Renormalization Hopf algebras and gauge theories

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The story begins...

- *On the Hopf algebra structure of perturbative quantum field theories* [Kreimer, ATMP 1998]:

  > *"We show that the process of renormalization encapsules a Hopf algebra structure in a natural manner."*

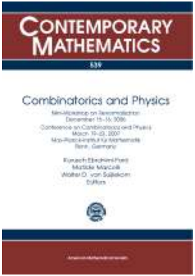
- Then quickly followed by the works of Connes and Kreimer on Renormalization Hopf algebras and Birkhoff decomposion.
- From the start, it was clear that the Hopf algebraic structure of renormalization for gauge theories is quite rich [Broadhurst–Kreimer, Kreimer–Delbourgo 1999]

- This gained in momentum with *Anatomy of a gauge theory* [Kreimer, 2006] containing the following closed expression for the coproduct on Green's functions in terms of the grafting operator:

$$\Delta(B_+^{k;r}(X_{k,r})) = B_+^{k;r}(X_{k,r}) \otimes \mathrm{I} + (\mathrm{id} \otimes B_+^{k;r})\Delta(X_{k,r})$$

- This gauge theory theorem is crucially based on Slavnov–Taylor identities, formed the basis for much research [Kreimer–Yeats 2006, vS, ...]

- Meanwhile, as a postdoc at MPI Bonn (meeting Kurusch there) I started to work on the Hopf algebra of Feynman graphs in QED [vS 2006].



- 'First contact' with Dirk, and somewhat later, I managed to prove that Slavnov–Taylor identities generate Hopf ideals, expressing compatibility of renormalization with gauge symmetries.
- For me, this was the start of a fruitful period of interaction and collaboration with Dirk...

IHÉS 2009

IHÉS 2010

Berlin!

Berlin!

The Les Houches schools

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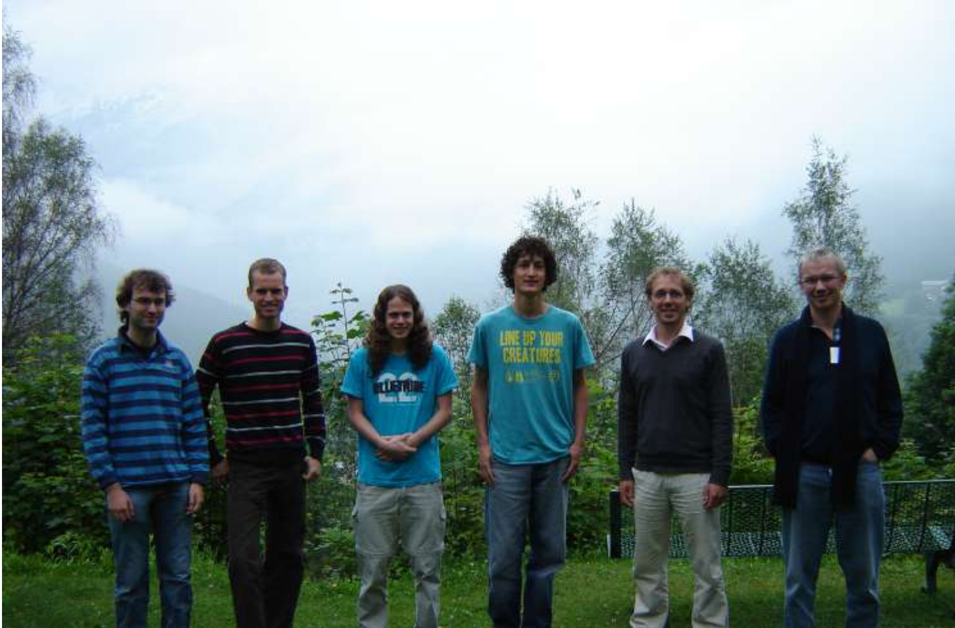
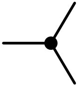


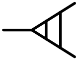
Feynman graphs

Graphs built from a fixed set $\{v_1, \ldots, v_k\}$ of types of vertices (possibly $k = \infty$ [Bloch–Kreimer]) and a fixed set $\{e_1, \ldots, e_N\}$ of types of edges.

