Categorical Interactions in Algebra, Geometry and Physics: Cubical Structures and Truncations

Ralph Kaufmann

Purdue University

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Plan

Dedication

Celebrating 60 years of achievement

Happy birthday Dirk!

Personal note

Ever since I was a post-doc back in 1998 at the IHES the conversations and discussions with Dirk have been very influential for me. Much of the latest developments of what I will be presenting would not be in the same form, or exist, if these would not have taken place.
Thank you!

Ideas and Goals

Main idea

- Several structures are most naturally regarded in a categorical context $\rightsquigarrow$ Feynman categories.
- There are natural structures on *morphisms* that are the "raison d'être" for many other more complicated structures.

Main Objective

Find common categorical background for various constructions in Algebra, Geometry and Physics, which allows for

1. Deeper theoretical understanding.
2. Calculations.
3. Links between Algebra, Geometry and Physics.

Automatic relation between different structures

Functors, push-forward, pull-back, six functor formalism.

Constuctions

Combinatorics $\rightsquigarrow$ Algebra

The morphisms directly yield colored or partial algebras and co-algebras. Hopf and bialgebras ensue.

Representations $\rightsquigarrow$ Combinatorics, Algebra and Geometry

Look at representations. These are functors into a target category $\mathcal{C}$. This can have combinatorial, algebraic or geometric flavor.

Different flavours

Can also enrich, this makes the morphisms directly linear or geometric. Spaces of morphisms, dg morphisms etc.

Constructions

Constructions $\rightsquigarrow$ Geometry, enrichment

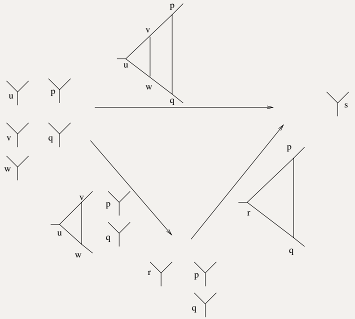
1. Grothendieck construction. E.g. gives various types of graphs.
2. W construction $\rightsquigarrow$ cubical complexes. E.g. of graphs $\rightsquigarrow$ moduli spaces.
3. Plus construction. Interpolates between different theories. $\rightsquigarrow$ hierarchies:
   - object,simplicial, planar rooted trees (non-sigma operadic), nestings of these (hyper–) ...
   - object, crossed-simplicial (symmetric) aka. NCSet, rooted trees (operads), nestings of these ...
   - Cyclic objects, planar trees, nestings, ...
   - Graphs, nestings (hyper–), ...

Example for a graphical Feynman category

Objects and morphisms

- Objects are stars: labeled vertices, with marked flags/tails.
- Morphisms: Graphs, with identification data for the source and target.

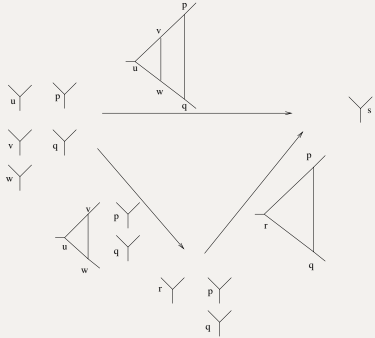
ϕ^3/restriction to 3–valent vertices



Note: “everything” is labelled and tracked.

Example

ϕ^3/restriction to 3–valent vertices

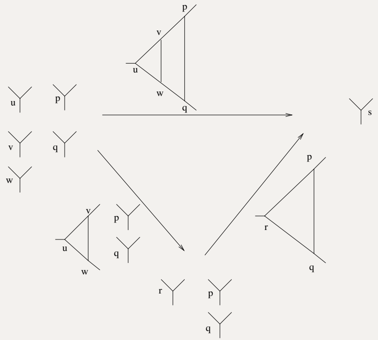


Rules

- Composition of morphisms $\widehat{=}$ insertion.
- Contraction of subgraph $\widehat{=}$ factorization of morphisms.

Example

ϕ^3/restriction to 3–valent vertices



Subtleties

- Isomorphisms and automorphisms of objects $\rightsquigarrow$ groupoid.
- Isomorphisms and automorphisms of morphisms $\rightsquigarrow$ multiplicities and compatibilities.

Example cubical complex

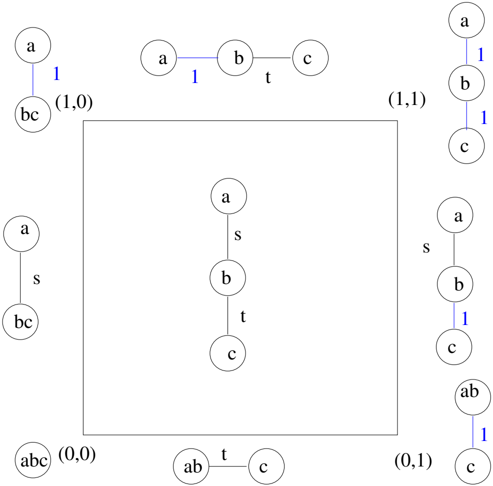


Figure: The cubical structure in the case of $n = 3$. One can think of the edges marked by 1 as cut.

Feynman categories

Data

1. $\mathcal{V}$ a groupoid
2. $\mathcal{F}$ a symmetric monoidal category
3. $\imath : \mathcal{V} \to \mathcal{F}$ a functor.

Notation

$\mathcal{V}^{\otimes}$ the free symmetric category on $\mathcal{V}$ (words in $\mathcal{V}$).

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\imath} & \mathcal{F} \\ & {\scriptstyle \jmath} \searrow \quad \nearrow {\scriptstyle \imath^{\otimes}} & \\ & \mathcal{V}^{\otimes} & \end{array}$$

Feynman category

Definition ([KW17])

Such a triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ is called a Feynman category if

i. $\imath^{\otimes}$ induces an equivalence of symmetric monoidal categories between $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$.
ii. $\imath$ and $\imath^{\otimes}$ induce an equivalence of symmetric monoidal categories between $Iso(\mathcal{F} \downarrow \mathcal{V})^{\otimes}$ and $Iso(\mathcal{F} \downarrow \mathcal{F})$.
iii. For any $* \in \mathcal{V}$, $(\mathcal{F} \downarrow *)$ is essentially small.

