two dimensional theory

This version allows one to connect with four dimensional Gross-Neveu-Yukawa (GNY) models through ultraviolet completion [Zinn-Justin]

Extension of Gross-Neveu model

The phase transition of the recent investigation into the fractionalized Gross-Neveu model is driven by a variation on the original Gross-Neveu model called the chiral Heisenberg-Gross-Neveu (cHGN) given by

$$L^{\text{cHGN}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i - m\bar{\psi}^i\psi^i + \frac{g^2}{2}\left(\bar{\psi}^i\lambda^a\psi^i\right)^2$$

or

$$L^{\text{cHGN}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i - m\bar{\psi}^i\psi^i + g\pi^a\bar{\psi}^i\lambda^a\psi^i - \frac{1}{2}\pi^a\pi^a$$

The ultraviolet completion to four dimensions is called the chiral Heisenberg-Gross-Neveu (cHGNY) model with Lagrangian

$$L^{\text{cHGNY}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i - m\bar{\psi}^i\psi^i + \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a + \frac{1}{2}g_1\pi^a\bar{\psi}^i\lambda^a\psi^i + \frac{1}{24}g_2^2\left(\pi^a\pi^a\right)^2$$

For the fractionalized Gross-Neveu model the λ^a correspond to $SO(3)$ matrices that satisfy

$$\lambda^a_{bc}\lambda^a_{de} ~=~ \delta_{be}\delta_{cd} ~-~ \delta_{bd}\delta_{ce}$$

The chiral Heisenberg Gross-Neveu model was also the basis for the phase transition in graphene from a conductor to a Mott-insulator and large N exponents were computed to $O(1/N^3)$

In that case the symmetry group was $SU(2)$ and λ^a corresponded to the Pauli matrices

Can generalize the $SU(3)$ computations to the case where λ^a are the group generators of a non-abelian Lie group T^a so that, for example,

$$T^a T^a ~=~ C_F ~~~,~~~ f^{acd} f^{bcd} ~=~ C_A \delta^{ab}$$

Then the Mott-insulator transition corresponds to $SU(2)$ in the fundamental and the fractionalized Gross-Neveu is adjoint $SO(3)$

Large N expansion of generalized non-abelian GN model

Large N critical point method of Vasil'ev et al determines the critical exponents of the universality class as a function of $d = 2\mu$

Key ingredient is the universal interaction which is common in all the theories of the universality class whose critical dimension is $2n$ where n is an integer

The ϵ expansion of the large N critical exponents are in complete agreement with the ϵ expansion of each theory in the tower where $d = 2n - 2\epsilon$

For the generalized non-abelian Gross-Neveu universality class the critical point Lagrangian is

$$L = i\bar{\psi}^i \partial\!\!\!/ \psi^i + \pi^a \bar{\psi}^i T^a \psi^i + f(\pi^a)$$

where the fermion kinetic term and the interaction define the canonical dimensions of the fields where $f(\pi^a)$ corresponds to the spectator sector

In coordinate space the propagators in the asymptotic limit to the fixed point take the scaling forms, $(d = 2\mu)$,

$$\psi(x) \sim \frac{A\not{x}}{(x^2)^\alpha}\left[1 + A'(x^2)^\lambda\right] \quad , \quad \pi(x) \sim \frac{C}{(x^2)^\gamma}\left[1 + C'(x^2)^\lambda\right]$$

where corrections to scaling are included and

$$\alpha = \mu + \tfrac{1}{2}\eta \quad , \quad \gamma = 1 - \eta - \chi_\pi$$

which imply

$$2\alpha + \gamma = 2\mu + 1 - \chi_\pi$$

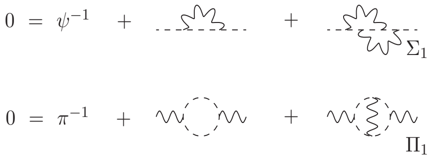
The anomalous dimension of ψ is η and χ_π is the vertex anomalous dimension and have expansions

$$\eta(\mu) = \sum_{n=1}^{\infty} \frac{\eta_n(\mu)}{N^n} \quad ,$$

Explicit expressions for η and χ_π can be found by algebraically solving the skeleton Schwinger-Dyson equations in the approach to criticality

