Bogoliubov type recursions for renormalisation in Regularity Structures

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Singular SPDEs

We consider a singular SPDE of the form:

$$\partial_t u - \Delta u = F(u, \nabla u, \xi), \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d,$$

where ξ is a space-time noise. Local expansion of the solution u:

$$u(y) = u(x) + \sum_{\tau \in \mathcal{T}_{\mathbf{eq}}} \Upsilon[\tau](x)(\Pi_x \tau)(y) + R(x,y),$$

where Π_x : *Decorated trees* $\rightarrow$ *recentered iterated integrals*.

- Π_x is a recentering map. With Γ_{xy} such that $\Pi_x \Gamma_{xy} = \Pi_y$, (Π, Γ) is a model.
- $\Pi_x \tau$ could be ill-defined: renormalisation maps M such that $\Pi_x^M = \Pi_x M$.

Dispersive PDEs

We consider nonlinear dispersive equations of the form

$$i\partial_t u(t,x) + \mathcal{L}u(t,x) = p\left(u(t,x), \overline{u}(t,x)\right)$$
$$u(0,x) = v(x), \quad (t,x) \in \mathbb{R}_+ \times \mathbf{T}^d$$

where $\mathcal{L}$ is a differential operator and p is a polynomial nonlinearity.

Resonance scheme U_k^r of order r:

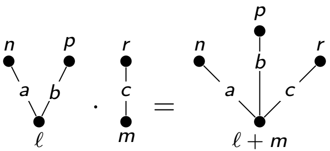
$$U_k^r(t,v) = \sum_{\tau \in \mathcal{V}_k^r} \Upsilon[\tau](v) \left(\hat{\Pi}^r \tau\right)(t)$$

where $\mathcal{V}_k^r$ are decorated trees of order r,
$\hat{\Pi}^r$: *Decorated trees* $\rightarrow$ *approximation of iterated integrals.*

Decorated trees

- $\mathcal{T}$ = set of non-planar rooted trees.
- $\mathfrak{L}$ finite set and $\mathcal{D} = \mathfrak{L} \times \mathbf{N}^{d+1}$.
- Decorated trees $\mathcal{T}^{\mathcal{D}}$ are elements of the form $T_{\mathfrak{e}}^{\mathfrak{n}} = (T, \mathfrak{n}, \mathfrak{e})$ with $T \in \mathcal{T}$, $\mathfrak{n} : N_T \to \mathbf{N}^{d+1}$, $\mathfrak{e} : E_T \to \mathcal{D}$. We set $\mathcal{H} = \langle \mathcal{T}^{\mathcal{D}} \rangle$

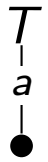
Tree product: $(T, \mathfrak{n}, \mathfrak{e}) \cdot (\bar{T}, \bar{\mathfrak{n}}, \bar{\mathfrak{e}}) = (T \cdot \bar{T}, \mathfrak{n} + \bar{\mathfrak{n}}, \mathfrak{e} + \bar{\mathfrak{e}})$.



Symbolic notations

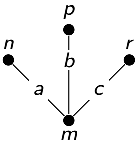
Symbols:

- Grafting operator $\mathcal{I}_a : \mathcal{T}^{\mathcal{D}} \to \mathcal{T}^{\mathcal{D}}$, $T \in \mathcal{T}^{\mathcal{D}}$, one has:

$\mathcal{I}_a(T) =$ 

- Monomial $X^k, k \in \mathbf{N}^{d+1}$ encodes by $\bullet_k$.

Tree product:



$= X^m \mathcal{I}_a(X^n)\mathcal{I}_b(X^p)\mathcal{I}_c(X^r)$

Degree of a tree

We suppose given $|\cdot|_1 : \mathfrak{L} \to \mathbf{R}$ and $|\cdot|_2 : \mathbf{N}^{d+1} \to \mathbf{R}$. Then the degree of a decorated tree $T_{\mathfrak{e}}^{\mathfrak{n}}$ is

$$\deg(T_{\mathfrak{e}}^{\mathfrak{n}}) = \sum_{u \in N_T} |\mathfrak{n}(u)|_2 + \sum_{e \in E_T} \left(|\mathfrak{e}_1(e)|_1 - |\mathfrak{e}_2(e)|_2\right).$$

We consider the space of "positive" decorated trees $\mathcal{T}_+$

$$\mathcal{T}_+^{\mathcal{D}} = \{X^k \prod_i \mathcal{I}_{a_i}(\tau_i) | \deg(\mathcal{I}_{a_i}(\tau_i)) \geq 0\}.$$

with the natural projection $\pi_+ : \mathcal{T}^{\mathcal{D}} \to \mathcal{T}_+^{\mathcal{D}}$, $\mathcal{H}_+ = \langle \mathcal{T}_+^{\mathcal{D}} \rangle$.

