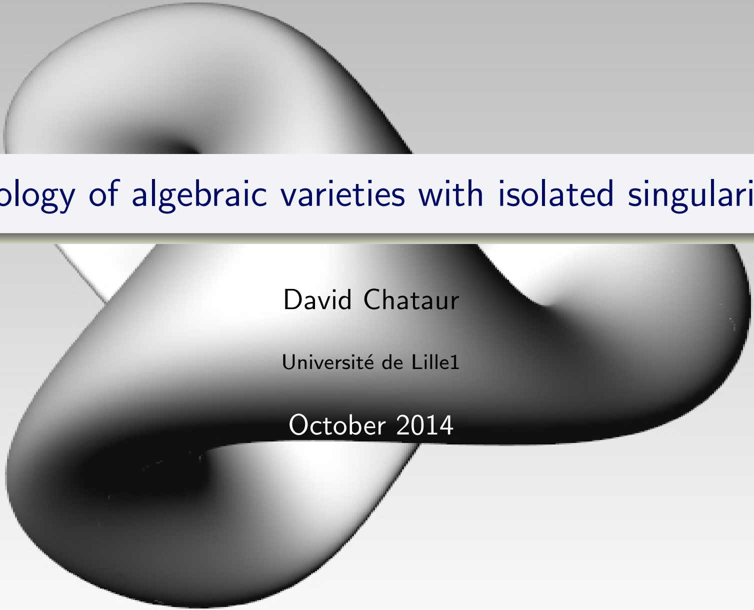


Topology of algebraic varieties with isolated singularities

David Chataur

Université de Lille1

October 2014

Aim of the talk

Give an overview of "intersection homotopy theory" in the case of complex projective varieties with isolated singularities. Based on joint work with Joana Cirici (F.U. Berlin).

Starting point

Mark Goresky's question (1984):

The results in this paper on Steenrod operations and Wu classes may be considered as part of a program to describe ways in which the intersection homology groups of certain singular spaces behave like the ordinary homology groups of a nonsingular space ([CGM] §1). It remains as open question whether there is an intersection homology – analogue to the rational homotopy theory of Sullivan. For example, one would like to know when Massey triple products are defined in intersection homology and whether they always vanish on a (singular) projective algebraic variety (see [DGMS]).