Graphs

Graphs for to determine η_1 and η_2 are



No dressing of internal lines since the inclusion of the anomalous dimension in the propagator exponent would overcount those contributions

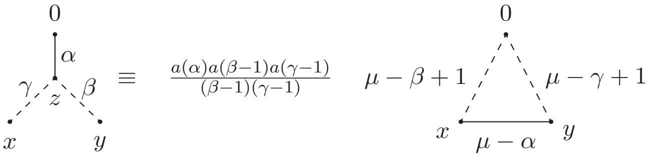
Two loop graphs are *analytically* regularized by Δ which is introduced by

$$\chi_\pi \;\rightarrow\; \chi_\pi \;+\; \Delta$$

Two loop and other graphs are computed using conformal integration or uniqueness methods

Uniqueness

In coordinate space when the sum of the exponents at a 3-point vertex sum to $d + 1$ then the integration over z can be carried out



where $\alpha + \beta + \gamma = 2\mu + 1$ and

$$a(\alpha) \;=\; \frac{\Gamma(\mu - \alpha)}{\Gamma(\alpha)}$$

There is a similar rule for a purely scalar vertex but does not extend to the quark-gluon vertex of QCD

Leading order results

From the first few orders of 2-point functions in $1/N$ find

$$\eta_1 = -\frac{2\Gamma(2\mu-1)C_F}{\mu\Gamma(1-\mu)\Gamma(\mu-1)\Gamma^2(\mu)T_F} \quad , \quad \chi_{\pi\,1} = \frac{(2C_F - C_A)\mu}{2(\mu-1)C_F}\eta_1$$

$$\eta_2 = \left[\frac{(2\mu-1)C_F}{(\mu-1)}\Psi(\mu) - \frac{\mu C_A}{2(\mu-1)}\Psi(\mu) + \frac{(4\mu-1)(2\mu-1)C_F}{2\mu(\mu-1)^2} - \frac{3\mu C_A}{2(\mu-1)^2}\right]\frac{\eta_1^2}{C_F}$$

where

$$\Psi(\mu) = \psi(2\mu-1) - \psi(1) + \psi(2-\mu) - \psi(\mu)$$

The vertex anomalous dimension is determined after renormalization at $O(1/N^2)$ from ensuring that there are no $\ln(x^2)$ terms to preserve the scaling behaviour in the critical limit

The exponent $1/\nu$ is determined from the correction to scaling part of the asymptotic propagators by setting $1/\nu = 2\lambda$ where $\lambda_0 = \mu - 1$

This requires the corrections to the inverse propagator scaling functions where are determined from the inverse Fourier transform of the momentum space 2-point function scaling form

Explicitly we have

$$\psi^{-1}(x) \sim \frac{r(\alpha-1)\not{x}}{A(x^2)^{2\mu-\alpha+1}}\left[1 - A's(\alpha-1)(x^2)^{\lambda}\right]$$
$$\pi^{-1}(x) \sim \frac{p(\gamma)}{C(x^2)^{2\mu-\gamma}}\left[1 - C'q(\gamma)(x^2)^{\lambda}\right]$$

and

$$a(\alpha) = \frac{\Gamma(\mu-\alpha)}{\Gamma(\alpha)} \quad , \quad p(\gamma) = \frac{a(\gamma-\mu)}{a(\gamma)} \quad , \quad r(\alpha) = \frac{\alpha p(\alpha)}{(\mu-\alpha)}$$
$$q(\gamma) = \frac{a(\gamma-\mu+\lambda)a(\gamma-\lambda)}{a(\gamma-\mu)a(\gamma)} \quad , \quad s(\alpha) = \frac{\alpha(\alpha-\mu)q(\alpha)}{(\alpha-\mu+\lambda)(\alpha-\lambda)}$$

With the exponents

$$\alpha ~=~ \mu ~+~ \frac{1}{2}\eta \quad , \quad \gamma ~=~ 1 ~-~ \eta ~-~ \chi_\pi \quad , \quad \lambda ~=~ \mu ~-~ 1 ~+~ \tilde{\lambda}$$