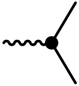
Examples:

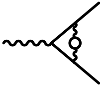
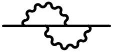
- Scalar ϕ^3-theory:

  vertex:  , edge: .

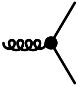
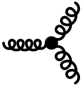
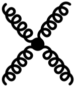
  and one constructs graphs such as  , 

- Electrodynamics:

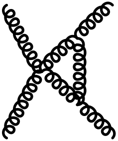
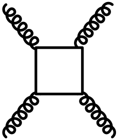
  vertex:  , edges: , .

  and one constructs graphs such as  , 

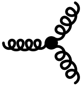
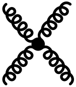
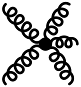
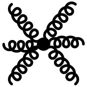
- Yang–Mills theory:

vertices:  ,  ,  , $\cdots$

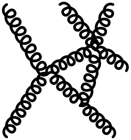
edges: ,  , $\cdots$

and one constructs graphs such as  , 

- (Gravity):

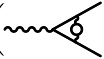
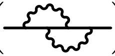
vertices:  ,  ,  ,  , $\cdots$

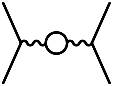
edges:  , $\cdots$

and one constructs graphs such as 

Hopf algebra of Feynman graphs

Define:

- One-particle irreducible graphs (example **not 1PI**: )
- Residue of a graph: $\mathrm{res}\left(\right.$  $\left.\right) =$  and $\mathrm{res}\left(\right.$  $\left.\right) =$
- M the free commutative algebra generated by *all* Feynman graphs (given the set R) including trees.
- $H \subset M$ the subalgebra generated by all 1PI graphs with residue in R.

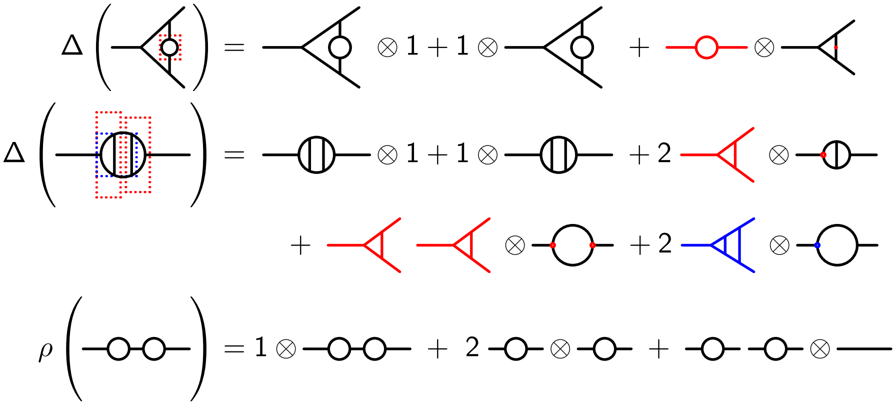
  Eg. a graph in M but not in H: 

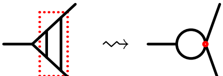
Consider the map $\rho : M \to H \otimes M$ defined by $\rho(\Gamma) = \sum_{\emptyset \subseteq \gamma \subseteq \Gamma} \gamma \otimes \Gamma/\gamma$ where the sum is over (disjoint unions of) 1PI subgraphs with residue v_i or e_j.
Then

- $\Delta := \rho|_H$ and $\epsilon(\Gamma) = \delta_{\Gamma,\emptyset}$ makes H a Hopf algebra [Connes–Kreimer]
- For this coproduct, M is a left H-comodule algebra.

