From complementations on lattices to locality
Or from renormalisation to quantum logic

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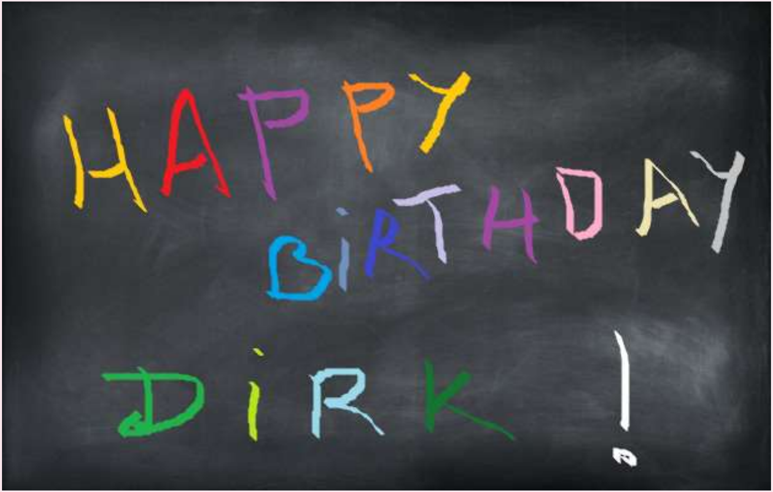
joint work with Pierre Clavier, Li Guo and Bin Zhang

Bures sur Yvette, November 17th 2020


HAPPY
BIRTHDAY
DIRK !

Motivations

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Renormalisation and locality

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- a (locality) algebra $(\mathcal{A}, \top_A, m_A)$ (Feynman graphs, trees, cones)

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Our aim

We want to generalise Laurent expansions to meromorphic germs in **several variables**, so on $\mathcal{M}(\mathbb{C}^\infty)$, we need a **separating device** on the underlying spaces $V = \mathbb{C}^k$ to distinguish the **polar part** from the **holomorphic part**.

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Orthogonal complements are useful to separate polar parts from holomorphic parts of meromorphic germs.

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to a 1-1 correspondence on a class of locality lattices $(L, \top)$

$$\top \longleftrightarrow \Psi^\top$$

with "orthocomplementations" $\Psi^\top$.

We expect that:

$$U \top W \Leftrightarrow W \in \downarrow U^\top \quad \text{and} \quad \Psi^\top(U) = \max U^\top.$$

Orthogonality in Laurent expansions

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Meromorphic germs with linear poles

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separates two meromorphic germs.
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such that the following Laurent expansion holds

$$f = \left[\sum_{j \in J} S_j \right] \oplus^Q h =: \mathfrak{L}_{\mathcal{C}}(f).$$

Here, $\mathcal{C} = \{(C(S_j)), j \in J\}$, is a properly positioned family of simplicial cones.

Decomposition of meromorphic germs

The inner product Q and the orthocomplementation Ψ^Q play a central role in the following decomposition.

Theorem

(L. Guo, S.P., B. Zhang PJM 2020)Given a meromorphic germ $f \in \mathcal{M}(\mathbb{C}^k)$, there exists a finite set of polar germs $\mathcal{M}_-^Q(\mathbb{C}^k) \ni \left\{ S = \frac{h_j}{L_{j1}^{s_{j1}} \cdots L_{jn_j}^{s_{jn_j}}} \right\}_{j \in J}$

i) that are properly positioned ;ii) whose denominators are pairwise not proportional ; iii) and a holomorphic germ h,

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Warning: The holomorphic germ h is unique yet the decomposition is not unique: $\frac{1}{L_1 L_2} = \frac{1}{L_1(L_1+L_2)} + \frac{1}{L_2(L_1+L_2)}$.

Orthogonality as a locality relation

Orthogonality and locality

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- (Recall) **Dependence** set $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$, $\ell_i, L_j \in \mathcal{L}(\mathbb{C}^k)$.