Basic consequences

1. $X \simeq \bigotimes_{v \in I} *_v$
2. $\phi : Y \to X$, $\phi \simeq \bigotimes_{v \in I} \phi_v$, $\phi_v : Y_v \to *_v$, $Y \simeq \bigotimes_{v \in I} Y_v$. The morphisms $\phi_v : Y \to *_v$ are called basic or one–comma generators.

Example

Diagram

Graph related Feynman categories. Why Feynman?

Math

Basic graphs, full subcategory of Borisov-Manin category of graphs whose objects are aggregates of corollas (no edges). The morphisms have an underlying graph, the ghost graph.

Physics (connected case)

Objects of $\mathcal{V}$ are the vertices of the theory. The morphims of $\mathcal{F}$ "are" the possible Feynman graphs. Both can be read off the Lagrangian or actions.
The source of a morphisms ϕ_Γ "is" the set of vertices $V(\Gamma)$ and the target of a basic morphism is the external leg structure $\Gamma/E(\Gamma)$.
The terms in the S matrix corresponding to the external leg structure $*$ is $(\mathcal{F} \downarrow *_v)$. "Dressed vertex".

Examples

Roughly (in the connected case and up to isomorphism)

The source of a morphism are the vertices of the ghost graph $\mathbb{\Gamma}$ and the target is the vertex obtained from $\mathbb{\Gamma}$ obtained by contracting all edges. If $\mathbb{\Gamma}$ is not connected, one also needs to merge vertices according to ϕ_V.

Composition corresponds to insertion of ghost graphs into vertices.

$$X \xrightarrow{\phi_2} Y \xrightarrow{\phi_1} *$$

$$\phi_0$$

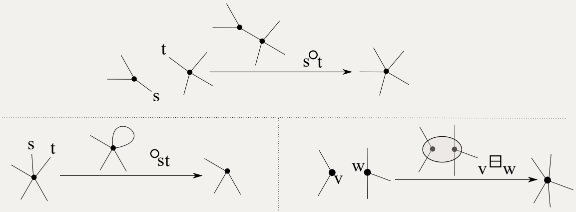
up to isomorphisms (if $\mathbb{\Gamma}_0, \mathbb{\Gamma}_1$ are connected) corresponds to inserting $\mathbb{\Gamma}_v$ into $*_v$ of $\mathbb{\Gamma}_1$ to obtain $\mathbb{\Gamma}_0$.

$$\amalg_v \amalg_{w \in V_v} *_w \xrightarrow{\amalg_v \mathbb{\Gamma}_v} \amalg_v *_v \xrightarrow{\mathbb{\Gamma}_1} *$$

$$\mathbb{\Gamma}_0$$

Basic graph morphisms

Basic morphisms



Names

- Non-self gluing/(virtual) edge contraction
- Self–gluing/(virtual) loop contraction (exclude for trees/forests)
- Merger (exclude for connected).

"Representations" of Feynman categories: $\mathcal{O}ps$ and $\mathcal{M}ods$

Definition

Fix a symmetric cocomplete monoidal category $\mathcal{C}$, where colimits and tensor commute, and $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ a Feynman category.

- Consider the category of strong symmetric monoidal functors $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} := Fun_{\otimes}(\mathcal{F}, \mathcal{C})$ which we will call $\mathcal{F}$–ops in $\mathcal{C}$
- $\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}} := Fun(\mathcal{V}, \mathcal{C})$ will be called $\mathcal{V}$-modules in $\mathcal{C}$ with elements being called a $\mathcal{V}$–mod in $\mathcal{C}$.

Trival *op*

Let $\mathcal{T} : \mathcal{F} \to \mathcal{C}$ be the functor that assigns $\mathbb{I} \in Obj(\mathcal{C})$ to any object, and which sends morphisms to the identity of the unit.

Remark

$\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ is again a symmetric monoidal category.

$\mathcal{O}ps$ as "Feynman rules"

Space of fields and propagators

Fix a linear target category for general graph type FCs: $\mathcal{V}ect_m/k$; k–vector spaces with a non–degenerate (super)-symmetric bilinear form.

Picking a basis ϕ_i (of fields), the quadratic form $g_{ij} = \langle \phi_i, \phi_j \rangle$ will give the Casimir (propagator)

$$C = \sum_{ij} g^{ij} \phi_i \otimes \phi_j$$

$\mathcal{O}ps$ as "Feynman rules"

Vertex factors from $\mathcal{O} \in \mathcal{O}ps$

In the simplest case: for each vertex $*_S$, we have an element, the vertex factor:

$$Y_S := \mathcal{O}(*_S) : W^{\otimes S} \to k$$

For a graph morphisms ϕ, we get the morphism given by contacting tensors with the propagator

$$\bigotimes_{v \in V} \mathcal{O}(*_v) \otimes C^{\otimes E} \to \check{W}^{\otimes F} \otimes W^{\otimes E} \to k$$

where V are the vertices, F are the flags or half edges, and E are the edges of the graph.

$\mathcal{O}ps$ as "Feynman rules"

Remarks

Can do different field lines as colors. Can do more elaborate propagators.

Examples based on $\mathfrak{G}$: morphisms have underlying graphs

$\mathfrak{F}$	Feynman category for	condition on graphs additional decoration
$\mathfrak{O}$	operads	rooted trees
$\mathfrak{O}_{mult}$	operads with mult.	b/w rooted trees.
$\mathfrak{C}$	cyclic operads	trees
$\mathfrak{G}$	unmarked nc modular operads	graphs
$\mathfrak{G}^{ctd}$	unmarked modular operads	connected graphs
$\mathfrak{M}$	modular operads	connected + genus marking
$\mathfrak{M}^{nc,}$	nc modular operads	genus marking
$\mathfrak{D}$	dioperads	connected directed graphs w/o directed loops or parallel edges
$\mathfrak{P}$	PROPs	directed graphs w/o directed loops
$\mathfrak{P}^{ctd}$	properads	connected directed graphs w/o directed loops
$\mathfrak{D}^{\circlearrowleft}$	wheeled dioperads	directed graphs w/o parallel edges
$\mathfrak{P}^{\circlearrowleft,ctd}$	wheeled properads	connected directed graphs
$\mathfrak{P}^{\circlearrowleft}$	wheeled props	directed graphs

Table: List of Feynman categories with conditions and decorations on the graphs, yielding the zoo of examples

Structure Theorems

Theorem

The forgetful functor $G : \mathcal{O}ps \to \mathcal{M}ods$ has a left adjoint F (free functor) and this adjunction is monadic. The endofunctor $\mathbb{T} = GF$ is a monad (triple) and $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$, algebras over the triple .