Examples of the coaction with $v =$  and $e =$

$$\boxed{\rho(\Gamma) = \textstyle\sum_{\emptyset \subseteq \gamma \subseteq \Gamma} \gamma \otimes \Gamma/\gamma \textbf{ and } \Delta = \rho|_H}$$



not allowed: 

Renormalization as a decomposition in G

- The above Hopf algebra H is the algebraic structure underlying the recursive procedure of renormalization.
- In fact, for a character $U_z : M \to \mathbb{C}$, there exists a character $C_z : H \to \mathbb{C}$ ('counterterm') defined for $z \neq 0$ as

$$C_z(\Gamma) = -T\left[U_z(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C_z(\gamma) U_z(\Gamma/\gamma)\right]$$

  with T (eg.) the projection onto the pole part, so that $R_z = C_z * U_z$ is finite at $z = 0$ [Connes and Kreimer 2000].
- Even though C_z is defined only on H, the map R_z is defined as a map from $M \to \mathbb{C}$: it gives the renormalized Feynman rules on all Feynman graphs.

Gauge theories

- The physical (renormalized) 1PI Green's functions are given by

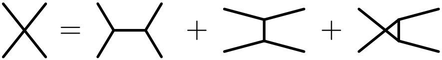
$$\phi_r(p,\mu,\alpha,\pm,\dots)\; R_{z=0}(G^r)(p,\mu,\alpha,\pm,\dots)$$

with $r = v_i, e_j$ and ϕ_r the corresponding formfactors (depending on momenta, Lorentz and spinor indices, chiralities *et cetera*) and

$$G^{v_i} = 1 + \sum_{\mathrm{res}(\Gamma)=v_i} \frac{\Gamma}{|\mathrm{Aut}(\Gamma)|} \in H, \qquad G^{e_j} = 1 - \sum_{\mathrm{res}(\Gamma)=e_j} \frac{\Gamma}{|\mathrm{Aut}(\Gamma)|} \in H$$

- Gauge symmetries imply certain identities between these formfactors, such as in pure Yang–Mills theories:

$$\text{“}\,\phi_{\times} = \phi_{\prec} \frac{1}{\phi_{\sim}} \phi_{\prec}\,\text{”}, \text{ or}$$



- For renormalizability of gauge theories it is essential for these identities to hold at any loop order: the Slavnov–Taylor identities for the couplings

$$R_{z=0}\left(G^{\times} G^{\sim}\right) = R_{z=0}\left((G^{\prec})^2\right), \quad \dots$$

- Thus, we first need an expression for the coproduct on the G^r's.

Structure of H

Gradings

- Grading by loop number $l(\Gamma) = h^1(\Gamma)$:

$$H = \bigoplus_{l \in \mathbb{Z}_{\geq 0}} H^l, \qquad q_l : H \to H^l$$

- Multigrading by number of vertices:

$$d_i(\Gamma) = \#\text{vertices } v_i \text{ in } \Gamma - \delta_{v_i, \mathrm{res}(\Gamma)}$$

with

$$H = \bigoplus_{n_1, \ldots, n_k \in \mathbb{Z}^k} H^{n_1, \ldots, n_k}, \qquad p_{n_1, \ldots, n_k} : H \to H^{n_1, \ldots, n_k}$$

- These are related via $\sum_{i=1}^k (\mathrm{val}(v_i) - 2) d_i = 2l$.

N.B. Connected Hopf algebra: $H^0 = H^{0,\ldots,0} = \mathbb{C}1$.

