Moduli spaces for realizing unstable coalgebras

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The paper

"The realization space of an unstable coalgebra"

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Let $\mathbb{F}$ be a prime field. We write

$$H_* = H_*(-, \mathbb{F})$$

for the homology of a space with coefficients in $\mathbb{F}$. This yields a functor

$$H_* \colon \mathcal{S} \to \mathcal{CA},$$

where

- $\mathcal{S}$ the category of spaces (actually simplicial sets) and
- $\mathcal{CA}$ for the category of unstable coalgebras.

Question: For a given unstable coalgebra C, does there exist a space X together with an isomorphism $H_*(X) \cong C$ in $\mathcal{CA}$?

Let $\mathcal{M}(C)$ denote the moduli space of realizations of C. We define and study a tower of moduli spaces under $\mathcal{M}(C)$.

We follow an approach by Blanc-Dwyer-Goerss (Π-algebras) and Goerss-Hopkins (E_∞-spectra)

Let $(\mathcal{T}_2)_{\mathbb{Q}}$ denote the category of simply connected rational spaces .
Let DGL_1 denote the category of connected rational differential graded Lie algebras.
Let DGC_2 denote the category of simply connected rational differential graded coalgebras.

Theorem (Quillen '66, Sullivan '77)

There is an equivalence

$$\mathrm{Ho}((\mathcal{T}_2)_{\mathbb{Q}}) \cong \mathrm{Ho}(\mathrm{DGL}_1) \cong \mathrm{Ho}(\mathrm{DGC}_2)$$

of homotopy categories.

In particular, any simply connected rational unstable coalgebra is realizable.

Extensive work by many, in particular

- Félix, Halperin-Stasheff, Lemaire-Sigrist, Thomas (70s-90s)
- Schlessinger-Stasheff (late 70s, `arXiv:1211.1647`)

mod p (co-)homology, obstruction theory:

- Massey-Peterson case (free as unstable algebra over $\mathcal{A}_p$):
  Harper (late 70s)
- obstructions for existence and uniqueness of liftings of maps:
  Bousfield (1989)
- obstructions for existence of liftings of objects:
  Blanc (2001)

Other results in this context:

- Hopf-Invariant-1-Problem, Adams (1966)
- Segal conjecture, Miller (1984)
- Kuhn conjecture, Schwartz ('98), Gaudens-Schwartz, Kuhn ('08)

Definition (for $p = 2$)

An *unstable right $\mathcal{A}_2$-module* consists of

1. a graded $\mathbb{F}_2$-vector space M which is trivial in negative degrees.
2. a $\mathbb{F}_2$-linear right action $M \otimes \mathcal{A}_2 \to M$ satisfying *instability*:

$$x \, \mathrm{Sq}^n = 0 \quad \text{for} \quad |x| < 2n$$

Let $\mathcal{U}^r = \mathcal{U}$ be the category of unstable right $\mathcal{A}_2$-modules.

Definition (for $p = 2$)

An *unstable coalgebra* consists of

- an unstable right module C,
- which is also a graded cocommutative coalgebra
- such that C_0 is set-like.

The two structures satisfy the following compatibility conditions:

1. The comultiplication $\Delta : C \to C \otimes C$ is a map in $\mathcal{U}$.
2. The root map $\xi : C_{2n} \to C_n$ satisfies

$$\xi(x) = xSq^n \text{ for } |x| = 2n$$

Definition

- $\mathcal{G} = \{K(\mathbb{F}, n) \mid n \geq 0\}$, where $\mathbb{F}$ is the fixed prime field
- A GEM is a product of objects in $\mathcal{G}$.

Proposition

For any space X and any $G \in \mathcal{G}$ (in fact, any GEM) we have:

$$[X, G] \cong \mathrm{Hom}_{\mathcal{CA}}(H_*X, H_*G)$$

Proposition

$\{H_*G \mid G \in \mathcal{G}\}$ *is a set of cogenerators for $\mathcal{CA}$.*

The forgetful functor

$$J\colon \mathcal{CA} \to \mathrm{Vec}$$

has a right adjoint

$$G\colon (V_n)_{n\geq 0} \mapsto H_*(\prod_{n\geq 0} K(V_n, n)).$$

Definition

The unstable coalgebras in the image of G are called *cofree*.

Let $c\mathcal{CA}$ be the category of cosimplicial unstable coalgebras.

For $C^\bullet$ in $c\mathcal{CA}$ let

$$\pi^s C^\bullet = H^s N C^\bullet.$$

A map $C^\bullet \to D^\bullet$ is an *equivalence* if $\pi^s C^\bullet \to \pi^s D^\bullet$ is an isomorphism for all $s \geq 0$.

