WWU MÜNSTER

Solution of ϕ^4_4 on the Moyal Space

Alexander Hock, joint work with Harald Grosse and Raimar Wulkenhaar

arXiv:1906.04600 and arXiv:1908.04543

Algebraic Structures in Perturbative Quantum Field Theory

November 19, 2020

Outline

1. Quartic matrix model and the Moyal space
2. Renormalized 2-point function
3. Nontriviality on the 4D Moyal space

Quartic Matrix Model

Let H_N be the space of Hermitian $(N \times N)$-matrices, $E \in H_N$ positive with eigenvalues $(E_1, ..., E_N)$. The matrix E should be understood as **Laplacian** in momentum space. Define the partition function

$$\mathcal{Z} = \int_{H_N} d\phi \exp\Big[-N\mathrm{Tr}\big(E\phi^2 + \frac{\lambda}{4}\phi^4\big)\Big].$$

Quartic Matrix Model

Let H_N be the space of Hermitian $(N \times N)$-matrices, $E \in H_N$ positive with eigenvalues $(E_1, ..., E_N)$. The matrix E should be understood as **Laplacian** in momentum space. Define the partition function

$$\mathcal{Z} = \int_{H_N} d\phi \exp\Big[-N\mathrm{Tr}\big(E\phi^2 + \frac{\lambda}{4}\phi^4\big)\Big].$$

The **2-point correlation function** is by definition

$$G_{pq} := N\langle\phi_{pq}\phi_{qp}\rangle = \frac{N\int_{H_N} d\phi\, \phi_{pq}\phi_{qp} \exp\Big[-N\mathrm{Tr}\big(E\phi^2 + \frac{\lambda}{4}\phi^4\big)\Big]}{\int_{H_N} d\phi \exp\Big[-N\mathrm{Tr}\big(E\phi^2 + \frac{\lambda}{4}\phi^4\big)\Big]}.$$

From Moyal Space to Matrix Model

The action of the **noncommutative** ϕ^4_4 **QFT** on the Moyal space is

$$S[\phi] := \frac{1}{8\pi}\int_{\mathbb{R}^4} dx \left(\frac{1}{2}\phi\left(-\Delta + \Omega^2\|2\Theta^{-1}x\|^2 + \mu^2\right)\phi + \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\right)(x),$$

where Δ is the Laplacian, μ the mass, λ the coupling constant and a regulator $\Omega \in \mathbb{R}$.

From Moyal Space to Matrix Model

The action of the **noncommutative** ϕ^4_4 **QFT** on the Moyal space is

$$S[\phi] := \frac{1}{8\pi}\int_{\mathbb{R}^4} dx \left(\frac{1}{2}\phi\left(-\Delta + \Omega^2\|2\Theta^{-1}x\|^2 + \mu^2\right)\phi + \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\right)(x),$$

where Δ is the Laplacian, μ the mass, λ the coupling constant and a regulator $\Omega \in \mathbb{R}$.
The **Moyal $\star$-product** has a matrix base, which leads with renormalization constants to

$$S[\phi] = N\left(\sum_{n,m=1}^{N} E_n\, Z\phi_{nm}\phi_{mn} + \frac{\lambda_{bare}Z^2}{4}\sum_{n,m,k,l=1}^{N} \phi_{nm}\phi_{mk}\phi_{kl}\phi_{ln}\right).$$

The eigenvalues have multiplicities $(E_1, .., E_N) = (\underbrace{e_1, .., e_1}_{r_1}, ..., \underbrace{e_d, ..., e_d}_{r_d})$,

where the **distinct eigenvalues** are $e_n = \frac{\mu^2_{bare}}{2} + \frac{n}{\sqrt{N}}$ and $r_n = n$

Renormalized 2-Point Dyson-Schwinger Equation

The planar 2-point function obeys in a formal N expansion the **nonlinear equation**

$$\Big(\mu_{bare}^2+\frac{p}{\sqrt{N}}+\frac{q}{\sqrt{N}}+\frac{\lambda_{bare}}{N}\sum_m mZG_{pm}\Big)ZG_{pq}=1+\frac{\lambda_{bare}}{N}\sum_m mZ\frac{G_{mq}-G_{pq}}{\frac{m}{\sqrt{N}}-\frac{p}{\sqrt{N}}}.$$

Renormalized 2-Point Dyson-Schwinger Equation

The planar 2-point function obeys in a formal N expansion the **nonlinear equation**

$$\Big(\mu_{bare}^2+\frac{p}{\sqrt{N}}+\frac{q}{\sqrt{N}}+\frac{\lambda_{bare}}{N}\sum_m mZG_{pm}\Big)ZG_{pq}=1+\frac{\lambda_{bare}}{N}\sum_m mZ\frac{G_{mq}-G_{pq}}{\frac{m}{\sqrt{N}}-\frac{p}{\sqrt{N}}}.$$