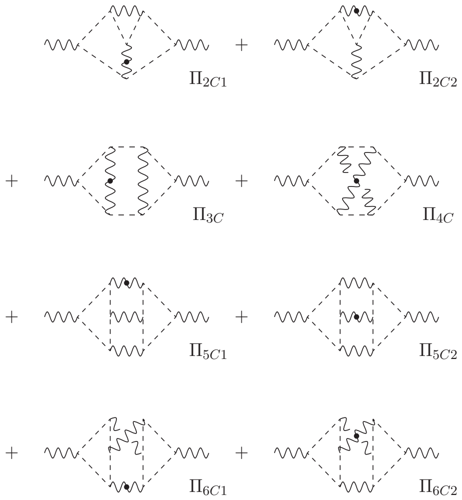
where $\tilde{\lambda}$ is $O(1/N)$ then

$$q(\gamma) ~=~ \frac{a(\gamma-\mu+\lambda)a(\gamma-\lambda)}{a(\gamma-\mu)a(\gamma)}$$

is $O(1/N)$ since

$$a(\alpha) ~=~ \frac{\Gamma(\mu-\alpha)}{\Gamma(\alpha)}$$

This means there is a reordering within the algebraic solution of the skeleton Dyson-Schwinger equations governing the correction to scaling so that higher order diagrams are needed to find λ_2


Π_{2C1}
Π_{2C2}
Π_{3C}
Π_{4C}
Π_{5C1}
Π_{5C2}
Π_{6C1}
Π_{6C2}

The higher order corrections contain light-by-light graphs which will give group factors like

$$\mathrm{Tr}\left(T^a T^c T^d T^e\right)\mathrm{Tr}\left(T^b T^e T^c T^d\right)$$

which are proportional to δ^{ab}

Have used the `color.h` routine of FORM to rationalize such group factors into a basis of Casimirs

In addition to C_A and C_F the rank 4 fully symmetric tensor

$$d_F^{abcd} = \frac{1}{6}\mathrm{Tr}\left(T^a T^{(b} T^c T^{d)}\right)$$

will appear but the rank 3 symmetric tensor

$$d^{abc} = \frac{1}{2}\mathrm{Tr}\left(T^a \{T^b, T^c\}\right)$$

is absent

Solving the corrections to the Schwinger-Dyson equation in the critical limit using the values for the master integrals at $O(1/N^2)$ gives

$$\begin{aligned}
\lambda_2 \quad = \quad & \left[\left[\frac{\mu(3\mu^2-6\mu+2)C_A^2C_F}{16T_F} + 4\mu\frac{d_F^{abcd}d_F^{abcd}C_F}{T_F^3N_A}\right]\frac{1}{(\mu-1)(\mu-2)^2\eta_1}\right. \\
& - \left[\frac{1}{24}\mu^2(2\mu-3)C_A^2 + 2\mu^2(2\mu-3)\frac{d_F^{abcd}d_F^{abcd}}{T_F^2N_A}\right]\frac{[\Psi^2(\mu)+\Phi(\mu)]}{(\mu-1)(\mu-2)} \\
& + \left[-(2\mu-1)^2(\mu+1)(\mu-1)(\mu-2)^2C_F^2\right. \\
& \quad + \mu(2\mu-1)(\mu-1)(\mu-2)^2C_FC_A \\
& \quad + \frac{1}{24}\mu^2(\mu-1)(6\mu^2-21\mu+20)C_A^2 \\
& \quad \left. - \mu^2(3\mu-5)(2\mu-5)\frac{d_F^{abcd}d_F^{abcd}}{T_F^2N_A}\right]\frac{\Psi(\mu)}{(\mu-1)^2(\mu-2)^2} \quad + \quad \ldots
\end{aligned}$$

