Random loops and T-algebras

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Stochastic quantisation

Basic idea: Consider discrete approximation to "Euclidean QFT" $e^{-\beta S(\varphi)}\, D\varphi$ so φ belongs to finite-dimensional vector space. This is invariant for stochastic evolution

$$d\varphi = -\nabla S(\varphi)\, dt + \sqrt{2/\beta}\, dW\ ,$$

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$1D$ σ-model: Field configurations given by loops on Riemannian manifold: $u\colon S^1 \to \mathcal{M}$, $S(u) = \int_{S^1} g_u(\partial_x u, \partial_x u)\,dx$, usual Dirichlet energy.

Formal Gibbs measure

Brownian loop measure on manifold $(\mathcal{M}, g)$ formally given (for some c) by

$$\mathbf{P}(Du) \propto \exp\Big(-\int_{S^1} \big(\frac{1}{2} g_u(\partial_x u, \partial_x u) - cR(u)\big)\, dx\Big) \text{ “}Du\text{”} .$$

Scalar curvature

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$$\boxed{\partial_t u = \nabla_{\partial_x u} \partial_x u + c \nabla R(u) + \sqrt{2g(u)}\xi \ .}$$

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In local coordinates

$$\boxed{(\partial_t - \partial_x^2) u^\alpha = \Gamma^\alpha_{\beta\gamma}(u)\, \partial_x u^\beta \partial_x u^\gamma + c g^{\alpha\beta}(u) \partial_\beta R(u) + \sqrt{2} \sigma_i^\alpha(u) \xi_i \;,}$$

with $\sigma_i^\alpha \sigma_i^\beta = g^{\alpha\beta}$, Γ Christoffel symbols for Levi-Civita.

A general result

Given $H \in \mathcal{C}^\infty(\mathbf{R}^d, \mathbf{R}^d)$, write $U^\varepsilon(\Gamma, \sigma, H)$ for some (formal) ε-approximation to

$$(\partial_t - \partial_x^2)u^\alpha = \Gamma^\alpha_{\beta\gamma}(u)\, \partial_x u^\beta \partial_x u^\gamma + H^\alpha(u) + \sigma_i^\alpha(u)\xi_i\ ,$$

RST yields a collection $\mathcal{S} = \{$  $, \ldots\}$ of **54** symbols and a valuation map $\Upsilon_{\Gamma,\sigma} \colon \mathcal{S} \to \mathcal{C}^\infty(\mathbf{R}^d, \mathbf{R}^d)$ s.t.:

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Comodule structure for two Hopf algebras, allowing to recenter in probability space (renormalisation) and in real space (Taylor-like expansions).

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$\Upsilon_{\Gamma,\sigma}(\bullet\bullet) = \sum_i \sigma_i^\alpha \sigma_i^\beta$, $\Upsilon_{\Gamma,\sigma}(\vee) = \Gamma^\gamma_{\alpha\beta}$,
incoming line = derivative,
joining lines = contraction of indices.

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Theorem A (H., Bruned, Chandra, Chevyrev, Zambotti): For every choice of Γ, σ, H and every truncation of heat kernel there exist constants $C_\varepsilon^{\text{BPHZ}} \in \langle \mathcal{S} \rangle$ such that

$$U(\Gamma, \sigma, H) = \lim_{\varepsilon \to 0} U^\varepsilon(\Gamma, \sigma, H + \Upsilon_{\Gamma,\sigma} C_\varepsilon^{\text{BPHZ}}) \;.$$

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Preservation of symmetries

Metatheorem: If, for some approximation procedure, U^ε satisfies a symmetry, then one can find constants C_ε such that

$$U^{\mathrm{sym}}(\Gamma, \sigma, H) = \lim_{\varepsilon \to 0} U^\varepsilon(\Gamma, \sigma, H + \Upsilon_{\Gamma,\sigma} C_\varepsilon)$$

also satisfies the symmetry in question. (Also $C_\varepsilon - C_\varepsilon^{\mathrm{BPHZ}} \to \mathrm{const.}$)