Perform the **continuum limit** (a scaling limit of the N and the number of eigenvalues with constant ratio Λ^2) with $\frac{p}{\sqrt{N}}\mapsto x\in[0,\Lambda^2]$ and $G_{pq}\mapsto G(x,y)$

$$\Big(\mu_{bare}^2+x+y+\frac{\lambda_{bare}}{N}Z\int_0^{\Lambda^2}dt\,t\,G(x,t)\Big)ZG(x,y)=1+\frac{\lambda_{bare}}{N}Z\int_0^{\Lambda^2}dt\,t\,\frac{G(x,t)-G(x,y)}{t-x},$$

where $\mu_{bare},\lambda_{bare},Z$ depend on Λ^2, the **cut-off**.

Spectral Dimension

We define the spectral dimension by the asymtotic behaviour of the eigenvalues of E in the scaling limit.
More concretely, let $\varrho(x)dx$ be the the spectral measure, then

$$D := \inf\Big\{p : \int_0^\infty dt \frac{\varrho(t)}{(1+t)^{p/2}} < \infty\Big\}.$$

On the $D = 4$ Moyal space, we have $\varrho(x) = x$
On the $D = 2$ Moyal space, we have $\varrho(x) = 1$

Implicit Equation on the 4D Moyal Space

Remember Raimar's talk, the solution is constructed by the implicitly defined function

$$R(z) := z - \frac{\lambda}{N} \sum_{k=1}^{d} \frac{r_k}{R'(\varepsilon_k)(z+\varepsilon_k)}, \quad e_n = R(\varepsilon_n), \quad \lim_{\lambda \to 0} \varepsilon_n = e_n.$$

Implicit Equation on the 4D Moyal Space

Remember Raimar's talk, the solution is constructed by the implicitly defined function

$$R(z) := z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{r_k}{R'(\varepsilon_k)(z+\varepsilon_k)}, \qquad e_n = R(\varepsilon_n), \qquad \lim_{\lambda\to 0}\varepsilon_n = e_n.$$

This implicit equation converges in the continuum limit on the 4D Moyal space (after renormalization) to a **linear integral equation**

$$\boxed{R(z) = z - \lambda z^2 \int_0^\infty \frac{dt\, R(t)}{(\mu^2+t)^2(\mu^2+t+z)}}$$

where μ^2 is fixed by boundary conditions.
On $D=4$ Moyal space, we observe the effective spectral measure $d\varrho_\lambda(t) = dt\, R(t)$

Solution of the Integral Equation

Compute with HyperInt the first ten orders of $R(z)=z-\lambda z^2\int_0^\infty \frac{dt\,R(t)}{(\mu^2+t)^2(\mu^2+t+z)}$ and guess for $f(x):=\frac{R(\mu^2x)}{\mu^2x(1+x)}$ the **series expansion**

$$
\begin{aligned}
f(x)=&c_\lambda\frac{\arcsin(\lambda\pi)}{\lambda\pi(1+x)}\sum_{n=0}^\infty \mathrm{Hlog}(x,[\underbrace{0,-1,\dots,0,-1}_{n}])\left(\frac{\arcsin(\lambda\pi)}{\pi}\right)^{2n}\\
&-\lambda c_\lambda\frac{\arcsin(\lambda\pi)^2}{x(\lambda\pi)^2}\sum_{n=0}^\infty \mathrm{Hlog}(x,[-1,\underbrace{0,-1,\dots,0,-1}_{n}])\left(\frac{\arcsin(\lambda\pi)}{\pi}\right)^{2n}
\end{aligned}
$$

where $\mathrm{Hlog}(x,[a_1,..,a_n])=\int_0^x\frac{dy_1}{y_1-a_1}\dots\int_0^{y_{n-1}}\frac{dy_n}{y_n-a_n}$, and the **boundary conditions** have a natural choice $\mu^2=\frac{\arcsin(\lambda\pi)}{\lambda\pi}-\lambda\left(\frac{\arcsin(\lambda\pi)}{\lambda\pi}\right)^2$ and $c_\lambda=\frac{1}{\mu^2}$.

Solution of the Integral Equation

Identify the differential equation of second order of the power series, which is solved by

Theorem (Grosse, AH, Wulkenhaar)

The self-dual ϕ^4 model on the 4D Moyal space (with infinite deformation parameter) has the initial solution

$$R(x) = x\,{}_2F_1\left(\begin{matrix}\alpha_\lambda, & 1-\alpha_\lambda \\ & 2\end{matrix}\,\middle|\, -\frac{x}{\mu^2}\right),$$

where $\alpha_\lambda = \frac{\arcsin(\lambda\pi)}{\pi}$.