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- $(z_1 - z_2) \perp^Q (z_1 + z_2)$ with Q: canonical inner product on $\mathbb{R}^2$.

The lattice $G(V)$

The poset $(G(V), \leq)$ is a (non distributive) bounded lattice

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- Given a finite dimensional vector space V, $(G(V), \leq)$ is a non distributive lattice equipped with the sum $\vee = +$ and the intersection $\wedge = \cap$ as lattice operations. It is bounded by $0 = \{0\}$ and $1 = V$.
- In a lattice $(L, \leq)$, the set $\downarrow a := \{b \leq a,\, b \in L\}$ is a sub-lattice (even a lattice ideal) of L.

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Modular lattices (conditional distributivity)

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Example: Modularity $\Rightarrow$ $\oplus$-modularity so $G(V)$ is $\oplus$ modular but it does not satisfy the $\oplus$-distributivity condition.

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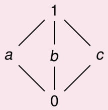
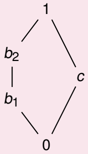
Remark: $\oplus$-modularity (resp. $\oplus$-cancellation) combined with sectional completeness implies modularity.

Special lattices

Distributivity and modularity are hereditary properties.

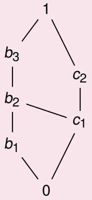


diamond lattice



pentagon lattice

The diamond lattice is modular and the pentagon lattice is not modular. They are both non distributive, non ⊕ distributive, non ⊕-modular and have no orthocomplementation.



extended pentagon lattice

The extended pentagon lattice ⊕-modular but not modular.

Orthomodular lattices

- A bounded lattice $(L, \leq, 0, 1)$ is **complemented** if $\forall a \in L, \exists b \in L, a \oplus b = 1$.

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Orthomodular lattices

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3. Given a Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$, the map $\Psi_{\langle\cdot,\cdot\rangle} : W \longmapsto W^\perp := \{v \in V,\ \langle v, w\rangle = 0\ \forall w \in W\}$ defines an orthocomplement map on $G(V)$. $(G(V), \leq, \cap, +, \psi_{\langle\cdot,\cdot\rangle})$ is an **orthomodular** lattice.

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We call $(P, \leq, \top)$ a (or weak degenerate orthogonal) **locality poset**.

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Example

Given a Hilbert (finite or infinite dimensional) vector space (V, Q), the locality relation $U \perp^Q W$ defines a lattice locality relation.

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3. (completeness) the set $a^\top$ admits a maximal element $\max(a^\top)$ for any $a \in L$.

In this case, we say that $(L, \leq, 0, \top)$ is a **separated locality** (or complete orthogonality poset) lattice. Recall that $\downarrow a \subset (a^\top)^\top$ since $\top$ is a locality relation on the poset $(L, \leq)$. If moreover,

- $\downarrow a = (a^\top)^\top$ or equivalently, if $\max\left((a^\top)^\top\right) = a$ for any $a \in L$,

we call the relation **strongly separating** and the lattice strongly separated.

Example

Given a Hilbert (finite or infinite dimensional) vector space $(V, \langle \cdot, \cdot \rangle)$, the poset $G(V)$ is a strongly separated locality lattice for $W_1 \top W_1 \Longleftrightarrow W_1 \perp^Q W_2$. For three subspaces W, U_1, U_2 in V we have $\left(\forall W \subseteq V,\ W \perp^Q U_1 \Rightarrow W \perp^Q U_2\right) \Longrightarrow U_2 \preceq U_1$.

Locality versus complements

From locality to complements and back

Main result (P. Clavier, L.Guo, S.P., B. Zhang (2020), G. Cattaneo, A. Mania (74!))

Let L be a bounded lattice.There is a one-to-one correspondence

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If the lattice is $\oplus$-modular, this yields a one-to-one correspondence

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Example: $L = G(V)$

This generalises the correspondence orthogonality $\longleftrightarrow$ orthogonal complement on vector spaces.