Plan of the talk

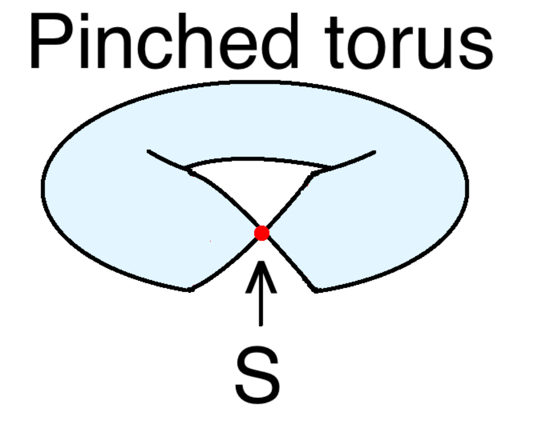
1. Intersection cohomology

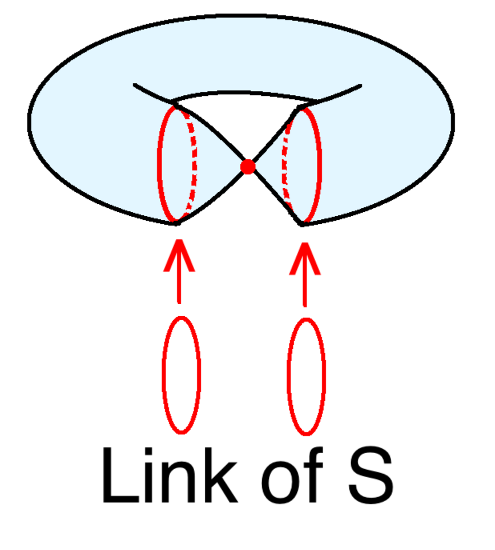
Plan of the talk

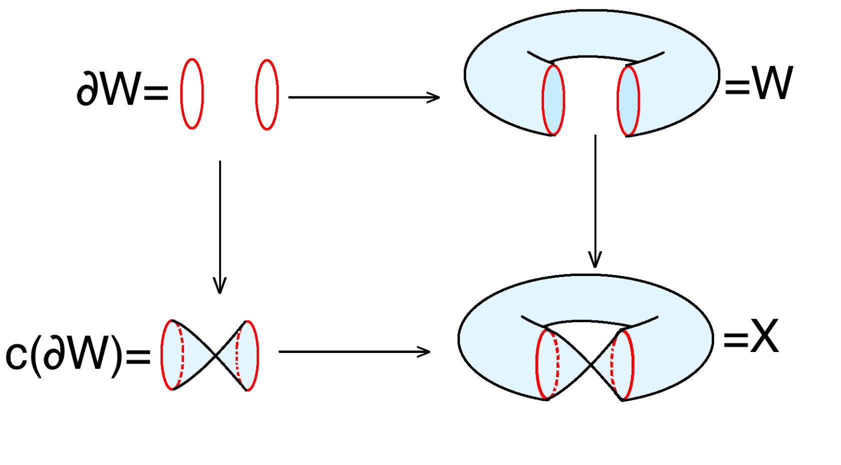
1. Intersection cohomology
2. Intersection homotopy theory

Plan of the talk

1. Intersection cohomology
2. Intersection homotopy theory
3. Formality and Hodge theory for intersection cohomology


Pinched torus
S


Link of S


∂W=
=W
c(∂W)=
=X

Convention for the talk

We only consider normal singular spaces with one isolated singularity:

$$X = W \cup_L cL$$

the link L is supposed to be connected.

Intersection cohomology

C. R. Acad. Sc. Paris, t. 284 (27 juin 1977) Série A — 1549

TOPOLOGIE. — *La dualité de Poincaré pour les espaces singuliers.*

Note (*) de **Mark Goresky** et **Robert MacPherson**, présenté par M. Israël M. Gelfand.

On introduit pour une pseudovariété X éventuellement singulière les groupes $IH_i^{\bar{p}}(X)$ appelé les groupes d'homologie d'intersection. Ces groupes ont un produit d'intersection et satisfont à une dualité. C'est une généralisation de la théorie des intersections de Lefschetz et la dualité de Poincaré dans l'homologie ordinaire d'un espace non singulier.

IH^*-cohomology

Let $p \in \mathbb{N}$ we have a functor

$$IH^*_{(p)}(-,\mathbb{F}) : \textbf{SingIsol} \to \mathbb{F} - \textbf{gr} - \textbf{vectorspaces}$$

such that:

$$IH^*_{(p)}(X,\mathbb{F}) = \begin{cases} H^*(W,\mathbb{F}) \ * \leq p, \\ Ker(H^*(W,\mathbb{F}) \to H^*(L,\mathbb{F})) \ * = p+1, \\ H^*(X,\mathbb{F}) \ * > p+1. \end{cases}$$

Case of a cone: the local situation

Let L be a 3-dimensional manifold, we consider $cL = L \times [0,1]/L \times \{0\}$.

$* \setminus (p)$	(0)	(1)	(2)	(3)	(4)
4	0	0	0	0	0
3	0	0	0	$H^3(L)$	$H^3(L)$
2	0	0	$H^2(L)$	$H^2(L)$	$H^2(L)$
1	0	$H^1(L)$	$H^1(L)$	$H^1(L)$	$H^1(L)$
0	$H^0(L)$	$H^0(L)$	$H^0(L)$	$H^0(L)$	$H^0(L)$

Properties

The collection $\{IH^*_{(p)}(X, \mathbb{F})\}_{p \in \mathbb{N}}$ is a perverse algebra:

Properties

The collection $\{IH^*_{(p)}(X, \mathbb{F})\}_{p \in \mathbb{N}}$ is a perverse algebra:
- when $p \leq p'$ we have a morphism:

$$IH^*_{(p)}(X, \mathbb{F}) \to IH^*_{(p')}(X, \mathbb{F})$$

Properties

The collection $\{IH^*_{(p)}(X,\mathbb{F})\}_{p\in\mathbb{N}}$ is a perverse algebra:

- when $p \leq p'$ we have a morphism:

$$IH^*_{(p)}(X,\mathbb{F}) \to IH^*_{(p')}(X,\mathbb{F})$$

- together with a graded commutative cup product.

$$\begin{array}{ccc} IH^*_p(X,\mathbb{F}) \otimes IH^*_q(X,\mathbb{F}) & \to & IH^*_{p+q}(X,\mathbb{F}) \\ \downarrow & & \downarrow \\ IH^*_{p'}(X,\mathbb{F}) \otimes IH^*_{q'}(X,\mathbb{F}) & \to & IH^*_{p'+q'}(X,\mathbb{F}) \end{array}$$

Properties

- Intersection cohomology is not a homotopy invariant,

Properties

- Intersection cohomology is not a homotopy invariant,
- Let X be a singular space with isolated singularities s.t. $dim(X) = n$ then $IH^*_{(p)}(X, \mathbb{F})$ is a topological invariant when $p < n-1$,

Properties

- Intersection cohomology is not a homotopy invariant,
- Let X be a singular space with isolated singularities s.t. $dim(X) = n$ then $IH^*_{(p)}(X, \mathbb{F})$ is a topological invariant when $p < n - 1$,
- $IH^*_{(0)}(X, \mathbb{F}) \cong H^*(X, \mathbb{F})$.