Deformed Butcher-Connes-Kreimer coproduct

$$\Delta^+ \mathbf{1} = \mathbf{1} \otimes \mathbf{1}, \quad \Delta^+ X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i,$$
$$\Delta^+ \mathcal{I}_a(\tau) = (\mathcal{I}_a \otimes \mathrm{id})\, \Delta^+ + \sum_{\ell \in \mathbf{N}^{d+1}} \frac{X^\ell}{\ell!} \otimes \mathcal{I}_{a+\ell}(\tau)$$

- Coproduct $\Delta^+ : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, a bigrading is required for the infinite sum (B.-Hairer-Zambotti).
- Deformed plugging + Guin-Oudom procedure (B.-Manchon).
- Coaction $\hat{\Delta}^+ = (\mathrm{id} \otimes \pi_+)\Delta^+ : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_+$, finite subtraction determined by the degree.
- Coproduct $\bar{\Delta}^+ = (\pi_+ \otimes \pi_+)\Delta^+ : \mathcal{H}_+ \to \mathcal{H}_+ \otimes \mathcal{H}_+$.
- Other projections than π_+ in Numerical Analysis (B.-Schratz).

Main properties

Theorem (B., Hairer, Zambotti)

- *There exits an algebra morphism* $\mathcal{A}_+ : \mathcal{H} \to \mathcal{H}$ *such that* $(\mathcal{H}, \cdot, \Delta^+, \mathbf{1}, \mathbf{1}^\star, \mathcal{A}_+)$ *is a Hopf algebra.*
- *There exits an algebra morphism* $\bar{\mathcal{A}}_+ : \mathcal{H}_+ \to \mathcal{H}_+$ *such that* $(\mathcal{H}_+, \cdot, \bar{\Delta}^+, \mathbf{1}, \mathbf{1}^\star, \bar{\mathcal{A}}_+)$ *is a Hopf algebra.*
- $(\mathcal{H}, \hat{\Delta}^+)$ *is a right-comodule for* $\mathcal{H}_+$*.*

An important map is the twisted antipode $\tilde{\mathcal{A}}_+ : \mathcal{H}_+ \to \mathcal{H}$ defined by:

$$\tilde{\mathcal{A}}_+ X_i = -X_i,$$

$$\tilde{\mathcal{A}}_+ \mathcal{I}_a(\tau) = - \sum_{|\ell|_2 \leq \deg(\mathcal{I}_a(\tau))} \frac{(-X)^\ell}{\ell!} \mathcal{M} \left(\mathcal{I}_{a+\ell} \otimes \tilde{\mathcal{A}}_+ \right) \hat{\Delta}^+ \tau.$$

Framework

- Let H be a connected graded Hopf algebra with coproduct Δ.
- $\hat{H}$ a right-comodule over H with coaction $\hat{\Delta} : \hat{H} \to \hat{H} \otimes H$. We suppose also given an injection $\iota : H \to \hat{H}$.
- $\mathrm{char}(H, A)$ is the set of characters from the Hopf algebra H into a commutative A.
- Rota-Baxter map $Q : A \to A$ satisfying for any $f, g \in A$:

$$Q(f) \cdot_A Q(g) + Q(f \cdot_A g) = Q\left(Q(f) \cdot_A g + f \cdot_A Q(g)\right)$$

- The algebra A splits into two subalgebras $A_- := Q(A)$ and $A_+ := (\mathrm{id}_A - Q)(A)$: $A = A_- \oplus A_+$.