Theorem

Feynman categories form a 2–category and it has push–forwards $f_!$ and pull–backs f^* for $\mathcal{O}ps$ and $\mathcal{M}ods$.

Remarks

The push–forward is given by a left Kan extension $f_! = Lan_f$. Sometimes there is also a right adjoint $f_* = Ran_f$ which is "extension by zero" together with its adjoint $f^!$ will form part of a 6 functor formalism.

Other examples

Trivial $\mathcal{V}$: $\mathcal{V} = \underline{*}$

- $\mathcal{V}^{\otimes} \simeq \underline{N}$ in the *non–symmetric case*.
  $\mathcal{V}^{\otimes} \simeq \mathbb{S}$ in the *symmetric* case.
  Both categories have the natural numbers as objects and while $\underline{N}$ is discrete $Hom_{\mathbb{S}}(\underline{n}, \underline{n}) = \mathbb{S}_n$.
- $\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}}$ are simply objects of $\mathcal{C}$.

$\mathcal{F} = \mathcal{V}^{\otimes}$, groupoid reps

$\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} = \mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}} = Rep(\mathcal{V})$, that is groupoid representation.
Special case $\mathcal{V} = \underline{G}$:

- $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ are representations.
- Morphisms induced by $f : H \to G \rightsquigarrow \underline{f} : \underline{H} \to \underline{G}$.
- $\underline{f}^*$ is restriction, $\underline{f}_!$ is induction, adjointness is Frobenius reciprocity.

"Smaller" examples

$\mathcal{S}urj$, (commutative) Algebras

$\mathcal{F} = \mathcal{S}urj$ is finite sets with surjections. $Iso(sk(\mathcal{S}urj)) = \mathbb{S}$.
$\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ are commutative algebra objects in $\mathcal{C}$. Let $\mathcal{O} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$.

- $A = \mathcal{O}(\underline{1})$.
- $\mathcal{O}(\underline{n}) = A^{\otimes n}$ ($\mathcal{O}$ is monoidal).
- $\pi : \underline{2} \to \underline{1}$ gives the multiplication $\mu = \mathcal{O}(\pi) : A^{\otimes 2} \to A$.
- $\pi \circ (\pi \amalg id) = (\pi \circ id) \amalg \pi = \pi_3 : \underline{3} \to \underline{1}$; product is associative.

$$\begin{array}{ccc} \underline{3} & \xrightarrow{\pi \amalg id} & \underline{2} \\ {\scriptstyle id \amalg \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \underline{2} & \xrightarrow[\pi]{} & \underline{1} \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} A^{\otimes 3} & \xrightarrow{\mu \otimes id} & A^{\otimes 2} \\ {\scriptstyle id \otimes \mu_2}\downarrow & & \downarrow{\scriptstyle \mu} \\ A^{\otimes 2} & \xrightarrow[\mu]{} & A \end{array}$$

- $(12) \circ \pi = \pi$; so product is commutative.

More exampes, see e.g. [Kau20]

Other versions

1. If once considers the non–symmetric analogue, one obtains ordered sets, with order preserving surjections and associative algebras.
2. The $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ for $\mathcal{F}in\mathcal{S}et$ are unital commutative algebras.

More examples of this type

1. Finite sets and injections $\rightsquigarrow$ FI algebras, Church-Farb-Ellenberg.
2. $\Delta_+ S$ crossed simplicial group. There are the skeleton of non-commutative sets: order on the fibers of morphisims

Hopf algebras [GCKT20a, GCKT20b]

Basic structures

Assume $\mathcal{F}$ is decomposition finite. Consider $\mathcal{B} = Hom(Mor(\mathcal{F}), \mathbb{Z})$. Let μ be the tensor product with unit $id_{\mathbb{I}}$.

$$\Delta(\phi) = \sum_{(\phi_0,\phi_1):\phi=\phi_1\circ\phi_0} \phi_0 \otimes \phi_1$$

and $\epsilon(\phi) = 1$ if $\phi = id_X$ and 0 else.

Quotient by symmetries/isomorphisms

We let $f \sim g$ is there are isomorphisms σ, σ' such that $f = \sigma^{-1} g \sigma$ and we set $\mathcal{B}^{iso} = \mathcal{B}/\sim$.

Theorem (Galvez-Carrillo, K , Tonks)

For a Feynman category, on $\mathcal{B}^{iso}$ the structure above together with the multiplication $\mu = \otimes$ induce a bi–algebra structure. Under certain explicitly checkable assumptions, a canonical quotient is a Hopf algebra.

Theorem

In the non–Σ case, if the monoidal structure is strict, $\mathcal{B}$ as above is already a bi-algebra.

The quotient in the $\neg\Sigma$ case

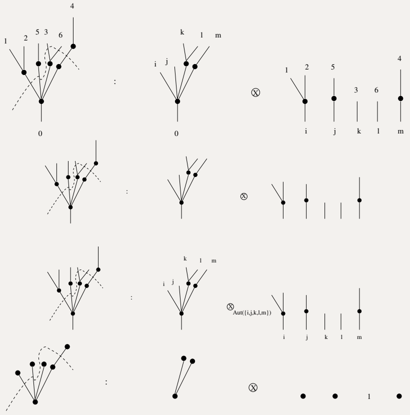
Let $\mathcal{I} = \langle id_X - 1\rangle$, then $\mathcal{H} = \mathcal{B}/\mathcal{I}$. $1 = id_{\mathbb{I}_{\mathcal{F}}}$.
"Reason:" $\Delta(\phi) = id_X \otimes \phi + \phi \otimes id_Y + \dots$.
In the symmetric case this is more complicated.