Properties

- Intersection cohomology is not a homotopy invariant,
- Let X be a singular space with isolated singularities s.t. $dim(X) = n$ then $IH^*_{(p)}(X, \mathbb{F})$ is a topological invariant when $p < n - 1$,
- $IH^*_{(0)}(X, \mathbb{F}) \cong H^*(X, \mathbb{F})$.
- if X is a manifold then $IH^*_{(p)}(X) \cong H^*(X, \mathbb{F})$,

Properties

- Intersection cohomology is not a homotopy invariant,
- Let X be a singular space with isolated singularities s.t. $dim(X) = n$ then $IH^*_{(p)}(X, \mathbb{F})$ is a topological invariant when $p < n-1$,
- $IH^*_{(0)}(X, \mathbb{F}) \cong H^*(X, \mathbb{F})$.
- if X is a manifold then $IH^*_{(p)}(X) \cong H^*(X, \mathbb{F})$,
- (Poincaré duality) If W is $\mathbb{F}$-oriented, we have a non-degenerate pairing

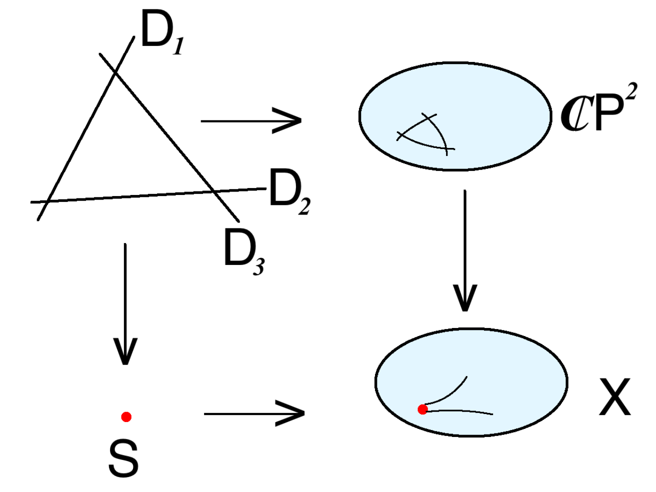
$$IH^*_{(p)}(X, \mathbb{F}) \otimes IH^{n-*}_{(n-2-p)}(X, \mathbb{F}) \to \mathbb{F}.$$

Example

Let D_1, D_2 and D_3, be three lines in general position in $\mathbb{C}P^2$, we contract these lines to a point:

Example

Let D_1, D_2 and D_3, be three lines in general position in $\mathbb{C}P^2$, we contract these lines to a point:



Intersection cohomology of X

$* \setminus \overline{p}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
4	$\mathbb{Q}.t$	$\mathbb{Q}.t$	$\mathbb{Q}.t$
3	$\mathbb{Q}.v_1, v_2$	$\mathbb{Q}.v_1, v_2$	0
2	$\mathbb{Q}.\alpha$	0	$\mathbb{Q}.D\alpha$
1	0	$\mathbb{Q}.Dv_1, Dv_2$	$\mathbb{Q}.Dv_1, Dv_2$
0	$\mathbb{Q}.1$	$\mathbb{Q}.1$	$\mathbb{Q}.1$

Geometric and topological applications of IH^*

- Signature and characteristic classes for singular spaces.

Geometric and topological applications of IH^*

- Signature and characteristic classes for singular spaces.
- IH^* is useful in the study of complex algebraic 3-folds (Markus Banagl, Maxim Laurentiu). IH^* can detect some information about deformations of hypersurfaces.

Geometric and topological applications of IH^*

- Signature and characteristic classes for singular spaces.
- IH^* is useful in the study of complex algebraic 3-folds (Markus Banagl, Maxim Laurentiu). IH^* can detect some information about deformations of hypersurfaces.
- IH^* detects some local information: we can recover the Betti numbers of the link and detect homology manifolds among singular spaces with isolated singularities.