Theorem

There is a proper simplicial model structure on $c\mathcal{CA}$ with these equivalences and the pair

$$J\colon\ c\mathcal{CA} \rightleftarrows c\mathrm{Vec} : G$$

is a Quillen adjunction.

Let $c\mathcal{S}$ be the category of cosimplicial spaces.

A map $X^\bullet \to Y^\bullet$ in $c\mathcal{S}$ is called a $\mathcal{G}$-equivalence (resolution equivalence) if

$$\pi^s H_* X^\bullet \cong \pi^s H_* Y^\bullet$$

for all $s \geq 0$.

Theorem (Dwyer/Kan/Stover, Bousfield)

There is a proper simplicial model structure on $c\mathcal{S}$ whose equivalences are the $\mathcal{G}$-equivalences.

There is a second quadrant spectral sequence (HSS)

$$E_2^{s,t} = \pi^s H_t X^\bullet \implies H_{t-s}\mathrm{Tot} X^\bullet.$$

Maps that induce isomorphisms on the E_2-term of the HSS are exactly the $\mathcal{G}$-equivalences.

Goal: Given C, build a cosimplicial space $X^\bullet$ such that

$$cC \to H_*(X^\bullet)$$

is a weak equivalence. Then the HSS collapses proving

$$H_* \mathrm{Tot} X^\bullet \cong C.$$

Such an $X^\bullet$ is called an ∞-*stage* for C.

Fibrant models for ∞-stages are Adams resolutions.

Definition

For a cosimplicial space $X^\bullet$ and $G \in \mathcal{G}$ we define *E_2-homotopy groups*

$$\pi_s[X^\bullet, G]_{\mathcal{S}} \qquad \Big(\cong \pi_s H^n(X^\bullet) \ \text{ for } G = K(\mathbb{F}, n) \\ \mathbb{F}\text{-dual to } \pi^s H_n(X^\bullet) \Big)$$

and *natural homotopy groups*

$$\pi_s^\natural(X^\bullet, G) = \pi_s \mathrm{Hom}_{\mathcal{S}}(X^\bullet, G) \cong \pi_s \mathrm{map}(X^\bullet, cG) \\ \cong [X^\bullet, \Omega^s_{\mathrm{ext}} cG]_{\mathcal{G}}.$$

Observe:

$$\pi_s^\natural(\mathrm{sk}_{n+1} X^\bullet, G) \cong \pi_s \,\mathrm{cosk}_{n+1} \mathrm{Hom}_{\mathcal{S}}(X^\bullet, cG) = \begin{cases} \pi_s^\natural(X^\bullet, G) & , \text{ for } s \leq n \\ 0 & , \text{ else} \end{cases}$$

Theorem

There is an isomorphism

$$\pi_0^\natural(X^\bullet, G) \cong \pi_0[X^\bullet, G] =: \pi_0$$

of $\mathcal{H}$-algebras and a long exact sequence of π_0-modules

$$\ldots \to \pi_{s-1}^\natural(X^\bullet, \Omega G) \to \pi_s^\natural(X^\bullet, G) \to \pi_s[X^\bullet, G] \to \pi_{s-2}^\natural(X^\bullet, \Omega G) \to \ldots$$
$$\ldots \to \pi_2[X^\bullet, G] \to \pi_0^\natural(X^\bullet, \Omega G) \to \pi_1^\natural(X^\bullet, G) \to \pi_1[X^\bullet, G] \to 0,$$

where Ω is the internal loop functor from $\mathcal{S}$.

Theorem

The d_2 of the cohomology spectral sequence

$$E_2^{s,*} = \pi_s H^* X^\bullet \implies H^{s+*}(\mathrm{Tot} X^\bullet)$$

is a morphism of $\pi_0 H^* X^\bullet$-modules. The d_2 of the HSS is a morphism of $\pi^0 H_* X^\bullet$-comodules.

Potential n-stages

Definition

Let $C \in \mathcal{CA}$. A potential n-stage for C is a cosimplicial space $X^\bullet$ such that there are natural isomorphisms

$$\pi^s H_* X^\bullet \cong \begin{cases} C & , \text{ for } s = 0 \\ C[n+1] & , \text{ for } s = n+2 \\ 0 & , \text{ else} \end{cases}$$

as unstable coalgebras for $s = 0$ and as C-comodules for $s = n+2$.

If $X_n^\bullet$ is a potential n-stage then $\mathrm{sk}_{k+1} X_n^\bullet$ is a potential k-stage for $0 \leq k \leq n$.