Bogoliubov type recursion

For every $\varphi \in \mathrm{char}(\hat{H}, A)$, there are unique algebra morphisms $\varphi_- : H \to A_-$ and $\varphi_+ : \hat{H} \to A$ such that for every $\tau \in H$:

$$\varphi_-(\tau) = -Q\left(\bar{\varphi}(\iota(\tau))\right)$$
$$\bar{\varphi}(\iota(\tau)) = \varphi(\iota(\tau)) + \sum_{(\iota(\tau))}{}' \varphi(\iota(\tau)') \cdot_A \varphi_-(\iota(\tau)'')$$
$$\varphi_+ = \varphi \star \varphi_- = m_A\left(\varphi \otimes \varphi_-\right)\hat{\Delta},$$

where the reduced coaction is given by

$$\hat{\Delta}' \circ \iota(\tau) = \sum_{(\iota(\tau))}{}' \iota(\tau)' \otimes \iota(\tau)'' = \hat{\Delta} \circ \iota(\tau) - \iota(\tau) \otimes \mathbf{1} - \mathbf{1} \otimes \iota(\tau)$$

Moreover, the map $\varphi_+ \circ \iota : H \to A$ takes values in A_+.

Applications

The previous recursion is enough in two cases:

- Renormalisation of the model: Q is given by $\mathbb{E}(\cdot)$.
- Numerical analysis: Q projects according to the frequencies.

The Hopf algebra $\mathcal{H}_+$ is not connected because of X^k. One needs to find the reduced coaction.

Cointeraction between renormalisation and recentering corresponds to two Bogoliubov recursions in "cointeraction".

Reduced coaction

Classical reduced coproduct $\tilde{\Delta}$:

$$\tilde{\Delta}\tau = \Delta\tau - \tau \otimes \mathbf{1} - \mathbf{1} \otimes \tau.$$

The reduced coaction $\hat{\Delta}^+_{\mathrm{red}}$ is given for $\tau = X^k \prod_{i=1}^n \mathcal{I}_{a_i}(\tau_i)$ by

$$\begin{aligned}\hat{\Delta}^+_{\mathrm{red}}\tau &= \hat{\Delta}^+\tau - \tau \otimes \mathbf{1} \\ &\quad - \sum_{\substack{\ell_1,\dots,\ell_n \\ \ell_i,k \in \mathbf{N}^{d+1}}} \Big(\prod_{i=1}^n \frac{1}{\ell_i!}\Big)\binom{k_0}{k} X^{k+\sum_i \ell_i} \otimes X^{k_0-k} \prod_{i=1}^n \pi_+ \mathcal{I}_{a_i+\ell_i}(\tau_i)\end{aligned}$$

One has $\Delta^+_{\mathrm{red}} X^k = 0$.

Splitting and Rota-Baxter maps

We consider $A = \mathcal{C}^\infty(\mathbf{R}^{d+1}, \mathbf{R})$. We fix $x \in \mathbf{R}^{d+1}$, then

$$A = A_x^+ \oplus A_x^- ,$$

where A_x^+ contains the functions vanishing at x and A_x^- consists of polynomial functions whose coefficients are functions of x.

We set for $\alpha \in \mathbf{R}_+$, $x, y \in \mathbf{R}^{d+1}$ and $f \in A$

$$\mathrm{T}_{\alpha,x,y} f \stackrel{\text{def}}{=} \sum_{\substack{\ell \in \mathbf{R}^{d+1} \\ |\ell|_2 < \alpha}} \frac{(y-x)^\ell}{\ell!} (D^\ell f)(x).$$

For $f, g \in A$, one has the Rota-Baxter identity:

$$(\mathrm{T}_{\alpha,x,\cdot} f)(\mathrm{T}_{\beta,x,\cdot} g) = \mathrm{T}_{\alpha+\beta,x,\cdot}[(\mathrm{T}_{\alpha,x,\cdot} f) g + f(\mathrm{T}_{\beta,x,\cdot} g) - fg].$$

Bogoliubov type recursion

We suppose given $\mathcal{F} = (\varphi_{\bar{x}})_{\bar{x}\in\mathbf{R}^{d+1}}$, $\varphi_{\bar{x}} \in \mathrm{char}(\mathcal{H}, A)$, the Bogoliubov recursion is given for $x, y \in \mathbf{R}^{d+1}$ by

$$\bar{\varphi}_{x,\bar{x},y}(\tau) = \varphi_{\bar{x},y}(\tau) + \sum_{(\tau)}^{+} \varphi_{\bar{x},y}(\tau')\varphi^{-}_{x,\bar{x},\bar{x}}(\tau'')$$

$$\varphi^{-}_{x,\bar{x},y}(\tau) = -\mathrm{T}_{\deg(\tau),x,y}\left(\bar{\varphi}_{x,\bar{x},\cdot}(\tau)\right)$$

$$\varphi^{+}_{x,\bar{x}} = \varphi_{\bar{x}} \star \varphi^{-}_{x,\bar{x},\cdot_{|\bar{x}}} = \left(\varphi_{\bar{x}} \otimes \varphi^{-}_{x,\bar{x},\cdot_{|\bar{x}}}\right)\hat{\Delta}^{+}.$$

where $\sum^{+}$ are the Sweedler's notation for $\hat{\Delta}^{+}_{\mathrm{red}}$.