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Lattice (resp.**strongly separating**) locality relation on $G(V)$

Locality on lattices versus locality on vector spaces

Locality relations on vector spaces

- A vector space locality relation $\top$ on a linear space V is a set locality relation $\top$ on V such that the polar set $U^\top$ of any subset U is a linear subspace.
- The vector space locality $\top$ is
  1. non-degenerate if for any $v \in V$, $v \top v \Longrightarrow v = 0$ (this implies $V^\top = \{0\}$).
  2. strongly non-degenerate if it is non-degenerate and $U^\top = \{0\} \Rightarrow U = V$ for any subspace U of V.

Vector spaces versus lattices

Vector space **locality relation** on $V \rightsquigarrow$ Lattice **locality relation** on $G(V)$: $W_1 \top W_2 \Longleftrightarrow w_1 \top w_2 \quad \forall w_i \in W_i,\ i \in \{1, 2\}$. Lattice **locality relation** on $G(V) \rightsquigarrow$ Vector space **locality relation** on V: $w_1 \top w_2 \Longleftrightarrow \langle w_1 \rangle \top \langle w_2 \rangle$.

One to one correspondence

Lattice (resp. **strongly separating**) locality relation on $G(V)$

$\longleftrightarrow$ Vector space (resp. **strongly non-degenerate**) locality relation on V

Orthocomplements versus locality on vector spaces

Corollary

A locality vector space $(V, \top)$ *is* **strongly regular** *if, and only if* $(G(V), \top)$ *is* **orthocomplemented**.

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(1) *Given a strongly regular locality relation on a vector space* $(V, \top)$*, then* $\Psi^\top$ *is an* **orthocomplement** *on* $G(V)$.

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(1) *Given a strongly regular locality relation on a vector space $(V, \top)$, then $\Psi^\top$ is an* **orthocomplement** *on $G(V)$. It is the unique map $\Psi : G(V) \to G(V)$ such that for any $W \in G(V)$:*

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(2) *Conversely, if $\Psi^{\top}$ defines an* **orthocomplement map** *on $G(V)$ then the locality relation*

$v_1 \top v_2 \iff v_1 \in \Psi^{\top}(\langle v_2 \rangle)$ *induces a strongly regular locality relation on V.*

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(2) *Conversely, if $\Psi^\top$ defines an* **orthocomplement map** *on $G(V)$ then the locality relation*

$v_1 \top v_2 \Longleftrightarrow v_1 \in \Psi^\top(\langle v_2 \rangle)$ induces a strongly regular locality relation on V.

Example

On a Hilbert space $(V, \langle\cdot,\cdot\rangle)$ this amounts to the correspondence we started from

$$\perp \quad \longleftrightarrow \quad (\Psi^\perp : U \mapsto U^\perp).$$

Example beyond orthogonality

Take $V := \mathbb{R}^2$,

$$G(\mathbb{R}^2) = \{\{0\}, \mathbb{R}^2\} \cup \{U_\theta := \mathbb{R}\, e^{i\theta} \,|\, \theta \in [0, \pi)\}.$$

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Any disjoint union $[0,\pi) = I' \sqcup I''$ and bijection $I' \to I''$ gives rise to an involutive map $\psi : [0,\pi) \to [0,\pi)$ with $\psi(I') = I''$ and an orthocomplement map

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on $G(\mathbb{R}^2)$.

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Any bijection $\psi : [0, \pi/2) \to [\pi/2, \pi)$ induces an involution $\psi : [0, \pi) \to [0, \pi)$, e.g.

$$\psi(\theta) = \pi - \theta, \quad \theta \in [0, \pi)$$

yields back $\Psi^\perp$ for the canonical inner product.

Openings

- generalise to orthocomplements beyond $\perp^Q$ the Laurent expansions for multi-variable meromorphic germs with linear poles built in [L. Guo, S.P., B. Zhang, to appear in PJM].

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- study the Galois group of transformations of multi-variable meromorphic germs with linear poles which stablise holomorphic germs at zero.

THANK YOU FOR YOUR ATTENTION!

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