On the Resurgent WKB Analysis

IHES, 12 June 2019

David Sauzin (CNRS - IMCCE, Paris Observatory - PSL University)

Joint work with Frédéric Fauvet (Strasbourg) and Ricardo Schiappa (Lisbon)

Exact WKB method and Parametric Resurgence

A. Voros 1983, J. Écalle 1984, 1994

Delabaere-Dillinger-Pham DDP 1993

... ...

Kawai-Takei 2005, T. Koike

... ...

Gaiotto-Moore-Neitzke 2008, Kontsevich-Soibelman 2008, Pasquetti-Schiappa 2010, Garoufalidis-Its-Kapaev-Mariño 2012, Iwaki-Nakanishi 2014, Ito-Mariño-Shu 2019

... ...

Our aim: explain where resurgence comes from, using Ecalle's inductive construction of elementary non-trivial resurgent functions, for which "mould formalism" is particularly efficient.

- Ecalle's moulds help elucidate the resurgent structure
- they help establish the bridge equations behind the Stokes automorphisms and the DDP formula
- they also induce recursive constructions of *n*-point correlation functions in the spirit of other recursions.

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = \eta^2 Q(x)\psi \qquad \text{(SE)}$$

originates with quantum mechanics: $\eta = 1/\hbar$ (or a complex variable $\sim \infty$)
$Q(x) \in \mathbb{C}[x]$ "energy − potential" (or $Q = Q_0(x) + \eta^{-1}Q_1(x) + \cdots$).

Data: a meromorphic quadratic differential $Q(x)\mathrm{d}x^2$ on $\mathbb{P}^1$, regular on $\dot{\mathbb{C}} := \mathbb{P}^1 \backslash \{x_1, \ldots, x_M, \infty\}$ where x_j's = turning points = zeroes of Q.

Complex curve $\dot{\mathbb{C}}_2 := \{(x,p) \in \dot{\mathbb{C}} \times \mathbb{C}^* \mid p^2 = Q(x)\}$, double cover of $\dot{\mathbb{C}}$ on which $\lambda_0 := p\,\mathrm{d}x$ is a square root of $Q(x)\mathrm{d}x^2$.

WKB method produces an $\hbar$-extension of λ_0: a formal 1-form on $\dot{\mathbb{C}}_2$

$$\lambda = \mathcal{P}(x,\eta)\,\mathrm{d}x, \quad \mathcal{P}(x,\eta) = Q^{1/2}(x) + \eta^{-1}\mathcal{P}_1(x) + \eta^{-2}\mathcal{P}_2(x) + \cdots$$

($\mathcal{P}\mathrm{d}x$ invariant by change of coordinates, coeff. uniquely determined and holomorphic on $\dot{\mathbb{C}}_2$). *[Definition recalled later.]*

We want to study $\mathcal{P}(x,\eta)$ and its Borel transform w.r.t. parameter η

$$\mathcal{B}_{\eta\to\xi}\mathcal{P} = \hat{\mathcal{P}}(x,\xi) = Q^{1/2}(x)\delta(\xi) + \sum_{k\geqslant 1} \mathcal{P}_k(x)\xi^{k-1}/(k-1)!.$$

Resurgence: convergent for $|\xi|$ small enough and endlessly continuable (isolated singularities).

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = \eta^2 Q(x)\psi \qquad \text{(SE)}$$

WKB formal solutions:

$$\psi^+ = \mathrm{e}^{\mathrm{i}\eta A(x)}\, Q^{-1/4}(x)\, \varphi^+(x,\eta), \quad \psi^- = \mathrm{e}^{-\mathrm{i}\eta A(x)}\, Q^{-1/4}(x)\, \varphi^-(x,\eta)$$

with $A(x) = \int_{x_0}^{x} Q^{1/2}(x')\,\mathrm{d}x'$ multivalued on $\dot{\mathbb{C}}_2$ and

$$\varphi^\pm(x,\eta) = 1 + \eta^{-1}\varphi_0^\pm(x) + \eta^{-2}\varphi_1^\pm(x) + \cdots$$

multivalued and determined up to a multiplicative x-independent factor.