Hopf algebras/(co)operads/Feynman category

H_{Gont}	$Inj_{*,*} = Surj^*$	$\mathfrak{F}_{Surj}$
H_{CK}	leaf labelled trees	$\mathfrak{F}_{Surj,\mathcal{O}}$
$H_{CK,graphs}$	graphs	$\mathfrak{F}_{graphs}$
H_{Baues}	$Inj^{gr}_{*,*}$	$\mathfrak{F}_{Surj,odd}$

Examples

Connes–Kreimer tree versions/admissible cuts



DIFFERENT VERSIONS: Fully labelled, planar or non planar, equivariant, contracting legs aka. quotienting out by ideal.

Remarks

Upshot: Common framewok

In this fashion, we can reproduce Connes–Kreimer's Hopf algebra, the Hopf algebras of Goncharov and a Hopf algebra of Baues that he defined for double loop spaces. This is a non–commutative graded version.

There is a three-fold hierarchy.

1. A non-commutative version
2. a commutative version and
3. an "amputated" version.

Decoration

Through decoration get for instance motic Hopf algebras of Brown.

Remarks

Remaks

1. Baues and Gontcharov: *"Simplices form an operad"*.
2. In fact use new notion of co–operads with multiplication.
3. Extension to not necessarily free case with multiplication. $\Delta = (id \otimes \mu^{\otimes n}) \circ \check{\gamma}$. Filtrations instead of grading. Developable and deformation of associated graded.
4. Iterated co-products correspond to elements in the nerve.

$$X_n \xrightarrow{\phi_n} X_{n-1} \cdots X_1 \xrightarrow{\phi_1} X_0$$

This is related to the $+$ construction and the CK-trees of sub-divergencies.

The graph case (core Hopf algebra)

Theorem

On isomorphism classes $\mathbb{\Gamma}$ in $\mathcal{A}gg^{ctd}$.

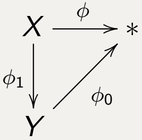
$$\Delta^{iso}(\mathbb{\Gamma}) = \sum_{\mathbb{\Gamma}_1 \subset \mathbb{\Gamma}} \mathbb{\Gamma}/\mathbb{\Gamma}_1 \otimes \mathbb{\Gamma}_1 = \sum_{\mathbb{\Gamma}_1 \subset \mathbb{\Gamma}} \mathbb{\Gamma}_0 \otimes \mathbb{\Gamma}_1 \tag{1}$$

Here $\mathbb{\Gamma}$ is the isomorphism class $\mathbb{\Gamma} = [\phi] = \mathbb{\Gamma}(\phi)$ and $\mathbb{\Gamma}_1 = \mathbb{\Gamma}(\phi_1)$ is a subgraph, which corresponds to the isomorphism class of a decomposition $[(\phi_0, \phi_1)]$ where then necessarily $\mathbb{\Gamma}(\phi_0) = \mathbb{\Gamma}(\phi)/\mathbb{\Gamma}_1$. Moreover if $\mathbb{\Gamma}$ is connected, so is $\mathbb{\Gamma}_0$. — both are isomorphism classes in $(\mathcal{A}gg^{ctd} \downarrow \mathcal{C}rl)$.

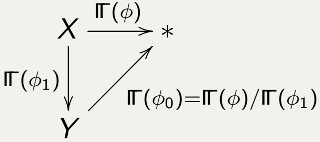
Basic factorization/ comodule structure B^+ operator

Basic morphisms

The corresponding factorization of a morphism in $(\mathcal{F} \downarrow \mathcal{V})$ is



and on iso classes



where $\mathbb{\Gamma}(\phi_1)a$ is a subgraph, $\mathbb{\Gamma}(\phi)/\mathbb{\Gamma}(\phi_0)$ is sometimes called the co–graph and $*$ is the residue in the physics nomenclature.

Remark

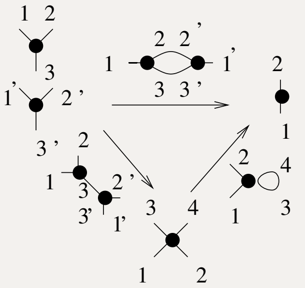
This is also the place for restrictions/generalization: non-core/ linear, infinite and B_+, where $B_+^{\phi_0}(\phi_1) = \phi_0 \circ \phi_1 = \phi$.

Example with a multiplicity

The co–product of a graph.

The factor of 2 is there, since there are two distinct subgraphs —given by the two distinct edges— which give rise to two factorizations whose abstract graphs coincide.

Details: label everything. One decomposition:



ϕ_1: $\phi_1^F(3) = 1', \phi_1^F(4) = 2'$ is a non-trivial identification of flags. There is only one choice for the vertex maps and the involution is the one given by the ghost graph.

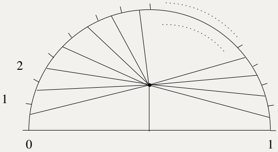
Relating Gontcharov's picture and CK trees/corollas

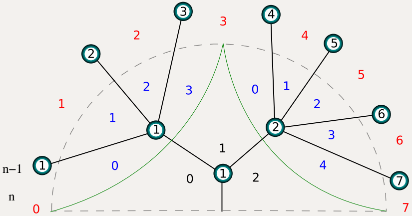
Joyal duality

Order preserving surjections/double base point preserving injections. Joyal duality.

$$Hom_{smCat}([n],[m]) = Hom_{*,*}([m+1],[n+1])$$

Pictorial representation,





Constructions yielding Feynman categories

A partial list

1. $\mathcal{F}_{dec\mathcal{O}}$: non–Sigma and dihedral versions. It also yields all graph decorations. “Vertex decoration”.
2. $+$ construction. Gives hierarchies. Twisted modular operads, twisted versions of any of the previous structures. Generalizations w/ M. Monaco.
3. Enrichment via $+$ construction. $\mathfrak{F}_{\mathcal{O}}$ for $\mathcal{O} : \mathfrak{F}^{+} \to \mathcal{C}$. “Edge decoration”.
4. Non–connected construction $\mathfrak{F}^{nc}$, whose $\mathcal{F}^{nc}$-$\mathcal{O}ps$ are equivalent to lax monoidal functors of $\mathcal{F}$.
   This is also where the B_+ operator appears.
5. Cobar/bar, Feynman transforms in analogy to algebras and (modular) operads $\rightsquigarrow$ Master-equations.
6. W–construction.