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1. Yields equivariant ('Stratonovich') solution theories U^{geo} parametrised by a 15-dimensional affine subspace $\mathcal{S}^{\mathrm{geo}}$ of vector fields.
2. Yields ('Itô') solution theories $U^{\mathrm{Itô}}$ satisfying Itô isometry (law depends only on $\sigma_i^\alpha \sigma_i^\beta = g^{\alpha\beta}$) parametrised by a 19-dimensional affine subspace $\mathcal{S}^{\mathrm{Itô}}$.

Itô = Stratonovich!

Theorem (Bruned, Gabriel, H., Zambotti): There exists a *two*-parameter family of solution theories U satisfying both symmetries simultaneously. All of them coincide with existing notions of solution in all previously studied cases.

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Expect $\big(\Upsilon_{\Gamma,\sigma} C_{\varepsilon}\big)(u) = 0$ whenever $\Gamma(u) = 0$ and $(\partial\sigma)(u) = 0$ (pointwise).

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Theorem: There exists a *one*-parameter family of solution theories U satisfying 'equivariance / Stratonovich', 'Itô isometry', and 'minimality'.

Back to geometry

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Explains previously observed fact that different approximations to Brownian bridge measure are of form

$$\exp\Big(-\frac{1}{2}\int \big(g_u(\partial_x u, \partial_x u) + cR(u)\big)\, dt\Big)\, Du$$

for different c's: Onsager-Machlup ($-\frac{1}{6}$), DeWitt ($\frac{1}{6}, -\frac{1}{4}$), Dekker ($\frac{1}{4}$), Inoue, Maeda ($-\frac{1}{6}$), Andersson, Driver ($0, -\frac{1}{3}$), etc. Our choice suggests $c = -\frac{1}{4}$.

Main step in the proof

One shows that 'geometric' and 'Itô' solutions differ by a counterterm in $\mathcal{S}^{\text{both}}$: terms $\tau \in \langle\mathcal{S}\rangle$ such that $(\Upsilon_{\Gamma,\sigma} - \Upsilon_{\Gamma,\bar\sigma})\tau$ is a vector field. One obviously has $\mathcal{S}^{\text{Itô}} + \mathcal{S}^{\text{geo}} \subset \mathcal{S}^{\text{both}}$. Non-trivial fact:

$$\boxed{\mathcal{S}^{\text{both}} = \mathcal{S}^{\text{Itô}} + \mathcal{S}^{\text{geo}}\ .}$$

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For SDEs, $\mathcal{S} = \{\bullet\!\!-\!\!\bullet\}$ and $\Upsilon_\sigma \bullet\!\!-\!\!\bullet = \sigma_i\, D\sigma_i$, so $\mathcal{S}^{\text{Itô}} = \mathcal{S}^{\text{geo}} = 0$. But $(\Upsilon_\sigma - \Upsilon_{\bar{\sigma}}) \bullet\!\!-\!\!\bullet = \nabla_{\sigma_i}\sigma_i - \nabla_{\bar{\sigma}_i}\bar{\sigma}_i$, so $\mathcal{S}^{\text{both}} = \mathcal{S}$!

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For SPDEs, one has $\mathcal{S}_{(2)} = \{$, $\}$ and

$\mathcal{S}^{\text{Itô}}_{(2)} = \langle$$\rangle$, $\mathcal{S}^{\text{geo}}_{(2)} = \langle$ $+$ $\rangle$, $\mathcal{S}^{\text{both}}_{(2)} = \mathcal{S}_{(2)}$.

Much harder to check at level 4, requires systematic approach.

T-algebras

Motivation: abstraction of functions with multiple ‘upper’ and ‘lower’ indices.

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Definition: A T-algebra is a bigraded vector space $\mathcal{V} = \bigoplus\{\mathcal{V}^u_\ell : u, \ell \geq 0\}$ with

- An action of $\mathrm{Sym}(u, \ell) = \mathrm{Sym}(u) \times \mathrm{Sym}(\ell)$ on each $\mathcal{V}^u_\ell$.