We'll be interested in the Borel transforms w.r.t. parameter η

$$\mathcal{B}_{\eta\to\xi}\varphi^\pm = \hat{\varphi}^\pm(x,\xi) = \delta(\xi) + \sum_{k\geqslant 0} \varphi_k^\pm(x)\xi^k/k!$$

at least for certain choices of normalization.

Resurgence: convergent for $|\xi|$ small enough and endlessly continuable (isolated singularities).

Simplest example: one turning point of multiplicity m and nothing else

$$Q(x) = c^2 x^m$$

(TP at $x_1 = 0$, pole at $x_\infty = \infty$). Then one finds

$$\psi^\pm = \mathrm{e}^{\pm \mathrm{i}\eta A(x)} Q^{-1/4}(x)\, \varphi^\pm(x,\eta)$$

with $A(x) = c\, x^{\frac{m+2}{2}}$ (choosing $x_0 = 0$), $Q^{-1/4}(x) = c^{-1/2} x^{-m/4}$, and

$$\mathcal{B}\big[\eta^{-\mu-1}\varphi^+\big] = \tfrac{\xi^\mu}{\Gamma(\mu+1)}\big(1 - \tfrac{\xi}{2\mathrm{i}A(x)}\big)^\mu, \quad \mathcal{B}\big[\eta^{-\mu-1}\varphi^-\big] = \tfrac{\xi^\mu}{\Gamma(\mu+1)}\big(1 + \tfrac{\xi}{2\mathrm{i}A(x)}\big)^\mu$$

with $\mu := -\frac{1}{2}\frac{m}{m+2}$.

Resurgence: The singularity of $\mathcal{B}\varphi^+$ at $\xi = 2\mathrm{i}A(x)$ is proportional to $\mathcal{B}\varphi^-$ and vice-versa *[blackboard].*

Exercise in alien calculus:

$$\Delta_{2\mathrm{i}A(x)}\varphi^+ = 2\mathrm{i}\sin(\pi\mu)\,\varphi^-, \quad \Delta_{-2\mathrm{i}A(x)}\varphi^- = 2\mathrm{i}\sin(\pi\mu)\,\varphi^+.$$

[Ecalle 1981, Vol. 2]

Airy $m = 1 \rightsquigarrow$ Stokes const $S = -\mathrm{i}$. Harm. osc. $m = 2 \rightsquigarrow S = -\mathrm{i}\sqrt{2}$.

Associated Riccati equation

Set $P = \frac{1}{\mathrm{i}\eta}\partial_x \log\psi$. Since $\mathrm{i}\eta P = \frac{\partial_x\psi}{\psi} \;\Rightarrow\; \mathrm{i}\eta\,\partial_x P = \frac{\partial_x^2\psi}{\psi} - (\mathrm{i}\eta)^2 P^2$,

$$\frac{\partial_x^2\psi}{\psi} = (\mathrm{i}\eta)^2 Q \qquad \text{(SE)} \qquad \Longleftrightarrow \qquad P^2 = Q(x) + \mathrm{i}\eta^{-1}\partial_x P \qquad \text{(RE)}$$

There are only two formal solutions, $P_+(x,\eta)$ and $P_-(x,\eta) = P_+(x^*,\eta)$

$$P_\pm(x,\eta) = \pm Q^{1/2}(x) + \frac{\mathrm{i}}{4}\eta^{-1}\partial_x \log Q(x) \pm \eta^{-2}\left(\frac{5}{32}\frac{(\partial_x Q)^2}{Q^{5/2}} - \frac{1}{8}\frac{\partial_x^2 Q}{Q^{3/2}}\right) + \cdots$$

from which we recover all WKB formal solutions to (SE):

$$\psi^\pm = C(\eta)\exp\left(\mathrm{i}\eta\left(\frac{\partial}{\partial x}\right)^{-1} P_\pm\right) = \tilde{C}(\eta)\,\mathrm{e}^{\pm\mathrm{i}\eta A(x)}\,Q^{-1/4}\varphi^\pm.$$