The skeletal filtration

$$\mathrm{sk}_1 X^\bullet \to \mathrm{sk}_2 X^\bullet \to \ldots \to X^\bullet$$

is an analogue of the Postnikov tower.

Let $X_n^\bullet$ be a Reedy cofibrant potential n-stage. Then the $\mathcal{G}$-homotopy cofiber of

$$X_{n-1}^\bullet = \mathrm{sk}_n X_n^\bullet \to X_n^\bullet$$

has exactly one non-vanishing natural homotopy group in degree n.

Definition

Let $C \in \mathcal{CA}$, M a C-comodule and $n \geq 1$. A cosimplicial space $L^\bullet$ is of *type* $L_C(M, n)$ if there are isomorphisms

$$\pi_s^\natural(L^\bullet, G) \cong \begin{cases} \mathrm{Hom}_{\mathcal{CA}}(C, H_* G) & , \text{ if } s = 0, \\ \mathrm{Hom}_{C/\mathcal{CA}}(\iota_C(M), H_* G) & , \text{ if } s = n, \\ 0 & , \text{ otherwise,} \end{cases}$$

The first isomorphism is an isomorphism of unstable algebras, the second one of modules over $C^\vee$.

They are “square-zero extensions” of C in the cosimplicial direction.
A space of type $L(C, 0)$ is the same as a potential 0-stage.

The maps $X^\bullet_{k-1} \to X^\bullet_k$ are "principal cofibrations": There are homotopy pushout squares:

$$\begin{array}{ccccc}
L_C(C[k], k+1) & \xrightarrow{w_k} & X^\bullet_{k-1} & & \\
\downarrow & & \downarrow & & \\
L(C, 0) & \longrightarrow & X^\bullet_k & \longrightarrow & L(C[k], k)
\end{array}$$

Definition

The map w_k is called the co-k-invariant of $X^\bullet_k$.

Theorem (Homotopy Excision for cosimplicial Spaces)

Consider in $c\mathcal{S}$ a homotopy pullback square in the $\mathcal{G}$-model structure

$$\begin{array}{ccc}
P^\bullet & \longrightarrow & Y^\bullet \\
\downarrow & & \downarrow {\scriptstyle g}\ {\scriptstyle n-\mathrm{con}} \\
X^\bullet & \underset{m-\mathrm{con}}{\xrightarrow{f}} & Z^\bullet
\end{array}$$

where f is m-connected and g is n-connected. Then the square is $(m+n)$-homotopy cocartesian.

Definition

A map $X^\bullet \to Y^\bullet$ in $c\mathcal{S}$ is *cosimplicially n-connected* if the induced map

$$\pi_s^\natural(Y^\bullet, G) \to \pi_s^\natural(X^\bullet, G)$$

is an isomorphism for $0 \leq s < n$ and surjective for $s = n$ for all $G \in \mathcal{G}$.

Equivalently, if $\pi^s H_* X^\bullet \to \pi^s H_* Y^\bullet$ is an isomorphism for $0 \leq s < n$ and injective for $s = n$.

Homotopy excision translates into the statement that the comparison map between the homotopy pullback $P^\bullet$ in $c\mathcal{CA}$ and the homotopy pullback in $c\mathrm{Vec}$ induces isomorphisms on π^s for $0 \leq s \leq m + n$ and is injective for $s = m + n + 1$.

Theorem

Let $cY \to Y^\bullet$ be $\mathcal{G}$-fibrant replacement with $D = H_*Y = \pi^0 H_* Y^\bullet$. Fix a morphism $C \to D$ in $\mathcal{CA}$, and let M be a C comodule. Then:

$$\pi_s \mathrm{map}_{L(C,0)}(L_C(M,n), Y^\bullet) \cong \begin{cases} \mathrm{AQ}_C^{n-s}(D,M), & \textit{for } 0 \leq s \leq n \\ 0, & \textit{else} \end{cases}$$

Conjecture

The derived Bousfield-Kan tower

$$\mathrm{im}\left\{\left(\mathrm{map}(X, \mathrm{Tot}_{s+1} Y^\bullet), f\right) \to \left(\mathrm{map}(X, \mathrm{Tot}_s Y^\bullet), f\right)\right\}_{s\geq 0}$$

maps to the tower

$$\left\{\mathrm{map}_{L(C,0)}(X_s^\bullet, Y^\bullet)\right\}_{s\geq 0}.$$

This induces an isomorphism of spectral sequences from the E_2-term onwards, the *unstable Adams spectral sequence*

$$E_2^{s,t} = \mathrm{AQ}_C^{s,t}(D,C) \implies \pi_{t-s}(\mathrm{map}(X, \mathrm{Tot}\, Y^\bullet), f),$$

where $C = H_*X, D = H_*Y, f\colon X \to Y$.