$\mathfrak{F}_{dec\mathcal{O}}$ w/ J. Lucas, C. Berger. Grothendieck construction

Theorem

Given an $\mathcal{O} \in \mathcal{F}$–$\mathcal{O}ps$, then there is a Feynman category $\mathcal{F}_{dec\mathcal{O}}$ which is indexed over $\mathcal{F}$.

- *It objects are pairs $(X, dec \in \mathcal{O}(X))$*
- *$Hom_{\mathcal{F}_{dec\mathcal{O}}}((X, dec), (X', dec'))$ is the set of $\phi : X \to X'$, s.t. $\mathcal{O}(\phi)(dec) = dec'$.*

(This construction works a priori for Cartesian $\mathcal{C}$, but with modifications it also works for the non–Cartesian case.)

Example

$\mathfrak{F} = \mathfrak{C}$, $\mathcal{O} = CycAss$, $CycAss(*_S) = \{\text{cyclic orders} \prec \text{on } S\}$. New basic objects of $\mathfrak{C}_{decCycAss}$ are planar corollas $*_{S,\prec}$. Morphisms "are planar trees".

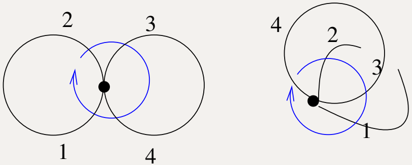
Some $\mathcal{O}ps$

Trivial $\mathcal{T} : \mathcal{F} \to \mathcal{C}$

$$\mathcal{T}(X) = \mathbb{I}_{\mathcal{C}}, \mathcal{T}(\phi) = id_{\mathcal{C}}$$

CycAss and decorating with it

$$CycAss(*_S) = \{\text{cyclic orders on } S\}$$



Results

Theorem

(2)
$$
\begin{array}{ccc}
\mathfrak{F}_{dec\mathcal{O}} & \xrightarrow{f^{\mathcal{O}}} & \mathfrak{F}'_{dec\, f_*(\mathcal{O})} \\
\downarrow{\scriptstyle forget} & & \downarrow{\scriptstyle forget'} \\
\mathfrak{F} & \xrightarrow{f} & \mathfrak{F}'
\end{array}
\qquad
\begin{array}{ccc}
\mathfrak{F}_{dec\mathcal{O}} & \xrightarrow{\sigma_{dec}} & \mathfrak{F}_{dec\mathcal{P}} \\
\downarrow{\scriptstyle f^{\mathcal{O}}} & & \downarrow{\scriptstyle f^{\mathcal{P}}} \\
\mathfrak{F}'_{decf_*(\mathcal{O})} & \xrightarrow{\sigma'_{dec}} & \mathfrak{F}'_{decf_*(\mathcal{P})}
\end{array}
$$

The squares above commute squares and are natural in $\mathcal{O}$. We get the induced diagram of adjoint functors.

(3)
$$
\begin{array}{ccc}
\mathcal{F}_{dec\mathcal{O}}\text{-}\mathcal{O}ps & \underset{f^{\mathcal{O}\,*}}{\overset{f^{\mathcal{O}}_*}{\rightleftarrows}} & \mathcal{F}'_{dec\, f_*(\mathcal{O})}\text{-}\mathcal{O}ps \\
{\scriptstyle forget_*}\downarrow\uparrow{\scriptstyle forget^*} & & {\scriptstyle forget'^*}\uparrow\downarrow{\scriptstyle forget'_*} \\
\mathcal{F}\text{-}\mathcal{O}ps & \underset{f^*}{\overset{f_*}{\rightleftarrows}} & \mathcal{F}'\text{-}\mathcal{O}ps
\end{array}
$$

Examples on $\mathfrak{G}$ with extra decorations

Decoration and restriction allows to generate the whole zoo and even new species

$\mathfrak{F}_{dec\mathcal{O}}$	Feynman category for	decorating $\mathcal{O}$	restriction
$\mathfrak{F}^{dir}$	directed version	$\mathbb{Z}/2\mathbb{Z}$ set	edges contain one input and one output flag
$\mathfrak{F}^{rooted}$	root	$\mathbb{Z}/2\mathbb{Z}$ set	vertices have one output flag.
$\mathfrak{F}^{genus}$	genus marked	$\mathbb{N}$	
$\mathfrak{F}^{c-col}$	colored version	c set	edges contain flags of same color
$\mathfrak{O}^{\neg\Sigma}$	non-Sigma-operads	*Ass*	
$\mathfrak{C}^{\neg\Sigma}$	non-Sigma-cyclic operads	*CycAss*	
$\mathfrak{M}^{\neg\Sigma}$	non–Signa-modular	*ModAss*	
$\mathfrak{C}^{dihed}$	dihedral	*Dihed*	
$\mathfrak{M}^{dihed}$	dihedral modular	*ModDihed*	

Table: List of decorates Feynman categories with decorating $\mathcal{O}$ and possible restriction. $\mathfrak{F}$ stands for an example based on $\mathfrak{G}$ in the list.

Example

Bootstrap

(4)
$$\begin{array}{ccccc}
\mathfrak{C}_{dec\,CycAss} = \mathfrak{C}^{\neg\Sigma} & \xrightarrow{i^{CycAss}} & \mathfrak{M}_{dec\,i_*(CycAss)} = \mathfrak{M}^{\neg\Sigma} \\
\downarrow \text{forget} & & \downarrow \text{forget} \\
\mathfrak{C} & \xrightarrow{i} & \mathfrak{M} = \mathfrak{G}^{ctd}_{j_*(\mathcal{T})} \\
& \searrow^{j} & \downarrow \text{forget} \\
& & \mathfrak{G}^{ctd}
\end{array}$$

1. $\mathfrak{C}$-$\mathcal{O}ps$ are cyclic operads. Basic graphs are trees.
2. $\mathfrak{G}^{ctd}$: Basic graphs are connected graphs.
3. $j_*(\mathcal{T})(*_S) = \amalg_{g\in\mathbb{N}} *$ hence elements of $\mathcal{V}$ for $\mathfrak{M}$ are of the form $*_{g,S}$ they can be thought of an oriented surface of genus g and S boundaries.