$\lambda = \mathcal{P}\,\mathrm{d}x$? Define "$\hbar$-extended momentum" $\mathcal{P} := \frac{1}{2}(P_+ - P_-)$.

$$P_\pm = \pm\mathcal{P} + \frac{1}{2}(P_+ + P_-) = \pm\mathcal{P} - \frac{1}{2\mathrm{i}\eta}\partial_x \log\mathcal{P}.$$

Help to handle normalization choices:

The coefficients of $P_\pm$ are uniquely determined and holomorphic on $\dot{\mathbb{C}}_2$, their Puiseux expansions at the x_j's and at $x_\infty = \infty$ do not contain the term $(x - x_j)^{-1}$.

Choice of normalization for $\psi^\pm$

$$= \text{selection of termwise primitive w.r.t. } x \text{ for } P_\pm(x,\eta)$$
$$= \text{selection of termwise primitive w.r.t. } x \text{ for } \mathcal{P}(x,\eta)$$

$$\mathcal{P} = Q^{1/2}\big(1 + \tfrac{1}{\mathrm{i}\eta}\mathcal{Y}\big), \quad \mathcal{Y} = -\frac{1}{\mathrm{i}\eta}\left(\frac{5}{32}\frac{(\partial_x Q)^2}{Q^3} - \frac{1}{8}\frac{\partial_x^2 Q}{Q^2}\right) + \cdots$$

$$P_\pm = \pm\mathcal{P} - \frac{1}{2\mathrm{i}\eta}\partial_x \log\mathcal{P} = \pm Q^{1/2}(x) - \frac{1}{2\mathrm{i}\eta}\partial_x \log\mathcal{P} \pm \frac{1}{\mathrm{i}\eta}Q^{1/2}\mathcal{Y}$$

"x_j-normalized solution" for $j \in \{1, \ldots, M, \infty\}$:

$$\psi_j^\pm := \mathrm{e}^{\pm\mathrm{i}\eta A(x)}\,\mathcal{P}^{-1/2}\exp\big(\pm\big[\tfrac{\partial}{\partial x}\big]_j^{-1}(Q^{1/2}\mathcal{Y})\big)$$

where $\big[\frac{\partial}{\partial x}\big]_j^{-1}$ indicates the primitive whose Puiseux expansion at x_j does not contain any constant term (equivalently: the average of the primitives $\int_a^x$ for a varying along the positive cycle around x_j in $\dot{\mathbb{C}}_2$).

Alternative notation: $\left[\frac{\partial}{\partial x}\right]_j^{-1} g(x) = \int_{\textcircled{x_j}}^{x} g(x')\,\mathrm{d}x'$

$$= \frac{1}{2}\int_{\gamma_{j,x}} g(x')\,\mathrm{d}x' \text{ if } g(x^*) = -g(x).$$

$$\psi_j^{\pm} = \mathrm{e}^{\pm \mathrm{i}\eta A(x)}\,\mathcal{P}^{-1/2} \exp\Big(\pm\, \mathrm{i}\eta \int_{\textcircled{x_j}}^{x} (\lambda - \lambda_0)\Big), \quad A(x) = \int_{x_0}^{x} \lambda_0.$$

Connection between different normalizations: "Voros coefficients"

$$\psi_k^{+}(x,\eta) = \mathrm{e}^{\Pi_{j,k}}\,\psi_j^{+}(x,\eta), \quad \psi_k^{-}(x,\eta) = \mathrm{e}^{-\Pi_{j,k}}\,\psi_j^{-}(x,\eta)$$

$$\Pi_{j,k}(\eta) = \int_{\textcircled{x_j}}^{\textcircled{x_k}} Q^{1/2}\mathcal{Y} = \frac{\mathrm{i}\eta}{2}\int_{\gamma_{j,k}} (\lambda - \lambda_0) \text{ quantum period}$$

$$\Pi_{j,k}(\eta) = \Pi_k(\eta) - \Pi_j(\eta), \quad \Pi_j(\eta) = \mathrm{i}\eta \int_{x_0}^{\textcircled{x_j}} (\lambda - \lambda_0).$$