Main theorems

Assumption on $\mathcal{F} = (\varphi_{\bar{x}})_{\bar{x}\in\mathbf{R}^{d+1}}$:

$$\varphi_{\bar{x},y}(X_i) = y_i - \bar{x}_i, \quad \varphi_{\bar{x},\cdot}(\mathcal{I}_{a+\ell}(\tau)) = D^{\ell}\varphi_{\bar{x},\cdot}(\mathcal{I}_a(\tau)).$$

Theorem (B.,Ebrahimi-Fard)

- *The map $\varphi^{-}_{x,\bar{x}}$ is a character from $\mathcal{H}_+$ into A^{-}_{x}.*
- *The map $\varphi^{+}_{x,\bar{x}}$ is a character from $\mathcal{H}$ into A.*
- *On has $\varphi^{-}_{x,\bar{x},\bar{x}} = \varphi_{\bar{x},x}\tilde{\mathcal{A}}_+$.*
- *One has for every $\tau \in \mathcal{H}$:*

$$\varphi^{+}_{\bar{x},x,y}(\tau) = \left(\bar{\varphi}_{x,\bar{x},y} - \mathrm{T}_{\deg(\cdot),x,y}\left(\bar{\varphi}_{x,\bar{x}}\right)\right)(\tau).$$

Main Theorems

Extra assumption on $\mathcal{F} = (\varphi_{\bar{x}})_{\bar{x}\in\mathbf{R}^{d+1}}$:

$$\varphi_{\bar{x},y}\big(\mathcal{I}_{(\mathfrak{l},k)}(\tau)\big) = \int_{\mathbf{R}^{d+1}} D^k K_{\mathfrak{l}}(y-z)\varphi_{\bar{x},z}(\tau)dz$$

where $(K_{\mathfrak{l}})_{\mathfrak{l}\in\mathfrak{L}}$ is a family of smooth kernels.

Theorem (B.,Ebrahimi-Fard)

- *One has $\bar{x}$-invariance for $\varphi^{+}_{x,\bar{x}}$: $\varphi^{+}_{x,0} := \varphi^{+}_{x,\bar{x}}$.*
- *The preparation map $\bar{\varphi}_{x,\bar{x}}$ is $\bar{x}$-invariant on planted trees:*

$$\bar{\varphi}_{x,\bar{x},y}\big(\mathcal{I}_{(\mathfrak{l},k)}(\tau)\big) = \big(D^k K_{\mathfrak{l}} * \varphi^{+}_{x}(\tau)\big)(y).$$

- *The map $\varphi^{-}_{x,\bar{x}}$ is $\bar{x}$-invariant on trees with zero node decoration at the root.*

Application to Regularity Structures

We consider the character $\boldsymbol{\Pi}^{(\bar{x})}$ given by: $\boldsymbol{\Pi}^{(\bar{x})} = \varphi_{\bar{x}}$.

The main characters used for defining the model are:

$$f_x^{(\bar{x})} = (\boldsymbol{\Pi}^{(\bar{x})} \tilde{\mathcal{A}}_+ \cdot)(x), \quad \Pi_x^{(\bar{x})} = \left(\boldsymbol{\Pi}^{(\bar{x})} \otimes f_x^{(\bar{x})} \right) \hat{\Delta}^+$$

Theorem (B.,Ebrahimi-Fard)

One has

$$\Pi_x^{(\bar{x})} = \varphi^+_{x,\bar{x}}, \quad f_x^{(\bar{x})} = \varphi^-_{x,\bar{x},\bar{x}}, \quad \Pi_x^{(\bar{x})} = \Pi_x, \quad \Gamma_{xy}^{(\bar{x})} = \Gamma_{xy}.$$

The model (Π, Γ) *is* $\bar{x}$*-invariant.*