Geometric and topological applications of IH^*

- Signature and characteristic classes for singular spaces.
- IH^* is useful in the study of complex algebraic 3-folds (Markus Banagl, Maxim Laurentiu). IH^* can detect some information about deformations of hypersurfaces.
- IH^* detects some local information: we can recover the Betti numbers of the link and detect homology manifolds among singular spaces with isolated singularities.
- IH^* can be used to distinguish homotopy equivalent but non-homeomorphic spaces: linkage spaces (Dirk Schutz)

Theorem (DC, Martin Saralegui (Lens), Daniel Tanré (Lille))

($Char(\mathbb{F}) = 0$). There is a functor

$$IA^*_{(*)} : \mathbf{Strat} \to \mathcal{P} - \mathbf{CDGA}$$

such that for any pseudomanifold X we have an isomorphism of perverse algebras

$$H^*(IA^*_{(*)}(X)) \cong IH^*_{(*)}(X, \mathbb{F}).$$

Theorem (DC, Martin Saralegui (Lens), Daniel Tanré (Lille))

($Char(\mathbb{F}) = 0$). There is a functor

$$IA^*_{(*)} : \mathbf{Strat} \to \mathcal{P} - \mathbf{CDGA}$$

such that for any pseudomanifold X we have an isomorphism of perverse algebras

$$H^*(IA^*_{(*)}(X)) \cong IH^*_{(*)}(X, \mathbb{F}).$$

Let $\mathcal{IA}$ be the sheafification of IA then in $D(X)$

$$\mathcal{IA}^*_{(*)} \simeq \mathcal{D}_{(*)}$$

where $\mathcal{D}_{(*)}$ is the Deligne's sheaf.

Theorem (DC, Martin Saralegui, Daniel Tanré)

The algebraic homotopy type:

$$[IA^*_{(*)}(X)] \in Ho(\mathcal{P} - (CDGA))$$

is a topological invariant.

Theorem (DC, Martin Saralegui, Daniel Tanré)

The algebraic homotopy type:

$$[IA^*_{(*)}(X)] \in Ho(\mathcal{P} - (CDGA))$$

is a topological invariant.

Theorem (DC, Martin Saralegui, Daniel Tanré)

There exists singular spaces X and X' such that
- they have the same homotopy type,
- the same intersection cohomology algebras,
- but different algebraic homotopy types.

Intersection formality

Let X be a singular space, we say that X is intersection formal if we have a zig-zag of quasi-isos of perverse CDGAs:

$$IA^*_{(*)}(X) \leftarrow \bullet \rightarrow IH^*_{(*)}(X, \mathbb{Q}).$$

Intersection formality

Let X be a singular space, we say that X is intersection formal if we have a zig-zag of quasi-isos of perverse CDGAs:

$$IA^*_{(*)}(X) \leftarrow \bullet \rightarrow IH^*_{(*)}(X, \mathbb{Q}).$$

Intersection formality $\Rightarrow$ formality (the converse is not true).

Intersection formality

Let X be a singular space, we say that X is intersection formal if we have a zig-zag of quasi-isos of perverse CDGAs:

$$IA^*_{(*)}(X) \leftarrow \bullet \rightarrow IH^*_{(*)}(X, \mathbb{Q}).$$

Intersection formality $\Rightarrow$ formality (the converse is not true).
Intersection formality $\Rightarrow$ vanishing of intersection Massey products.

Formality of projective varieties

In 1975, Pierre Deligne, Phillip Griffiths, John Morgan and Dennis Sullivan proved:

Main Theorem. (i) *Let M be a compact complex manifold for which the dd^c-lemma holds* (e.g. *M Kähler, or M a Moišezon space*). *Then the real homotopy type of M is a formal consequence of the cohomology ring $H^*(M;\mathbb{R})$.*

Formality of projective varieties

In 1975, Pierre Deligne, Phillip Griffiths, John Morgan and Dennis Sullivan proved:

Main Theorem. (i) *Let M be a compact complex manifold for which the dd^c-lemma holds (e.g. M Kähler, or M a Moišezon space). Then the real homotopy type of M is a formal consequence of the cohomology ring $H^*(M;\mathbb{R})$.*

The proof relies on the Hodge decomposition:

$$H^k(M,\mathbb{C}) \cong \oplus_{p+q=k} H^{p,q}(M)$$

$$\overline{H^{p,q}(M)} \cong H^{q,p}(M).$$

Hodge theory

- (Deligne) The cohomology of singular/quasi-projective complex varieties carries a MHS-structure, in particular a weight filtration

$$W_{-k}H^k \subset \ldots \subset W_{-1}H^k \subset W_0H^k \subset \ldots W_1H^k \subset \ldots \subset W_kH^k.$$

each piece: $W_iH^*/W_{i-1}H^*$ carries a Hodge decomposition. This filtration is obtained thanks to Hironaka's resolution of singularities.