For the x_∞-normalization, in view of the decay at ∞ of the coefficients of $Q^{1/2}\mathcal{Y}$, one can replace $\left[\frac{\partial}{\partial x}\right]_\infty^{-1}$ with usual $\int_{x_\infty}^x$.

$$P_\pm = \pm Q^{1/2}(x) - \frac{1}{4\mathrm{i}\eta}\partial_x \log Q(x) + \frac{1}{\mathrm{i}\eta} Q^{1/2} Y_\pm$$

$$\text{defines } Y_\pm = \eta^{-1} Y_0^\pm(x) + \eta^{-2} Y_1^\pm(x) + \cdots$$

One gets $\mathcal{Y} = \frac{1}{2}(Y_+ - Y_-)$ and

$$\psi_\infty^\pm = \mathrm{e}^{\pm\mathrm{i}\eta A(x)} Q^{-1/4} \varphi_\infty^\pm, \quad \varphi_\infty^\pm = \exp\int_\infty^x Q^{1/2} Y_\pm.$$

φ_∞^+ and φ_∞^- are characterized by the vanishing at ∞ of all coefficients except the 0^{th} one.

$$Y_\pm = Q^{-1/2}\partial_x \log \varphi_\infty^\pm$$

Why are $Y_\pm$ and the $\varphi_j^\pm$'s resurgent? Why is $\varphi_\infty^\pm$ resurgent?

We define a vector field ∂ on $\dot{\mathbb{C}}_2$ by the formula

$$\partial := Q(x)^{-1/2}\frac{\partial}{\partial x}$$

(dual to the square root of the quadratic differential $Q(x)\mathrm{d}x^2$).

This way, under the change $\psi^\pm = \mathrm{e}^{\pm \mathrm{i}\eta A(x)} Q^{-1/4}\varphi^\pm$, (SE) becomes

$$\partial^2\varphi^\pm \pm 2\mathrm{i}\eta\,\partial\varphi^\pm = K\varphi^\pm \tag{SE'}$$

with notation $H := \partial \log Q^{1/2} = \frac{1}{2}\frac{\partial_x Q}{Q^{3/2}}$ and $K := \frac{1}{2}\partial H + \frac{1}{4}H^2$.

Correspondingly, under $P_\pm = \pm Q^{1/2}(x) - \frac{1}{4\mathrm{i}\eta}\partial_x \log Q(x) + \frac{1}{\mathrm{i}\eta}Q^{1/2}Y_\pm$,

$$\partial Y_\pm \pm 2\mathrm{i}\eta Y_\pm = -Y_\pm^2 + K \tag{RE'}$$

and $Y_\pm = \partial \log \varphi^\pm$. The "$x_\infty$-normalized solution" is $\varphi_\infty^\pm = \exp\bigl(\partial_\infty^{-1} Y_\pm\bigr)$, where ∂_∞^{-1} denotes the "x_∞-based" right inverse to ∂.

Compute $\varphi^+ = \varphi^+_\infty$ and $\chi^+ = \partial\varphi^+_\infty$ (note that $Y_+ = \frac{\chi^+}{\psi^+}$)

as well as $\varphi^- = \varphi^-_\infty$ and $\chi^- = (\partial - 2\mathrm{i}\eta)\varphi^-_\infty$ (then $Y_- = \frac{\chi^-}{\psi^-} + 2\mathrm{i}\eta$)

by Neumann series:

$$\partial^2\varphi^+ + 2\mathrm{i}\eta\,\partial\varphi^+ = K\varphi^+ \Leftrightarrow \left| \begin{array}{ll} \partial\varphi^+ & = \chi^+ \\ (\partial + 2\mathrm{i}\eta)\chi^+ & = K\varphi^+ \end{array} \right. \Leftrightarrow \left| \begin{array}{l} \varphi^+ = 1 + \partial_\infty^{-1}\chi^+ \\ \chi^+ = (\partial + 2\mathrm{i}\eta)^{-1}\mathcal{K}\varphi^+ \end{array} \right.$$