$$
\begin{aligned}
&+ \left[-\frac{3}{2}\mu^2(2\mu+1)(\mu-2)C_F^2 + \frac{3}{4}\mu^2(2\mu+5)(\mu-2)C_F C_A \right. \\
&\quad \left. - \frac{11}{8}\mu^2(\mu-2)C_A^2 + 3\mu^2(5\mu-7)\frac{d_F^{abcd}d_F^{abcd}}{T_F^2 N_A} \right] \frac{\Theta(\mu)}{(\mu-1)(\mu-2)} \\
&+ \frac{3\mu(2\mu-1)}{4(\mu-1)^2} C_F C_A + \frac{(2\mu-1)^2(2\mu^3-4\mu^2-2\mu+1)}{2\mu(\mu-1)^2} C_F^2 \\
&- \frac{\mu^2(8\mu^4-42\mu^3+85\mu^2-75\mu+20)}{48(\mu-1)^3(\mu-2)^2} C_A^2 \\
&- \frac{\mu^2(4\mu^4-18\mu^3+26\mu^2-15\mu+7)}{2(\mu-1)^3(\mu-2)^2} \frac{d_F^{abcd}d_F^{abcd}}{T_F^2 N_A} \Bigg] \frac{\eta_1^2}{C_F^2}
\end{aligned}
$$

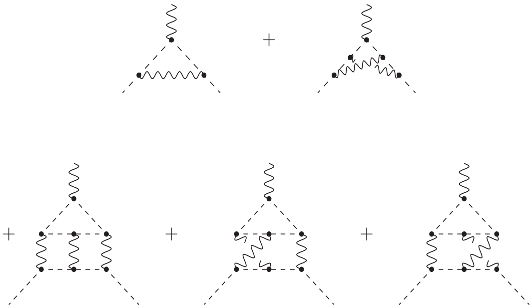
where

$$
\begin{aligned}
\Theta(\mu) &= \psi'(\mu) - \psi'(1) \\
\Phi(\mu) &= \psi'(2\mu-1) - \psi'(2-\mu) - \psi'(\mu) + \psi'(1)
\end{aligned}
$$

η_3

To proceed to next order need to use another technique which is the large N conformal bootstrap of Vasil'ev et al motivated by early work by Parisi et al

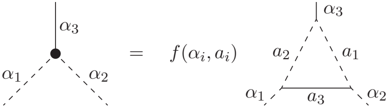
Same asymptotic scaling forms of the propagators are used but the 3-point function Schwinger-Dyson equation of primitive diagrams is solved



where each vertex is replaced by a conformal triangle

Conformal triangle

The conformal triangle vertex is



with all internal vertices unique and

$$\begin{aligned} a_1 + a_2 + \alpha_3 &= 2\mu + 1 \\ a_2 + a_3 + \alpha_1 &= 2\mu + 1 \\ a_3 + a_1 + \alpha_2 &= 2\mu + 1 \end{aligned}$$

where $f(\alpha_i, a_i)$ is the value of the vertex itself

Conformal transformation is

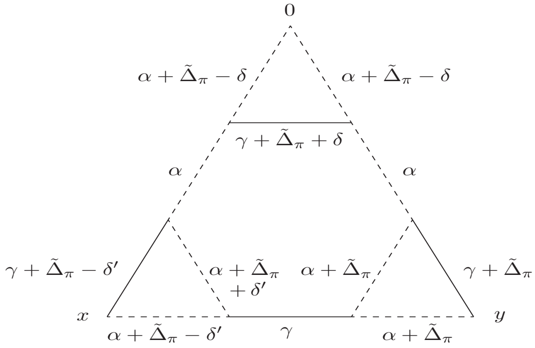
$$x_\mu \rightarrow \frac{x_\mu}{x^2}$$

which implies

$$(\not{x} - \not{y}) \rightarrow -\frac{\not{y}(\not{x} - \not{y})\not{x}}{x^2 y^2} = -\frac{\not{x}(\not{x} - \not{y})\not{y}}{x^2 y^2}$$