Structure of H

Hopf subalgebras

Example: scalar ϕ^3-theory (with one type of vertex $v = \prec$ and one type of edge $e = —$) :

Proposition

The elements $X = G^v (G^e)^{-3/2}$ and G^e generate a Hopf subalgebra in H:

$$\Delta(X) = \sum_{l=0}^{\infty} X^{2l+1} \otimes q_l(X), \qquad \Delta(G^e) = \sum_{l=0}^{\infty} G^e X^{2l} \otimes q_l(G^e)$$

- This is recognized as the Hopf algebra dual to (a subgroup of) the group $\mathbb{C}[[\lambda]]^\times \rtimes \overline{\mathrm{Diff}}(\mathbb{C}, 0)$. Namely, a character ϕ on this Hopf subalgebra defines
  1. An invertible formal power series by $\sum_{l=0}^{\infty} \phi(q_l(G^e))\lambda^l$
  2. A formal diffeomorphism on $\mathbb{C}$ by $\lambda \mapsto \sum_{l=0}^{\infty} \phi(q_l(X))\lambda^{l+1}$.

Structure of H

Hopf subalgebras and ideals

- In general (vertices $\{v_1, \ldots, v_k\}$ and edges $\{e_1, \ldots, e_N\}$), we define for each vertex v

$$X_v := \left(\frac{G^v}{\prod_i (G^{e_j})^{\mathrm{val}_j(v)/2}} \right)^{1/\mathrm{val}(v)-2}$$

Proposition (vS 2008)

The coproduct on the Green's functions reads

$$\Delta(G^r) = \sum_{n_1,\ldots,n_k} G^r (X_{v_1})^{n_1(\mathrm{val}(v_i)-2)} \cdots (X_{v_k})^{n_k(\mathrm{val}(v_k)-2)} \otimes p_{n_1,\ldots,n_k}(G^r),$$

- On the elements X_{v_i} we then have

$$\Delta(X_v) = \sum_{n_1,\ldots,n_k} X_v (X_{v_1})^{n_1(\mathrm{val}(v_i)-2)} \cdots (X_{v_k})^{n_k(\mathrm{val}(v_k)-2)} \otimes p_{n_1,\ldots,n_k}(X_v),$$

  Thus, X_v and G^e (equivalently, G^v and G^e) for all vertices v and edges e generate a Hopf subalgebra, when restricted to each multidegree...

Structure of H

Hopf subalgebras and ideals

Theorem (vS 2008)

1. *The elements G^{v_i} and G^{e_j} generate a Hopf subalgebra H' in H with dual group*

$$\boxed{G := \mathrm{Hom}_{\mathbb{C}}(H', \mathbb{C}) \subset \left(\mathbb{C}[[\lambda_1, \ldots, \lambda_k]]^\times\right)^N \rtimes \overline{\mathsf{Diff}}(\mathbb{C}^k)}$$

2. *The ideal $J := \langle X_{v_i} - X_{v_j} \rangle$ in H' is a Hopf ideal, i.e. H'/J is a Hopf algebra with dual group*

$$\boxed{\mathrm{Hom}_{\mathbb{C}}(H'/J, \mathbb{C}) \subset \left(\mathbb{C}[[\lambda]]^\times\right)^N \rtimes \overline{\mathsf{Diff}}(\mathbb{C})}$$

- The relations $X_{v_i} = X_{v_j}$ in the quotient Hopf algebra H'/J are called (generalized) Slavnov–Taylor identities for the couplings.

Hochschild cohomology of Hopf algebras

- Let H be a bialgebra and M an H-bicomodule, with cocommuting left and right coactions $\rho_L : M \to H \otimes M$ and $\rho_R : M \to M \otimes H$.
- Denote by $C^n(H, M)$ the space of linear maps $\phi : M \to H^{\otimes n}$
- The Hochschild coboundary map $b : C^n(H, M) \to C^{n+1}(H, M)$ is

$$b\phi = (\mathrm{id} \otimes \phi)\rho_L + \sum_{i=1}^{n} (-1)^n \Delta_i \phi + (-1)^{n+1} (\phi \otimes \mathrm{id})\rho_R.$$

where Δ_i denotes the application of the coproduct on the i'th factor.