Hodge theory

- (Deligne) The cohomology of singular/quasi-projective complex varieties carries a MHS-structure, in particular a weight filtration

$$W_{-k}H^k \subset \ldots \subset W_{-1}H^k \subset W_0 H^k \subset \ldots W_1 H^k \subset \ldots \subset W_k H^k.$$

each piece: $W_i H^*/W_{i-1}H^*$ carries a Hodge decomposition. This filtration is obtained thanks to Hironaka's resolution of singularities.
- (Morgan, Hain) The rational homotopy type of a quasi-projective variety carries a MHS. $A^*(X)$ has a weight filtration.

Hodge theory

- (Deligne) The cohomology of singular/quasi-projective complex varieties carries a MHS-structure, in particular a weight filtration

$$W_{-k}H^k \subset \ldots \subset W_{-1}H^k \subset W_0H^k \subset \ldots W_1H^k \subset \ldots \subset W_kH^k.$$

each piece: $W_iH^*/W_{i-1}H^*$ carries a Hodge decomposition. This filtration is obtained thanks to Hironaka's resolution of singularities.

- (Morgan, Hain) The rational homotopy type of a quasi-projective variety carries a MHS. $A^*(X)$ has a weight filtration.

- (Navarro-Aznar, Cirici) Thanks to Deligne's results we know that the spectral sequence associated to the weight filtration of $A^*(X)$ collapses at the E_2-term, in fact we have quasi-isos:

$$A^*(X) \leftarrow \bullet \rightarrow E_1(X).$$

This is E_1-formality.

Hodge theory and intersection cohomology

- (Beilinson-Bernstein-Deligne-Gabber, Saito) Intersection cohomology is endowed with a MHS, satisfies hard Lefschetz theorem. If $dim_{\mathbb{C}}(X) = n$ then for the middle perversity $IH^*_{(n/2)}(X)$ is pure.

Hodge theory and intersection cohomology

- (Beilinson-Bernstein-Deligne-Gabber, Saito) Intersection cohomology is endowed with a MHS, satisfies hard Lefschetz theorem. If $dim_{\mathbb{C}}(X) = n$ then for the middle perversity $IH^*_{(n/2)}(X)$ is pure.
- (Durfee-Hain) Let L be the link of a singular point (more generally a singular stratum), its cohomology is equipped with a MHS.

Hodge theory and intersection cohomology

- (Beilinson-Bernstein-Deligne-Gabber, Saito) Intersection cohomology is endowed with a MHS, satisfies hard Lefschetz theorem. If $dim_{\mathbb{C}}(X) = n$ then for the middle perversity $IH^*_{(n/2)}(X)$ is pure.
- (Durfee-Hain) Let L be the link of a singular point (more generally a singular stratum), its cohomology is equipped with a MHS.
- (Goresky-MacPherson) If X admits a small resolution $f : V \to X$ s.t. $dim(f^{-1}(S)) < n/2$ then $IH^*_{n/2}(X) \cong H^*(V, \mathbb{C})$ compatible with the Hodge structure.

Theorem (DC, Joana Cirici)

Let X be a complex projective variety with isolated singularities then $IA^*_{(*)}(X, \mathbb{C})$ carries a MHS, and we have E_1-formality.

Theorem (DC, Joana Cirici)

Let X be a complex projective variety with isolated singularities then $IA^*_{(*)}(X, \mathbb{C})$ carries a MHS, and we have E_1-formality.

Theorem (DC, Joana Cirici)

Let X be a complex projective variety with isolated singularities, then X is intersection formal if:

Theorem (DC, Joana Cirici)

Let X be a complex projective variety with isolated singularities then $IA^*_{(*)}(X, \mathbb{C})$ carries a MHS, and we have E_1-formality.

Theorem (DC, Joana Cirici)

Let X be a complex projective variety with isolated singularities, then X is intersection formal if:
- X is a $\mathbb{Q}$-homology manifold,

Theorem (DC, Joana Cirici)

Let X be a complex projective variety with isolated singularities then $IA^*_{(*)}(X, \mathbb{C})$ carries a MHS, and we have E_1-formality.