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- An associative product $\mathcal{V}_{\ell_1}^{u_1} \times \mathcal{V}_{\ell_2}^{u_2} \to \mathcal{V}_{\ell_1+\ell_2}^{u_1+u_2}$ satisfying

$$B \bullet A = S_{\ell_1,\ell_2}^{u_1,u_2}(A \bullet B)\,, \qquad \alpha_1 A \bullet \alpha_2 B = (\alpha_1 \cdot \alpha_2)(A \bullet B)\,.$$

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- A trace $\mathrm{tr}\colon \mathcal{V}_{\ell+1}^{u+1} \to \mathcal{V}_\ell^u$ with $\mathrm{tr}(A \bullet B) = A \bullet \mathrm{tr}\, B$ (if $\deg B \geq (1,1)$) and

$$\alpha\, \mathrm{tr}\, A = \mathrm{tr}\big((\alpha \cdot \mathrm{id}_1^1)A\big)\ , \qquad \mathrm{tr}^2 A = \mathrm{tr}^2\left((\mathrm{id}_\ell^u \cdot S_{1,1}^{1,1})A\right)\ .$$

Examples

Canonical example: Given a vector space V, set

$$\mathscr{V}[V]_\ell^u = (V^*)^{\otimes \ell} \otimes V^u \,.$$

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Additional structure: derivation $\partial \colon \mathcal{V}_\ell^u \to \mathcal{V}_{\ell+1}^u$ via

$$L(V, \mathscr{V}[V]_\ell^u) \simeq V^* \otimes \mathscr{V}[V]_\ell^u \simeq \mathscr{V}[V]_{\ell+1}^u \; ,$$

if V finite-dimensional. Satisfies $\partial^2 A = (S_{1,1} \cdot \mathrm{id}_\ell^u)\, \partial^2 A$, plus Leibniz rule and natural interaction with trace and symmetric group.

Free T-algebras

Given by 'T-graphs' with nodes decorated by generators.

Given $W = \bigoplus\{W_\ell^u \,:\, u, \ell \geq 0\}$ with action of symmetric group, generates a T-algebra $\mathrm{Tr}(W)$. Every T-graph g yields a subspace $\mathrm{Tr}_g(W) \subset \mathrm{Tr}(W)$.

Non-degeneracy result

Fix $W = \bigoplus\{W_\ell^u : u, \ell \geq 0\}$ locally finite-dimensional with action of symmetric group, $\hat{W}_\ell^u \subset W_\ell^u$ invariant, and finite collection G of connected anchored T-graphs.

Theorem: There exists V finite-dimensional, $\Phi, \bar{\Phi} \in \mathrm{Hom}(\mathrm{Tr}(W), \mathscr{V}[V])$ injective on $\mathrm{Tr}_G(W)$ such that, for $\tau \in \mathrm{Tr}_G(W)$, $\Phi\tau = \bar{\Phi}\tau$ if and only if $\tau \in \mathrm{Tr}(\hat{W})$.

If W (and therefore $\mathrm{Tr}(W)$) admits a derivation, same holds with $\mathscr{V}[V]$ replaced by $\mathscr{W}[V]$ and $\Phi, \bar{\Phi} \in \mathrm{Hom}_\partial(\mathrm{Tr}(W), \mathscr{W}[V])$.

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Remark: Φ certainly cannot be injective on all of $\mathrm{Tr}(W)$ since $\dim \mathrm{Tr}(W)_\ell^u = \infty$ but $\dim \mathscr{V}[V]_\ell^u < \infty$!

Some open questions

- Minimal dimension required for V in non-degeneracy result?
- More intrinsic "geometric" formulation of solution theory?
- Behaviour in sub-Riemannian case, notion of hypoellipticity?
- Large deviations between closed geodesics?
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Thank you for your attention!