with operators acting on $\mathcal{O}(\dot{\mathbb{C}}_2)[[\eta^{-1}]]$: $\mathcal{K} =$ multiplication operator and

$$(\partial + 2\mathrm{i}\eta)^{-1} = \sum_{k\geqslant 0} (-1)^k (2\mathrm{i}\eta)^{-k-1}\partial^k.$$

$$\varphi^+ = \sum_{n\geqslant 0} \big(\partial_\infty^{-1}(\partial + 2\mathrm{i}\eta)^{-1}\mathcal{K}\big)^n 1, \quad \chi^+ = \sum_{n\geqslant 0} (\partial + 2\mathrm{i}\eta)^{-1}\mathcal{K}\big(\partial_\infty^{-1}(\partial + 2\mathrm{i}\eta)^{-1}\mathcal{K}\big)^n 1$$

Similarly $\varphi^- = -2\mathrm{i}\eta \sum (\partial - 2\mathrm{i}\eta)^{-1}\big(\partial_\infty^{-1}\mathcal{K}(\partial - 2\mathrm{i}\eta)^{-1}\big)^n 1$

and $\chi^- = -2\mathrm{i}\eta \sum \big(\partial_\infty^{-1}\mathcal{K}(\partial + 2\mathrm{i}\eta)^{-1}\big)^n 1.$

A series of formal series in η^{-1}, each of which is resurgent.

Counterpart of $(\partial + 2\mathrm{i}\varepsilon\eta)^{-1}$ in the Borel plane: when applied to $g = g(x)$ or $G(x,\eta) = \sum g_k(x)\eta^{-k-1}$, we find

$$\mathcal{B}\big((\partial + 2\mathrm{i}\varepsilon\eta)^{-1} g\big) = \tfrac{1}{2\mathrm{i}\varepsilon}\mathcal{B}\Big(\sum \big(\tfrac{-1}{2\mathrm{i}\varepsilon}\big)^k \eta^{-k-1}\partial^k g\Big) = \tfrac{1}{2\mathrm{i}\varepsilon}\sum \tfrac{1}{k!}\big(\tfrac{-\xi}{2\mathrm{i}\varepsilon}\big)^k \partial^k g$$

$$= \tfrac{1}{2\mathrm{i}\varepsilon}\exp\big(\tfrac{-\xi}{2\mathrm{i}\varepsilon}\partial\big) g = g\big(\mathcal{T}^{-\xi/2\mathrm{i}\varepsilon}(x)\big),$$

where $t \mapsto \mathcal{T}^t$ is the time-t flow map of ∂.

$$\mathcal{B}\big((\partial + 2\mathrm{i}\varepsilon\eta)^{-1} G\big) = \tfrac{1}{2\mathrm{i}\varepsilon}\sum \tfrac{1}{k!}\big(\tfrac{-\xi}{2\mathrm{i}\varepsilon}\big)^k * \partial^k \hat{G}(x,\xi)$$

$$= \tfrac{1}{2\mathrm{i}\varepsilon}\int_0^\xi \exp\big(\tfrac{-\xi'}{2\mathrm{i}\varepsilon}\partial\big)\hat{G}(x,\xi-\xi')\,\mathrm{d}\xi'$$

$$= \tfrac{1}{2\mathrm{i}\varepsilon}\int_0^\xi \hat{G}\big(\mathcal{T}^{-\xi'/2\mathrm{i}\varepsilon}(x),\xi-\xi'\big)\,\mathrm{d}\xi'.$$