Theorem (DC, Joana Cirici)

Let X be a complex projective variety with isolated singularities, then X is intersection formal if:

- X is a $\mathbb{Q}$-homology manifold,

- $dim_{\mathbb{C}} X \leq 2$,

Theorem (DC, Joana Cirici)

Let X be a complex projective variety with isolated singularities then $IA^*_{(*)}(X, \mathbb{C})$ carries a MHS, and we have E_1-formality.

Theorem (DC, Joana Cirici)

Let X be a complex projective variety with isolated singularities, then X is intersection formal if:
- X is a $\mathbb{Q}$-homology manifold,
- $dim_{\mathbb{C}} X \leq 2$,
- an hypersurface (more generally a complete intersection).

Singular quartic

Let C be the nodal cubic:

$$\{y^2z - x^2z - x^3\} \subset \mathbb{C}P^2$$

and let D be a smooth quartic

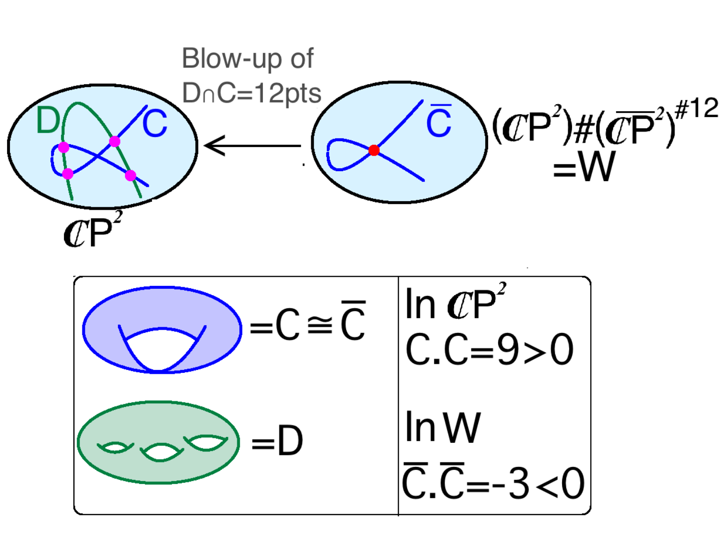
$$\{f(x,y,z) = 0\} \subset \mathbb{C}P^2$$

such that $C \pitchfork D$, we have $card(C \cap D) = 12$. And let us consider:

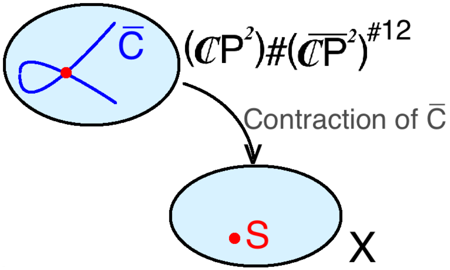
$$X = \{w(y^2z - x^2z - x^3) + f(x,y,z) = 0\} \subset \mathbb{C}P^3.$$

X has only one singular point $S = [1:0:0:0] \in X$.