Example

Bootstrap

(5)

$$\begin{array}{ccc}
\mathfrak{C}_{dec\ CycAss} = \mathfrak{C}^{\neg\Sigma} & \xrightarrow{i^{CycAss}} & \mathfrak{M}_{dec\ i_*(CycAss)} = \mathfrak{M}^{\neg\Sigma} \\
\downarrow \text{forget} & & \downarrow \text{forget} \\
\mathfrak{C} & \xrightarrow{i} & \mathfrak{M} = \mathfrak{G}^{ctd}_{j_*(\mathcal{T})} \\
& \searrow^{j} & \downarrow \text{forget} \\
& & \mathfrak{G}^{ctd}
\end{array}$$

4. $\mathfrak{M}^{\neg\Sigma}$ are non–sigma modular operads (Markl, K-Penner). Elements of $\mathcal{V}$ are $*_{g,s,S_1,\dots,S_b}$ where each S_i has a cyclic order. These can be thought of as oriented surfaces with genus g, s internal marked points, b boundaries where each boundary i has marked points labelled by S_i in the given cyclic order.

Example

Bootstrap

(6)
$$
\begin{array}{ccc}
\mathfrak{C}_{dec\ CycAss} = \mathfrak{C}^{\neg\Sigma} & \xrightarrow{i^{CycAss}} & \mathfrak{M}_{dec\ i_*(CycAss)} = \mathfrak{M}^{\neg\Sigma} \\
\downarrow{\scriptstyle forget} & & \downarrow{\scriptstyle forget} \\
\mathfrak{C} & \xrightarrow{i} & \mathfrak{M} = \mathfrak{G}^{ctd}_{j_*(\mathcal{T})} \\
& \searrow{\scriptstyle j} & \downarrow{\scriptstyle forget} \\
& & \mathfrak{G}^{ctd}
\end{array}
$$

5. This is now actually a *calculation*. A succinct proof uses the theorem that the spanning tree graph is connected and mutations act transitively. (Thanks Karen!)
6. This also re-proves well known results about TFT and OCTFT being defined by Frobenius algebras using adjunction.

W-construction

Input: Cubical Feynman categories in a nutshell

- Ex: $\phi_{e_1} \circ \phi_{e_2} = \phi_{e'_2} \circ \phi_{e'_1}$, commutative square for two consecutive edge contractions.
- Generators and relations for basic morphisms.
- Additive length function $l(\phi)$, $l(\phi) = 0$ equivalent to ϕ is iso.
- Quadratic relations and every morphism of length n has precisely $n!$ decompositions into morphisms of length 1 up to isomorphisms.

Definition

Let $\mathcal{P} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{T}op}$. For $Y \in ob(\mathcal{F})$ we define

$$W(\mathcal{P})(Y) := colim_{w(\mathfrak{F},Y)} \mathcal{P} \circ s(-)$$

Nontechnical version

Nontechnical version for graphs

Glue together cubes. One n-cube for each graph with n edges. There are two boundaries per edge. Contract or mark. Glue along these edges.

Remark (Kreimer)

This is exactly what happens in Cutkosky rules. Only instead of marking edge as fixed, forget (aka. cut) the edge.

Pictures for an algebra restriction

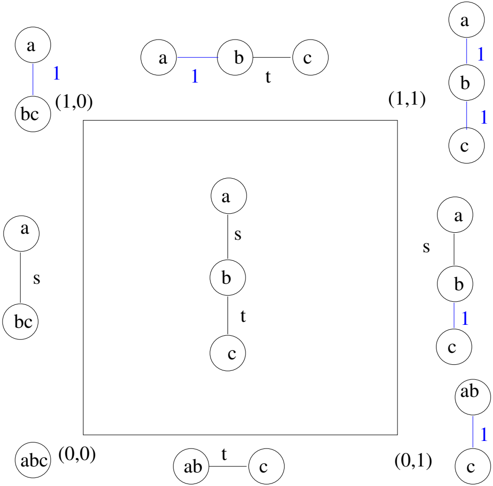


Figure: The cubical structure in the case of $n = 3$. One can think of the edges marked by 1 as cut.

Other interpretations of the same picture

Remark

The cubical structure also becomes apparent if we interpret $[n] = 0 \to 1 \to 2 \to \cdots \to n$ as the simplex.

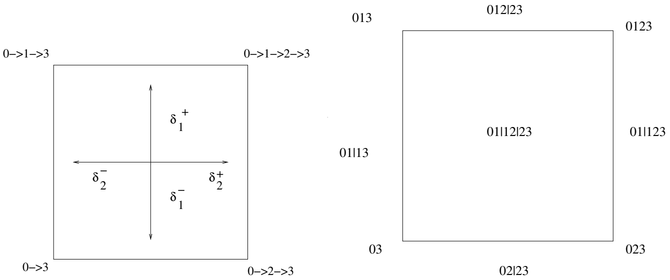


Figure: Two other renderings of the same square. Note: $0 \xrightarrow{a} 1 \xrightarrow{b} 2 \xrightarrow{c} 3$

Cubical decomposition of associahedra

$W(Ass)$

The associative operad $Ass(n) = regular(\mathbb{S}_n)$. $W(Ass)(n)$ is a cubical decomposition of the associahedron.

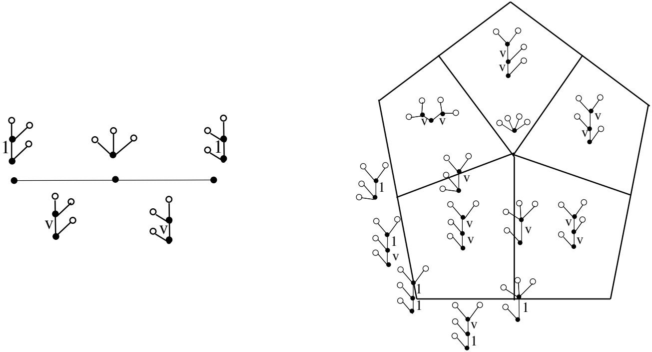


Figure: The cubical decomposition for K_3 and K_4, v indicates a variable height.