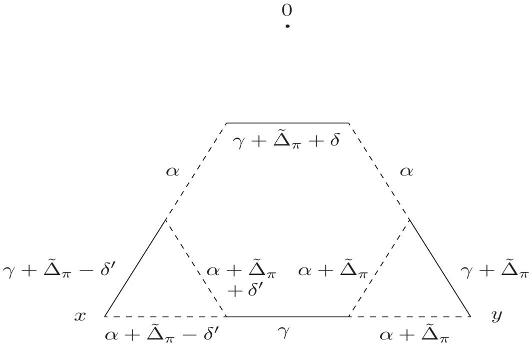
To determine η at $O(1/N^3)$ requires not only the evaluation of the two and higher loop graphs but also the expansion of the one loop ones to the same order

Regularization is required by shifting π^a and one ψ^i external leg dimension

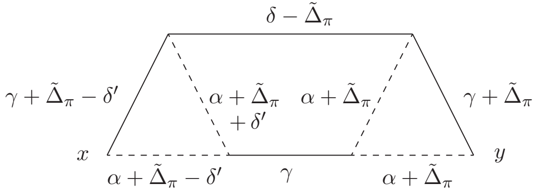


where δ and δ' are the regulators and $\chi_\pi = 2\tilde{\Delta}_\pi$

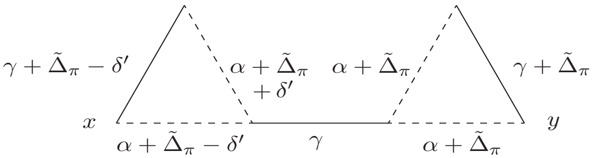
Applying the conformal transformation to the regularized 3-point vertex diagrams with conformal triangles as vertices produces



This reduces to the diagram to a d-dimensional 2-point function



which equates to



$\times \quad f_A(\delta - \tilde{\Delta}_\pi)$

where the graphs are simple to evaluate

The function $f_A(\epsilon - \tilde{\Delta}_\pi)$ can be written as

$$f_A(\delta - \tilde{\Delta}_\pi) = \exp\Big[x_1(\delta - \tilde{\Delta}_\pi) + x_2(\delta - \tilde{\Delta}_\pi)^2 + x_3(\delta - \tilde{\Delta}_\pi)\tilde{\Delta}_\pi + x_4(\delta - \tilde{\Delta}_\pi)\delta' + \ldots\Big]$$

to the order of the approximation needed for finding η_3

The parameters x_i are determined by repeating the process separately by choosing in turn each of the other two external points as the origin of the conformal transformation

For the external points x and y the discrepancy functions are $f_B(\delta' - \tilde{\Delta}_\pi)$ and $f_C(\tilde{\Delta}_\pi)$

Ultimately one loop graph can be written as

$$\Gamma_1 ~=~ - ~ \frac{Q_\pi}{\tilde{\Delta}_\pi[\tilde{\Delta}_\pi - \delta][\tilde{\Delta}_\pi - \delta']} \exp \left(F_{\Gamma_1}(\delta, \delta', \tilde{\Delta}_\pi) \right)$$

where

$$\begin{aligned} F_{\Gamma_1}(\delta, \delta', \tilde{\Delta}_\pi) ~&=~ \left[5B(\gamma) - 2B(\alpha - 1) - 3B(0) - \frac{2}{(\alpha - 1)} \right] \tilde{\Delta}_\pi \\ &+ ~ \left[B_0 - B_{\alpha-1} - \frac{1}{(\alpha - 1)} \right] \delta' \\ &- ~ [B_\gamma - B_0] \, \delta ~+~ \ldots \end{aligned}$$

with

$$B(0) ~=~ \psi(\mu) ~+~ \psi(1) ~~,~~ B(\alpha) ~=~ \psi(\alpha) ~+~ \psi(\mu - \alpha)$$

for $\alpha \neq 0$ or μ

In effect in this conformal bootstrap approach, one is carrying out perturbation theory in the vertex anomalous dimension $\chi_\pi = \frac{1}{2}\tilde{\Delta}_\pi$