Resolution

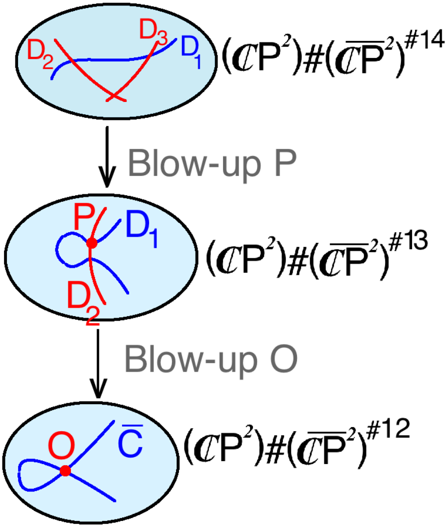


Resolution

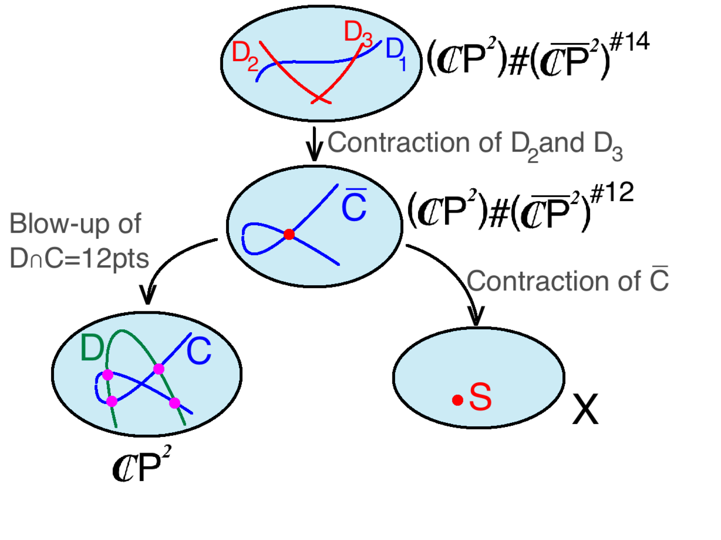


Castelnuovo criterion: as $\bar{C}.\bar{C}<0$,
we can contract $\bar{C}$ to a point.
X is a projective variety.

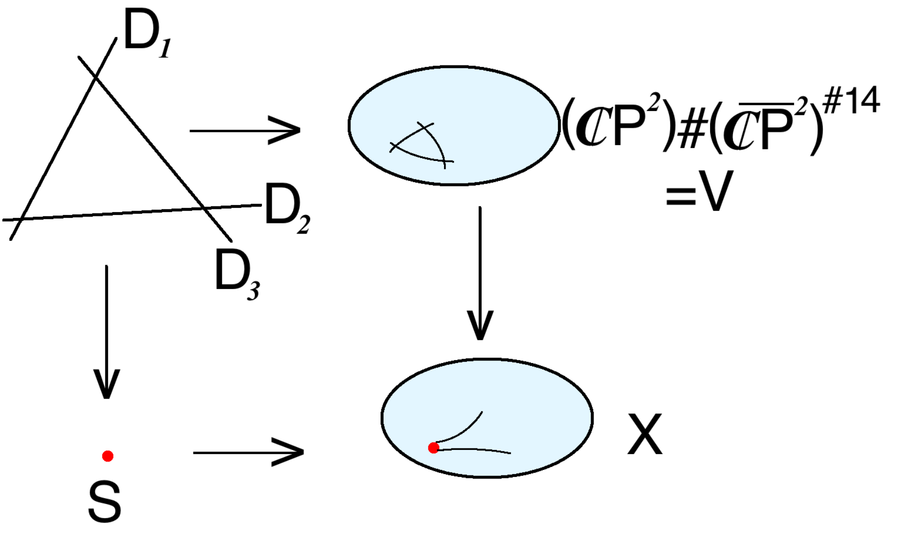
Resolution


D3
D2
D1
(ℂP²)#(ℂP²)#14
Blow-up P
P
D1
D2
(ℂP²)#(ℂP²)#13
Blow-up O
O
C̄
(ℂP²)#(ℂP²)#12

Resolution


D2
D3
D1
(CP2)#(CP2)#14
Contraction of D2 and D3
C̄
(CP2)#(CP2)#12
Blow-up of
D∩C=12pts
Contraction of C̄
D
C
S
X
CP2

Resolution



E_1-model

E_1-model of X:

$E_1^{*,0}$	$E_1^{*,1}$	$E_1^{*,2}$
$H^*(V) \oplus H^*(S)$	$\oplus_{i=1}^{3} H^*(D_i)$	$\oplus_{i \neq j} H^*(D_i \cap D_j)$

E_1-model

E_1-model of X:

$E_1^{*,0}$	$E_1^{*,1}$	$E_1^{*,2}$
$H^*(V) \oplus H^*(S)$	$\oplus_{i=1}^3 H^*(D_i)$	$\oplus_{i \neq j} H^*(D_i \cap D_j)$

The E_1-model for $IH^*(X)$ depends on:
- cohomology algebra $H^*(V)$,
- cohomology algebras $H^*(D_{i_1} \cap \ldots \cap D_{i_k})$,
- restriction maps,

$p : V \to X$ is a resolution and $p^{-1}(S) = \cup_i D_i$ is a normal crossing divisor.