Technical Details

The category $w(\mathfrak{F}, Y)$, for $Y \in \mathcal{F}$ Objects:

Objects are the set $\coprod_n C_n(X, Y) \times [0,1]^n$, where $C_n(X, Y)$ are chains of morphisms from X to Y with n degree ≥ 1 maps modulo contraction of isomorphisms.
An object in $w(\mathfrak{F}, Y)$ will be represented (uniquely up to contraction of isomorphisms) by a diagram

$$X \xrightarrow[f_1]{t_1} X_1 \xrightarrow[f_2]{t_2} X_2 \to \cdots \to X_{n-1} \xrightarrow[f_n]{t_n} Y$$

where each morphism is of positive degree and where $t_1, \ldots, t_n$ represents a point in $[0,1]^n$. These numbers will be called weights. Note that in this labeling scheme isomorphisms are always unweighted.

Setup: quadratic Feynman category $\mathfrak{F}$

The category $w(\mathfrak{F}, Y)$, for $Y \in \mathcal{F}$ Morphisms:

1. Levelwise commuting isomorphisms which fix Y, i.e.:

$$\begin{array}{ccccccccccc}
X & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \dots & \longrightarrow & X_n & \longrightarrow & Y \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & & & \downarrow \cong & \nearrow & \\
X' & \longrightarrow & X'_1 & \longrightarrow & X'_2 & \longrightarrow & \dots & \longrightarrow & X'_n & &
\end{array}$$

2. Simultaneous $\mathbb{S}_n$ action.
3. Truncation of 0 weights: morphisms of the form $(X_1 \xrightarrow{0} X_2 \to \cdots \to Y) \mapsto (X_2 \to \cdots \to Y)$.
4. Decomposition of identical weights: morphisms of the form $(\cdots \to X_i \xrightarrow{t} X_{i+2} \to \dots) \mapsto (\cdots \to X_i \xrightarrow{t} X_{i+1} \xrightarrow{t} X_{i+2} \to \dots)$ for each (composition preserving) decomposition of a morphism of degree ≥ 2 into two morphisms each of degree ≥ 1.

Models for moduli spaces and push–forwards

The square revisited

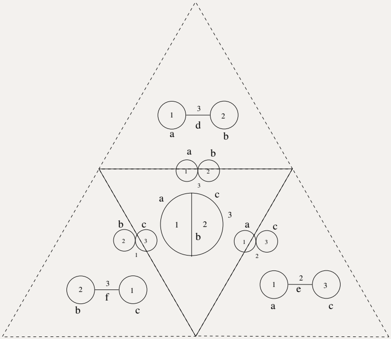
$$\begin{array}{ccc} \mathcal{F}_{dec\ CycAss} = \mathfrak{C}^{\neg\Sigma} & \xrightarrow{i^{CycAss}} & \mathfrak{M}_{dec\ i_*(CycAss)} = \mathfrak{M}^{\neg\Sigma} \\ \downarrow \text{forget} & & \downarrow \text{forget} \\ \mathfrak{C} & \xrightarrow{i} & \mathfrak{M} \end{array}$$

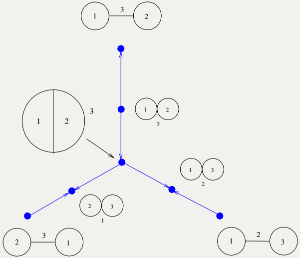
Work with C. Berger

1. $Wi_*(CycAss) = (*_{g,n}) = Cone(\bar{M}^{K/P}_{g,n}) \supset \bar{M}^{K/P}_{g,n} \supset M_{g,n}$, metric almost ribbon graphs (emtpy graph is allowed).
2. $i^{cycAss}_*(W\mathcal{T})(*_{g,s,S_1 \amalg \cdots \amalg S_b}) \simeq M_{g,s,S_1 \amalg \cdots \amalg S_b}$. This is a generalization of Igusa's theorem $M_{g,n} = Nerve(IgusaCat)$
3. $FT_{\mathfrak{M}^{\neg\Sigma}}(\mathcal{T})(*_{g,s,S_1,\dots S_b}) = CC_*(\bar{M}^{K/P}_{g,s,S_1 \amalg \cdots \amalg S_b})$.

$M_{0,3}$

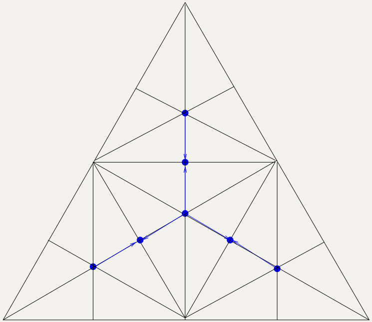
$M_{0,3}^{comb}$, its spine/nerve

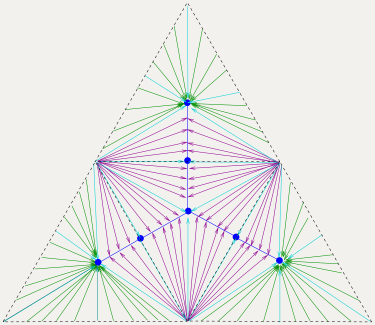




$M_{0,3}$

$M_{0,3}^{comb}$, its spine/nerve and the retraction





Cutkosky/Outer space, w/ C. Berger

The cube complex $j_*(W(CycAss))(*_S)$

Is the complex whose cubical cells are indexed by pairs (Γ, τ), where

- Γ is a graph with S–labelled tails and τ is a spanning forest.
- The cell has dimension $|E(\tau)|$
- the differential ∂_e^- contracts the edge
- ∂_e^+, removes the edge from the spanning forest.

Remark

This complex and the differential are not defined by hand, but automatic!