Sum of all 3-point graphs is denoted by $V(\bar{y},\alpha,\gamma;\delta,\delta')$ and the consistency equations for the vertex functions are

$$\begin{aligned} 1 &= V(\bar{y},\alpha,\gamma;0,0) \\ \frac{T_F N\, r(\alpha-1)}{C_F\, p(\gamma)} &= \left. \frac{\left[1 + 2\chi_\pi \frac{\partial}{\partial\delta'} V(\bar{y},\alpha,\gamma;\delta,\delta')\right]}{\left[1 + 2\chi_\pi \frac{\partial}{\partial\delta} V(\bar{y},\alpha,\gamma;\delta,\delta')\right]} \right|_{\delta=\delta'=0} \end{aligned}$$

which involve two unknown variables $\bar{y}$ and η

For $O(1/N^3)$ graphs only need the difference of

$$\left. \left[\frac{\partial}{\partial\delta'} V(\bar{y},\alpha,\gamma;\delta,\delta') \;-\; \frac{\partial}{\partial\delta} V(\bar{y},\alpha,\gamma;\delta,\delta') \right] \right|_{\delta=\delta'=0}$$

at leading order

Value of η_3

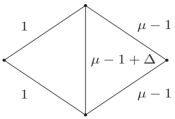
Solving the large N conformal bootstrap equations in d-dimensions yields

$$\begin{aligned}
\eta_3 &= \Bigg[\frac{(2\mu-1)(35\mu^3-43\mu^2+16\mu-2)}{4\mu^2(\mu-1)^4} C_F^2 - \frac{\mu^2(43\mu^2-35\mu+6)}{8\mu^2(\mu-1)^4} C_F C_A \\
&- \frac{\mu^4(4\mu^3-5\mu^2-9\mu-14)}{48\mu^2(\mu-1)^4} C_A^2 \\
&- \frac{\mu^4(4\mu^3-2\mu^2-3\mu+10)}{4\mu^2(\mu-1)^4} \frac{d_F^{abcd} d_F^{abcd}}{T_F^2 N_A} \\
&+ \Bigg[\frac{1}{2}(11\mu-3)(2\mu-1)^2 - \frac{1}{4}\mu(19\mu-2)(2\mu-1) C_F C_A \\
&- \frac{1}{24}\mu^3(2\mu^2-6\mu-23) C_A^2 \\
&- \frac{\mu^3(2\mu^2-3\mu+4) d_F^{abcd} d_F^{abcd}}{2 T_F^2 N_A} \Bigg] \frac{\Psi(\mu)}{\mu(\mu-1)^3} + \ldots
\end{aligned}$$

$$
\begin{aligned}
&\frac{(2\mu-1)(35\mu^3-43\mu^2+16\mu-2)}{4\mu^2(\mu-1)^4} C_F^2 - \frac{\mu^2(43\mu^2-35\mu+6)}{8\mu^2(\mu-1)^4} C_F C_A \\
&+ \frac{3(2C_F-4\mu C_F+\mu C_A)^2}{8(\mu-1)^2}\left[3\Psi^2(\mu)+\Phi(\mu)\right] \\
&+ \Bigg[(2\mu-1)(\mu+1)C_F^2 - \frac{1}{2}\mu(5\mu-1)C_F C_A - \frac{1}{24}\mu^2(\mu-4)C_A^2 \\
&\quad + \mu^2(\mu+8)\frac{d_F^{abcd}d_F^{abcd}}{T_F^2 N_A}\Bigg]\left[\Theta(\mu)+\frac{1}{(\mu-1)^2}\right]\frac{1}{4(\mu-1)^2} \\
&- \Bigg[\left[C_A^2 - 24\frac{d_F^{abcd}d_F^{abcd}}{N_A T_F^2}\right]\left[\Theta(\mu)+\frac{1}{(\mu-1)^2}\right] \\
&\quad \times \left[2\Psi(\mu)+\Xi(\mu)\right]\frac{\mu^2}{16(\mu-1)}\Bigg]\frac{\eta_1^3}{C_F^2}
\end{aligned}
$$

$\Xi(\mu)$ is a new function that appears at $O(1/N^3)$ in non-chirally symmetric theory

It is related to the derivative with respect to the regularizing parameter Δ of



Analytically it can be expressed as an ${}_4F_3$ function where the regulator derivative translates to derivatives of the parameters