Definition

The Hochschild cohomology $HH^\bullet(H, M)$ of the bialgebra H with values in M is defined as the cohomology of the complex $(C^\bullet(H, M), b)$.

Hochschild cohomology group $HH_\epsilon^\bullet(H)$

- $M = H$ as a comodule over itself, with $\rho_L = \Delta$ but with $\rho_R = (\mathrm{id} \otimes \epsilon)\Delta$ [Connes–Kreimer 1998]
- For example, $\phi \in HH_\epsilon^1(H)$ means:

$$\Delta\phi = (\mathrm{id} \otimes \phi)\Delta + (\phi \otimes \mathrm{I}).$$

  where $(\phi \otimes \mathrm{I})(h) \equiv \phi(h) \otimes 1$ for $h \in H$
- The (suitably normalized) grafting operator $B_+^\gamma : H \to H$ inserting graphs into a primitive graph γ satisfies [Kreimer 2006, vS 2011]

$$\Delta(B_+^\gamma(X_{k,r})) = B_+^\gamma(X_{k,r}) \otimes \mathrm{I} + (\mathrm{id} \otimes B_+^\gamma)\Delta(X_{k,r})$$

  where $X_{k,r} = G^r(X_v)^{2k} \in H/J$, independent of the choice of $v = \mathrm{res}\gamma$.

Dirk's gauge theory theorem

We define a linear map $B_+^{k;r} : H \to H$ by

$$B_+^{k;r} = \sum_{\substack{\gamma \text{ prim} \\ l(\gamma)=k,\ \mathrm{res}(\gamma)=r}} \frac{1}{\mathrm{Sym}(\gamma)} B_+^{\gamma}$$

Theorem (Kreimer 2005)

In the quotient Hopf algebra H/J, the following hold

1. $G^r = \sum_{k=0}^{\infty} B_+^{k;r}(X_{k,r})$
2. $\Delta(B_+^{k;r}(X_{k,r})) = B_+^{k;r}(X_{k,r}) \otimes \mathrm{I} + (\mathrm{id} \otimes B_+^{k;r})\Delta(X_{k,r})$.
3. $\Delta(G_k^r) = \sum_{j=0}^{k} \mathrm{Pol}_j^r(G) \otimes G_{k-j}^r$.

where $\mathrm{Pol}_j^r(G)$ is a polynomial in the G_m^r of degree j, determined as the order j term in the loop expansion of $G^r(X_v)^{2k-2j}$.

Dirk's unexpected influence on NCG

- Alain Connes' noncommutative geometry is based on a (noncommutative) algebra of coordinates $\mathcal{A}$ and a (generalization of a) Dirac operator D, both acting on a Hilbert space $\mathcal{H}$:

$$(\mathcal{A}, \mathcal{H}, D)$$

- Key example: $(C_0^\infty(\mathbb{R}^4), L^2(\mathbb{R}^4) \otimes \mathbb{C}^4, \partial\!\!\!/)$.
- Gauge fields are derived by inner fluctuations:

$$D \mapsto D + V; \qquad V = \sum_j a_j [D, b_j]$$

- Spectral action functional :

$$\mathrm{tr}\ f(D + V)$$

- My own quest: understand its structure and renormalizability properties

Expansion of the spectral action

Ongoing work with Teun van Nuland

- It turns out that we can write [vS 2011]

$$\mathrm{tr}\ f(D+V) - \mathrm{tr}\ f(D) = \sum_{n\geq 1} \frac{1}{n}\frac{1}{2\pi i}\mathrm{tr}\ \oint f'(z)\left(V(z-D)^{-1}\right)^n$$

- Let us write this in terms of

$$\langle V_1, V_2, \ldots, V_n\rangle = \frac{1}{2\pi i}\mathrm{tr}\ \oint f'(z)\prod_j \left(V_j(z-D)^{-1}\right)$$

and then Hochschild cochains $A^{n+1} \to \mathbb{C}$:

$$\phi_n(a^0, a^1, \ldots, a^n) = \langle a^0[D, a^1], [D, a^2], \ldots, [D, a^n]\rangle.$$

Lemma

We have $b\phi_n = \phi_{n+1}$ for odd n and we have $b\phi_n = 0$ for even n.