Blow-ups/Compactifications w/ J.J. Zuniga

Claim

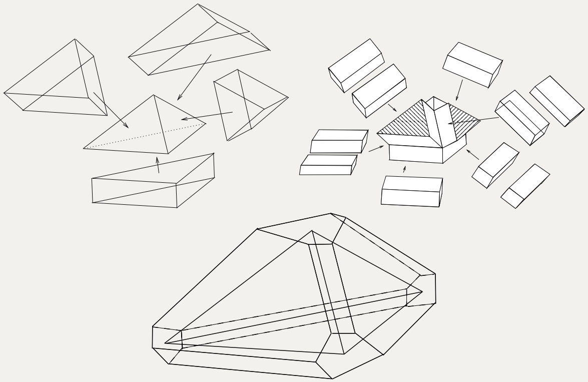
1. There is a natural blow–up of the W–construction above, which is induced by the cubical structure of the Feynman category. This leads to new compactificaion of the moduli space.
2. There is a sequence of blow–downs which terminates in the final blow–down $\overline{M}_{g,n}^{KSV} \to \overline{M}_{g,n}^{DM} \to \overline{M}_{g,n}^{comb}$.
3. This can be modeled on both the analytic/algebraic side and the combinatorial side, giving the desired orbircell decomposition to all spaces.

Remark

This is driven by master-equations [KWZn15] and is directly related to the Jewels of Vogtman et. al. and the truncations in QFT (Kreimer group).

Relative truncation/blow–up pictures

Blow-up the simplex to a cyclohedron to prove the A_∞ Deligne conjecture. W/ R. Schwell



More $\mathcal{F}_{dec\mathcal{O}}$

Theorem

If $\mathcal{T}$ is a terminal object for $\mathcal{F}$-$\mathcal{O}ps$ and forget : $\mathcal{F}_{dec\mathcal{O}} \to \mathcal{F}$ is the forgetful functor, then forget$^*(\mathcal{T})$ is a terminal object for $\mathcal{F}_{dec\mathcal{O}}$-$\mathcal{O}ps$. We have that forget$_*$forget$^*(\mathcal{T}) = \mathcal{O}$.

Definition

We call a morphism of Feynman categories $i : \mathfrak{F} \to \mathfrak{F}'$ a minimal extension over $\mathcal{C}$ if $\mathfrak{F}$-$\mathcal{O}ps_{\mathcal{C}}$ has a a terminal/trivial functor $\mathcal{T}$ and $i_*\mathcal{T}$ is a terminal/trivial functor in $\mathfrak{F}'$-$\mathcal{O}ps_{\mathcal{C}}$.

Proposition

If $f : \mathfrak{F} \to \mathfrak{F}'$ is a minimal extension over $\mathcal{C}$, then $f^{\mathcal{O}} : \mathfrak{F}_{dec\mathcal{O}} \to \mathfrak{F}'_{decf_*(\mathcal{O})}$ is as well.

Factorization

Theorem (w/ C. Berger)

Any morphisms of Feynman $f : \mathfrak{F} \to \mathfrak{F}'$ categories factors and a minimal extension followed by a decoration cover.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i} & \mathcal{F}'_{dec\, f_*(\mathcal{T})} \\ & \overset{f}{\searrow} & \downarrow \\ & & \mathcal{F}' \end{array}$$

Enrichment, algebras (modules) There is a construction $\mathfrak{F}^+$ which gives nice enrichments.

Theorem/Definition [paraphrased]

$\mathfrak{F}^+$-$\mathcal{O}ps_{\mathcal{C}}$ are the enrichments of $\mathcal{F}$ (over $\mathcal{C}$). Given $\mathcal{O} \in \mathfrak{F}^+$-$\mathcal{O}ps_{\mathcal{C}}$ we denote by $\mathfrak{F}_{\mathcal{O}}$ the enrichment of $\mathfrak{F}$ by $\mathcal{O}$.

$$Hom_{\mathcal{F}_{\mathcal{O}}}(X, Y) = \bigoplus_{\phi \in Hom_{\mathcal{F}}(X,Y)} \mathcal{O}(\phi)$$

By definition the $\mathcal{F}_{\mathcal{O}}$-$\mathcal{O}ps_{\mathcal{E}}$ will be the algebras (modules) over $\mathcal{O}$.

Examples

$Tr^+ = \mathcal{S}urj$ (non-symmetric)/Modules

A an algebra then $\mathcal{T}r_A^+$ has objects $\underline{n}$ with $Hom(\underline{n},\underline{n}) = A^{\otimes n}$ and hence we see that the $\mathcal{O}ps$ are just modules over A.

$\mathcal{S}urj^+ = \mathfrak{F}_{May}$/algebras over operads

$Hom_{\mathcal{S}urj_{\mathcal{O}}}(\underline{n},\underline{1}) = \mathcal{O}(n)$. Composition of morphisms $\underline{n} \stackrel{f}{\to} \underline{k} \to \underline{1}$

$$\gamma : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \to \mathcal{O}(n)$$

where $n_i = |f^{-1}(i)|$.
So $\mathcal{O}ps$ are algebras over the operad $\mathcal{O}$.

Master equations

General story [KWZn15]

The Feynman transform is quasi–free. An algebra over $F\mathcal{O}$ is dg–if and only if it satisfies an associated master equestion.

Examples [Gerstenhaber, Kapranov-Manin, Merkulov-Valette, Barannikov, KWZ]

Name of $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$	Algebraic Structure of $F\mathcal{O}$	Master Equation (ME)
operad	odd pre-Lie	$d(S) + S \circ S = 0$
cyclic operad	odd Lie	$d(S) + \frac{1}{2}[S, S] = 0$
modular operad	odd Lie + Δ	$d(S) + \frac{1}{2}[S, S] + \Delta(S) = 0$
properad	odd pre-Lie	$d(S) + S \circ S = 0$
wheeled properad	odd pre-Lie + Δ	$d(S) + S \circ S + \Delta(S) = 0$
wheeled prop	dgBV	$d(S) + \frac{1}{2}[S, S] + \Delta(S) = 0$

Extra Page

Free space

Next steps

Some things going on

- Non–connected case for bi–algebra.
- B_+ operator, many discussions with Dirk: thank you!
- Connect to Rota–Baxer, Dynkin-operators, ...
- Connect to quiver theories and to stability conditions. Wall crossing corresponds to contracting and expanding an edge. item Connect to Tannakian categories. E.g. find out the role of fibre functors or special large/small object.
- Truncation/bordification connect to old constructions.
- Understand motivic coaction in this framework more precisely.
- ...

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The end

Thank you!

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