The ϵ expansion of $\Xi(\mu)$ is known to all orders near 2 and 4 dimensions and contains $\zeta_{5,3}$ at high order in ϵ

In three dimensions [Vasil'ev et al]

$$\Xi\left(\frac{3}{2}\right) = \frac{3}{2\pi^2}\Psi''\left(\frac{1}{2}\right) + 2\ln(2) + \frac{4}{3}$$

Checks

As the exponent evaluation has been carried out for the general interaction $\pi^a \bar{\psi}^i T^a \psi^i$ taking certain limits should produce known results

The original Gross-Neveu interaction, $\sigma\bar{\psi}^i\psi^i$, corresponds to the limit

$$C_F \rightarrow 1 \ , \ T_F \rightarrow 1 \ , \ d_F^{abcd} d_F^{abcd} \rightarrow 1 \ , \ C_A \rightarrow 0$$

The Mott insulating phase corresponds to taking the colour group to be $SU(2)$

$$C_F \rightarrow \frac{3}{4} \ , \ T_F \rightarrow \frac{1}{2} \ , \ d_F^{abcd} d_F^{abcd} \rightarrow \frac{5}{64} \ , \ C_A \rightarrow 2$$

The fractionalized Gross-Neveu model is described by the exponents in the adjoint of $SO(3)$

$$C_F \rightarrow 2 \ , \ T_F \rightarrow 2 \ , \ d_F^{abcd} d_F^{abcd} \rightarrow \frac{20}{3} \ , \ C_A \rightarrow 2$$

All agree with direct large N evaluations and the ϵ expansion of all known three and four loop renormalization group functions near four dimensions

Three dimensions

In three dimensions the exponents take a simpler form

For instance in the adjoint representation where

$$C_F = C_A \quad , \quad T_F = C_A \quad , \quad d_F^{abcd} d_F^{abcd} = d_A^{abcd} d_A^{abcd}$$

then

$$\lambda = 1 - \frac{16}{3\pi^2 N} + \left[96 \frac{d_A^{abcd} d_A^{abcd}}{C_A^4 N_A} + \frac{5248}{\pi^2} - 432 \right] \frac{1}{27\pi^2 N^2}$$

$$\begin{aligned} \eta = & \frac{8}{3\pi^2 N} + \frac{1216}{27\pi^4 N^2} \\ & + \Big[[9072\zeta_3 - 864\pi^2 \ln 2][C_A^4 N_A - 24 d_A^{abcd} d_A^{abcd}] \\ & \quad + [25920\pi^2 - 435456] d_A^{abcd} d_A^{abcd} \\ & \quad + [151072 - 8760\pi^2] C_A^4 N_A \Big] \frac{1}{243\pi^6 C_A^4 N_A N^3} \end{aligned}$$

Conclusions

Have provided a set of general critical exponents for a scalar Yukawa type of interaction that includes a Lie group generator in the large N expansion to third order

The d-dimensional exponents agree with direct computations carried out in specific models by taking various limits

Essential for this was the `color` package written in FORM that expresses group generator combinations in terms of colour Casimirs

This allows one to track the presence and contribution of the so-called light-by-light diagrams; such graphs will arise in similar computations in QCD

A similar approach can be used for other universality classes with a different basic driving interaction at criticality but endowed with a Lie group symmetry

Gross-Neveu-Yukawa models

The corresponding renormalizable four dimensional Gross-Neveu-Yukawa (GNY) model is

$$L^{\mathrm{GNY}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i - m\bar{\psi}^i\psi^i + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}g_1\sigma\bar{\psi}^i\psi^i + \frac{1}{24}g_2^2\sigma^4$$

The quartic scalar interaction is only relevant in four dimensions due to power counting and is termed a spectator

The canonical dimensions of the fields at the Wilson-Fisher fixed point are

$$[\psi] = \tfrac{1}{2}d - \tfrac{1}{2} \quad , \quad [\sigma] = 1$$

with both couplings dimensionless in $d = 4$

Both the GN and GNY models lie in the same universality class at the Wilson-Fisher fixed point in $2 < d < 4$