Intersection Cohomology

$* \setminus (p)$	(0)	(1)	(2)
4	$\mathbb{C}.t$	$\mathbb{C}.t$	$\mathbb{C}.t$
3	0	0	0
2	$\mathbb{C}^{\oplus 12} \oplus \mathbb{C}.v$	$\mathbb{C}^{\oplus 12}$	$\mathbb{C}^{\oplus 12} \oplus \mathbb{C}.Dv$
1	0	0	0
0	$\mathbb{C}.1$	$\mathbb{C}.1$	$\mathbb{C}.1$

Intersection Cohomology

$* \setminus (p)$	(0)	(1)	(2)
4	$\mathbb{C}.t$	$\mathbb{C}.t$	$\mathbb{C}.t$
3	0	0	0
2	$\mathbb{C}^{\oplus 12} \oplus \mathbb{C}.v$	$\mathbb{C}^{\oplus 12}$	$\mathbb{C}^{\oplus 12} \oplus \mathbb{C}.Dv$
1	0	0	0
0	$\mathbb{C}.1$	$\mathbb{C}.1$	$\mathbb{C}.1$

$$W_0 IH^2_{(0)}(X, \mathbb{C}) \cong Ker : H^2(V, \mathbb{C}) \to H^2(D_1 \cup D_2 \cup D_3, \mathbb{C}) \cong \mathbb{C}^{\oplus 12}.$$

Intersection Cohomology

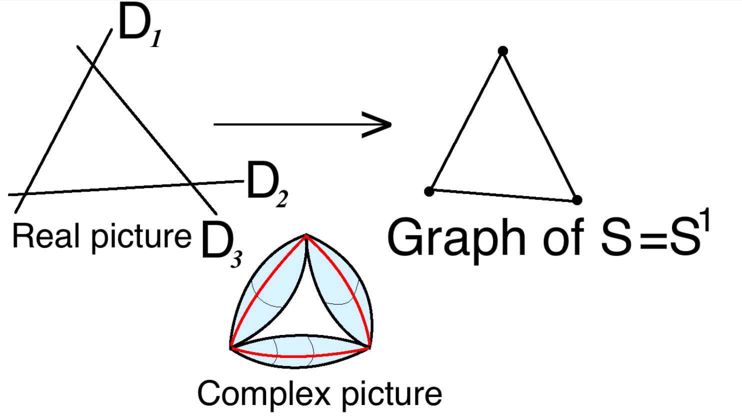
$* \setminus (p)$	(0)	(1)	(2)
4	$\mathbb{C}.t$	$\mathbb{C}.t$	$\mathbb{C}.t$
3	0	0	0
2	$\mathbb{C}^{\oplus 12} \oplus \mathbb{C}.v$	$\mathbb{C}^{\oplus 12}$	$\mathbb{C}^{\oplus 12} \oplus \mathbb{C}.Dv$
1	0	0	0
0	$\mathbb{C}.1$	$\mathbb{C}.1$	$\mathbb{C}.1$

$$W_0 IH^2_{(0)}(X, \mathbb{C}) \cong Ker : H^2(V, \mathbb{C}) \to H^2(D_1 \cup D_2 \cup D_3, \mathbb{C}) \cong \mathbb{C}^{\oplus 12}.$$

$$W(v) = 2, W(Dv) = -2, W(t) = 0.$$

Vanishing cycle

v comes from the exceptional divisor:



More fun

C: a nodal quartic of equation g, D: a smooth quintic of equation h. C and D are transverse.

More fun

C: a nodal quartic of equation g, D: a smooth quintic of equation h. C and D are transverse.

$$X = \{[w, x, y, z] : wg(x, y, z) + h(x, y, z) = 0\} \subset \mathbb{C}P^3$$

$* \setminus (p)$	(0)	(1)	(2)
4	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
3	0	0	0
2	$\mathbb{C}^{\oplus 20} \oplus \mathbb{C}_1^{\oplus 4} \oplus \mathbb{C}_2$	$\mathbb{C}^{\oplus 20}$	$\mathbb{C}^{\oplus 20} \oplus \mathbb{C}_{-1}^{\oplus 4} \oplus \mathbb{C}_{-2}$
1	0	0	0
0	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

More fun

C: a nodal quartic of equation g, D: a smooth quintic of equation h. C and D are transverse.

$$X = \{[w,x,y,z] : wg(x,y,z) + h(x,y,z) = 0\} \subset \mathbb{C}P^3$$

$* \setminus (p)$	(0)	(1)	(2)
4	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
3	0	0	0
2	$\mathbb{C}^{\oplus 20} \oplus \mathbb{C}_1^{\oplus 4} \oplus \mathbb{C}_2$	$\mathbb{C}^{\oplus 20}$	$\mathbb{C}^{\oplus 20} \oplus \mathbb{C}_{-1}^{\oplus 4} \oplus \mathbb{C}_{-2}$
1	0	0	0
0	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

$20 = Card(C \cap D)$
$4 = 2g(C)$
$dim(W_{-2})$ is the number of cycles in the graph of the singularity.

The End.
Thank You.