From $R(z)$ to $G(x,y)$

Define

$$I(z) := -R\big(-\mu^2 - R^{-1}(z)\big),$$

then is for $x \in \mathbb{C}\setminus\mathbb{R}_+$

$$\underbrace{\left[\mu_{bare}^2 + x + y + \frac{\lambda_{bare}}{N} Z \int_0^{\Lambda^2} dt\, t\left(G(x,t) + \frac{1}{t-x}\right)\right]}_{=y+I(x)} ZG(x,y) = 1 + \frac{\lambda_{bare}}{N} Z \int_0^{\Lambda^2} dt\, t\,\frac{G(x,t)}{t-x}.$$

Notice the hidden structure

$$-I(-I(z)) = R\big(-\mu^2 - R^{-1}(R(-\mu^2 - R^{-1}(z)))\big) \text{ " = " } z.$$

A singular integral equation remains, solution theory is known!

Example on the $D=2$ Moyal space [Panzer,Wulkenhaar '18]

Remeber on $D=2$, $\varrho(x)=1$, then the R-function is simply

$$R(z)=z+\lambda\log(1+z),\qquad R^{-1}(x)=\lambda W\Big(\frac{e^{\frac{1+x}{\lambda}}}{\lambda}\Big)-1,$$

where $W(x)e^{W(x)}=x$ is the Lambert-W function. Consequently,

$$I(z)=-R(-1-R^{-1}(z))=\lambda W\Big(\frac{e^{\frac{1+z}{\lambda}}}{\lambda}\Big)-\lambda\log\Big[1-W\Big(\frac{e^{\frac{1+z}{\lambda}}}{\lambda}\Big)\Big]$$

Found by Panzer and Wulkenhaar on Dirk's and Spencer's conference in **Les Houches 2018**

Back to 4D: Exact Solution of the 2-Point Function

Solving the **singular integral equation** (of Carleman type) yields $G(x,y)=\frac{\mu^2\exp(N(x,y))}{\mu^2+x+y}$

$$
\begin{aligned}
N(x,y) := \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} dt \Big\{ &\log\big(x - R(-\tfrac{\mu^2}{2}-\mathrm{i}t)\big)\frac{d}{dt}\log\big(y - R(-\tfrac{\mu^2}{2}+\mathrm{i}t)\big) \\
&-\log\big(-R(-\tfrac{\mu^2}{2}-\mathrm{i}t)\big)\frac{d}{dt}\log\big(-R(-\tfrac{\mu^2}{2}+\mathrm{i}t)\big) \\
&-\log\big(x-(-\tfrac{\mu^2}{2}-\mathrm{i}t)\big)\frac{d}{dt}\log\big(y-(-\tfrac{\mu^2}{2}+\mathrm{i}t)\big) \\
&+\log\big(-(-\tfrac{\mu^2}{2}-\mathrm{i}t)\big)\frac{d}{dt}\log\big(-(-\tfrac{\mu^2}{2}+\mathrm{i}t)\big)\Big\},
\end{aligned}
$$

Spectral Dimension of ϕ^4_4

The asymptotic of the hypergeometric functions

$$ {}_2F_1\left({a,\; 1-a \atop 2} \middle| -x \right) \overset{x\to\infty}{\sim} \frac{1}{x^a}. $$

Spectral Dimension of ϕ^4_4

The asymptotic of the hypergeometric functions

$$ {}_2F_1\left({a,\ 1-a \atop 2}\middle| -x\right) \overset{x\to\infty}{\sim} \frac{1}{x^a}. $$

The R-function defines an **effective measure**, which behaves asymptotically

$$ R(x) = x\, {}_2F_1\left({\alpha_\lambda,\ 1-\alpha_\lambda \atop 2}\middle| -\frac{x}{\mu^2}\right) \overset{x\to\infty}{\sim} x^{1-\alpha_\lambda}, $$

where $\alpha_\lambda = \frac{\arcsin(\lambda\pi)}{\pi}$.

Finally, the **spectral dimension** D has the asymptotics $x^{\frac{D}{2}-1} \quad \rightarrow \quad D = 4 - 2\frac{\arcsin(\lambda\pi)}{\pi}$.

Why does it avoid the Triviality Problem?

The inverse R^{-1} is an essential ingredient for the exact solution!
Would instead the solution be constructed by

$$\tilde{R}(x) = x - \lambda x^2 \int_0^\infty \frac{d\varrho_0(t)}{(\mu^2+t)^2(\mu^2+t+x)}, \qquad d\varrho_0(t) = dt\, t$$

$\Rightarrow$ no inverse exists globally on $\mathbb{R}_+$
$\Rightarrow$ $\tilde{R}$ has an upper bound behaving at $x_{max} = K \cdot e^{\frac{1}{\lambda}}$

Why does it avoid the Triviality Problem?