Definition

Let $\mathcal{V}$ subcategory of coabelian cogroup objects in $\mathcal{CA}$.

The inclusion $i\colon \mathcal{V}C \to C/\mathcal{CA}$ possesses a right adjoint

$$\mathrm{Coab}\colon C/\mathcal{CA} \to \mathcal{V}C.$$

Definition

Let $R\mathrm{Coab}$ be the right derived functor. For $M \in \mathcal{V}C$ and $n \geq 0$

$$\begin{aligned} AQ^n_C(C; M) &= [\Omega^n_C cM, R\mathrm{Coab}(C)] \\ &= [Li(\Omega^n_C cM), cC] \end{aligned}$$

is the n-th André-Quillen cohomology of C with coefficients in M.

For C let $\mathcal{M}(C)$ be the moduli space of realizations of C.

Theorem

If the HSS converges, then there is a weak equivalence

$$\mathcal{M}(C) \simeq \mathcal{M}_\infty(C),$$

where $\mathcal{M}_\infty(C)$ is the moduli space of ∞-stages for C.

Let $\mathcal{M}_k(C)$ be the moduli space of k-stages for C. There is a tower

$$\mathcal{M}_\infty(C) \to \ldots \to \mathcal{M}_k(C) \to \mathcal{M}_{k-1}(C) \to \ldots$$

Theorem

There is a weak equivalence

$$\mathcal{M}_\infty(C) \simeq \operatorname{holim}_k \mathcal{M}_k(C).$$

Theorem

There is a weak equivalence

$$\mathcal{M}_0(C) \simeq B\mathrm{Aut}(C).$$

Our results III

There is a pointed space $\mathcal{AQ}^k_C(C,M))$ with

$$\pi_s \mathcal{AQ}^k_C(C,M)) \cong \begin{cases} AQ^{k-s}_C(C,M)) & \text{, for } 0 \le s \le k \\ 0 & \text{, else} \end{cases}$$

together with a basepoint preserving action of $\mathrm{Aut}_C(M)$. We denote the homotopy fixed points by

$$\widetilde{\mathcal{AQ}}^k_C(C,M) = \mathcal{AQ}^k_C(C,M)_{h\mathrm{Aut}_C(M)}.$$

Theorem

There are homotopy pullback squares:

$$\begin{array}{ccccc} \mathcal{AQ}^{k+1}_C(C,C[k])) & \longrightarrow & \mathcal{M}_k(C) & \longrightarrow & B\mathrm{Aut}(C,C[k]) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \mathcal{M}_{k-1}(C) & \longrightarrow & \widetilde{\mathcal{AQ}}^{k+2}_C(C,C[k]) \end{array}$$

Theorem (Blanc)

Let $X^\bullet_{n-1}$ be a potential $(n-1)$-stage for C. There is an element

$$o_n(X^\bullet_{n-1}) \in AQ^{n+2}(C, C[n]), n \geq 1$$

induced by the k-invariants such that there exists a potential n-stage over $X^\bullet_{n-1}$ if and only if $o_n(X^\bullet_{n-1}) = 0$.

Theorem

Assume that $X^\bullet_{n-1}$ lifts to a potential n-stage. There is a certain group B acting on $AQ^{n+1}(C, C[n])$ and an element

$$d_n(X^\bullet_{n-1}) \in AQ^{n+1}(C, C[n]), n \geq 1$$

such that there is a unique potential n-stage up to $\mathcal{G}$-equivalence if $d_n(X^\bullet_{n-1})$ is in the B-orbit of 0.

Our results V

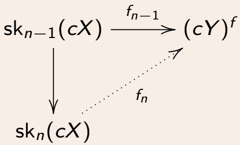
Given spaces X, Y and a map $\varphi\colon\ C = H_*X \to H_*Y = D$. Assume, the HSS converges for X and Y. There is always a map

$$\mathrm{sk}_1(cX) \to (cY)$$

inducing φ on $\pi^0 H_*$.

Theorem ((equivalent to) Bousfield)

Suppose that there is a lift f_{n-1} of φ:



There is an element

$$e_n(f_{n-1}) \in AQ_C^{n+1}(D, C[n]), n \geq 1$$

such that a map f_n inducing φ exists if and only if $e_n(f_{n-1}) = 0$.

There is a similar statement for uniqueness with obstructions in $AQ_C^n(D, C[n])$.