Hochschild cocycles

- For the first few terms in the expansion we find

$$\langle a[D,b]\rangle = \int_{\phi_1} A$$

$$\langle a[D,b], a[D,b]\rangle = \int_{\phi_2} A^2 + \int_{\phi_3} AdA$$

$$\langle a[D,b], a[D,b], a[D,b]\rangle = \int_{\phi_3} A^3 + \int_{\phi_4} AdAA + \int_{\phi_5} AdAdA$$

$$\langle a[D,b], a[D,b], a[D,b], a[D,b]\rangle = \int_{\phi_4} A^4 + \int_{\phi_5} (A^3 dA + AdAA^2) + \cdots$$

  with $A = adb$ the universal differential 1-form corresponding to $a[D,b]$.
- This can be recollected as

$$\frac{1}{2}\int_{\phi_2} \left(dA + A^2\right) + \frac{1}{4}\int_{\phi_4} \left(dA + A^2\right)^2 + \cdots$$

Hochschild and cyclic cocycles

- Also the remaining terms can be put in a nice form:

$$\mathrm{tr}\,(f(D+V)-f(D)) = \int_{\psi_1} A + \frac{1}{2}\int_{\phi_2}(dA+A^2) + \frac{1}{2}\int_{\psi_3}(AdA+\tfrac{2}{3}A^3)$$
$$+\frac{1}{4}\int_{\phi_4}(dA+A^2)^2 + \frac{1}{6}\int_{\psi_5}(A(dA)^2+\tfrac{3}{2}A^3dA+\tfrac{3}{5}A^5) + \frac{1}{6}\int_{\phi_6}(dA+A^2)^3$$
$$+\cdots$$

  where $V = a[D,b]$ and $A = adb$.
- The universal curvature 2-form $F = dA + A^2$ appear as Yang–Mills terms F, F^2, F^3, integrated against even Hochschild cocycles ϕ_2, ϕ_4, ϕ_6.
- We find Chern–Simons 1, 3 and 5-forms, integrated against odd cyclic cocycles ψ_1, ψ_3, ψ_5.
- An early instance of such an expression has been found for the scale-invariant part of the spectral action [Chamseddine–Connes 2006].

Hochschild and cyclic cocycles

This structure of the spectral action functional persists at all orders!

Definition

The Chern–Simons form of degree $2n-1$ is given by

$$cs_{2n-1}(A) = \int_0^1 A(F_t)^{n-1} dt; \qquad F_t = tdA + t^2 A^2.$$

Theorem

In terms of the universal curvature 2-form $F = dA + A^2$ of A we have

$$\mathrm{tr}\,(f(D+V) - f(D)) \sim \sum_{k=0}^{\infty} \left(\int_{\psi_{2k+1}} cs_{2k+1}(A) + \frac{1}{2k+2} \int_{\phi_{2k+2}} F^{k+1} \right)$$

where (ψ_{2k+1}) is a cyclic cocycle.

Gauge structure of the spectral action

in progress..

$$\mathrm{tr}\,\left(f(D+V)-f(D)\right)\sim\sum_{k=0}^{\infty}\left(\int_{\psi_{2k+1}}\mathrm{cs}_{2k+1}(A)+\frac{1}{2k+2}\int_{\phi_{2k+2}}F^{k+1}\right)$$

- This simple structure of the spectral action in terms of Chern–Simons and Yang–Mills forms, integrated against cyclic and Hochschild cocycles, respectively, invites for a study of the gauge structure
- BRST-analysis: gauge invariance of the counterterms

...with many thanks to Dirk for his continuing inspiration!