The inverse R^{-1} is an essential ingredient for the exact solution!
Would instead the solution be constructed by

$$\tilde{R}(x) = x - \lambda x^2 \int_0^\infty \frac{d\varrho_0(t)}{(\mu^2+t)^2(\mu^2+t+x)}, \qquad d\varrho_0(t) = dt\, t$$

$\Rightarrow$ no inverse exists globally on $\mathbb{R}_+$
$\Rightarrow$ $\tilde{R}$ has an upper bound behaving at $x_{max} = K \cdot e^{\frac{1}{\lambda}}$

The function $R(x)$ has a global inverse on $\mathbb{R}_+$!
The effective dimension drop is only **visible** on the level of exact solutions
Not accessible with perturbation theory!

Open Questions

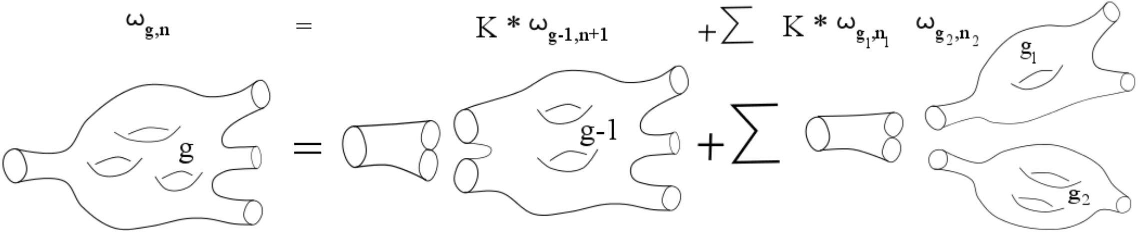
- The algebraic structure for finite N is in the continuum limit no longer algebraic
- **(Blobbed) Topological Recursion** on the 4D Moyal space (Raimar's talk)?
- Structure of generating series of iterated integrals
- **(Galois) coaction** on the correlation function $\rightarrow$ closeness condition?
  $\Rightarrow$ only possible with exact solutions

Thank you

Back-ups

Topological Recursion

The Euler-characteristic is $\chi = 2 - 2g - n$. Computing **meromorphic forms** $\omega_{g,n}$ by



The Initial Data $\omega_{0,1}$ and $\omega_{0,2}$ gives recursively in χ all other $\omega_{g,n}$

Topological Recursion

More precisely, let $J=\{z_1,..,z_{n-1}\}$, $\omega_{0,1}(z)=y(z)dx(z)$, $\omega_{0,2}(z_0,z_1)$ the fundamental of the second kind and $K(z_0,z)$ given by $\omega_{0,1},\omega_{0,2}$, then

$$\boxed{\omega_{g,n}(z_0,...,z_{n-1})=\operatorname*{Res}_{z\to a_i}\Big[K(z_0,z)\Big(\omega_{g-1,n+1}(z,\sigma_i(z),J)+\sum_{\substack{h+h'=g\\ I\uplus I'=J}}^{\prime}\omega_{h,|I|+1}(z,I)\omega_{h',|I'|+1}(\sigma_i(z),I')\Big)\Big],}$$

with $dx(z)=0\;\to\;z=a_i$ and $x(z)=x(\sigma_i(z))$ around a_i with $\sigma_i\neq\mathrm{id}$

$$K_i(z,q)=\frac{\frac{1}{2}\int_{\sigma_i(q)}^{q}\omega_{0,2}(z,q')}{\omega_{0,1}(q)-\omega_{0,1}(\sigma_i(q))}$$

$F(x,y)=0$ parametrized by $x(z),y(z)$ is called the **spectral curve**

Blobbed Topological Recursion

Decomposition

$$\omega_{g,n+1}(z,z_1,...,z_n) = \mathcal{H}_z\omega_{g,n+1}(z,z_1,...,z_n) + \mathcal{P}_z\omega_{g,n+1}(z,z_1,...,z_n),$$

where $\mathcal{P}_z\omega_{g,n+1}(z,z_1,...,z_n)$ is computed with the formula of TR.

We identify

$$x(z) = R(z), \quad y(z) = -R(-z), \quad \omega_{0,2}(u,z) = dz\,du\left(\frac{1}{(u-z)^2} + \frac{1}{(u+z)^2}\right).$$

For the correlation function, one needs the **inverse** $z = R^{-1}(x)$.
The $\omega_{g,n}$'s are **linear combinations** of our renormalized correlation functions.