CLAIM: $\hat{\varphi}^{\pm}$ and $\hat{\chi}^{\pm}$ are endlessly continuable, with only possible singularities located at $\xi = \omega(x)$ determined by

$$\mathcal{T}^{\mp\omega(x)/2\mathrm{i}}(x) \in \{x_1, \ldots, x_M\}.$$

Idea: The vector field $\partial = Q(x)^{-1/2} \frac{\partial}{\partial x}$ is straightened by the Liouville transformation $z = \int_{x_0}^{x} Q^{1/2}(x')\,\mathrm{d}x' = A(x) \iff x = x(z) = \mathcal{T}^{z}(x_0)$.

$$\mathcal{T}^{t}\big(x(z)\big) = x(z+t).$$

$z \mapsto \mathcal{T}^{z}(x_0)$ has periodic-multivalued (!) analytic continuation, singular at $\alpha_j := \int_{\Gamma_j} \lambda_0$ and at the replicas obtained by addition of classical periods $2(\alpha_k - \alpha_j)$. The function K is meromorphic on $\mathbb{C}$, with poles at the x_j's, hence in coordinate z it has endless analytic continuation with singularities at $\tilde{\alpha} \in \{\alpha_j + \sum_k 2m_k(\alpha_k - \alpha_j)\}$: all the possible values of A at the TPs. We get $z \mp \omega/2\mathrm{i} = \tilde{\alpha}$, i.e.

$$\omega(x) = \pm 2\mathrm{i}\big(A(x) - \tilde{\alpha}\big).$$

The condition $\mathcal{T}^{-\omega(x)/2\mathrm{i}}(x) \in \{x_1, \ldots, x_M\}$ for $\hat{\varphi}_\infty^+$ comes from

$$\varphi_\infty^+ = \sum \left(\partial_\infty^{-1}(\partial + 2\mathrm{i}\eta)^{-1}\mathcal{K}\right)^n 1$$

involving $\mathcal{B}\big((\partial + 2\mathrm{i}\eta)^{-1} g\big) = g\big(\mathcal{T}^{-\xi/2\mathrm{i}}(x)\big)$ and $\mathcal{B}\big((\partial + 2\mathrm{i}\eta)^{-1} G\big) = \ldots$
Similarly $\hat{\varphi}_\infty^- \rightsquigarrow \mathcal{B}(\partial - 2\mathrm{i}\eta)^{-1} \rightsquigarrow \mathcal{T}^{\omega(x)/2\mathrm{i}}(x) \in \{x_1, \ldots, x_M\}$

Next question: What are $\Delta_{2\mathrm{i}(A-\alpha_j)}\varphi^+$ and $\Delta_{-2\mathrm{i}(A-\alpha_j)}\varphi^-$?

$\Delta_{2\mathrm{i}(A-\alpha_j)}\varphi_j^+ = S_j\, \varphi_j^-,\ \Delta_{-2\mathrm{i}(A-\alpha_j)}\varphi_j^- = S_j\, \varphi_j^+$ with $S_j = -2\mathrm{i}\sin(\frac{\pi}{2}\frac{m_j}{m_j+2})$.

$$\varphi_\infty^+(x,\eta) = \mathrm{e}^{\Pi_{j,\infty}(\eta)}\, \varphi_j^+(x,\eta), \quad \varphi_\infty^-(x,\eta) = \mathrm{e}^{-\Pi_{j,\infty}(\eta)}\, \varphi_j^-(x,\eta)$$

$$\Delta_{2\mathrm{i}(A-\alpha_j)}\varphi_\infty^+ = S_j\, \mathrm{e}^{2\Pi_{j,\infty}}\, \varphi_\infty^-, \quad \Delta_{2\mathrm{i}(A-\alpha_j)}\varphi_\infty^- = S_j\, \mathrm{e}^{-2\Pi_{j,\infty}}\, \varphi_\infty^+.$$

One also gets formulas for $\Delta_{\pm 2\mathrm{i}(A-\alpha_j)} Y_\pm \ldots$

Next question: $\Delta_\omega \mathrm{e}^{\pm 2\Pi_{j,\infty}} = ?$
$\omega = 2\mathrm{i}(\alpha_k - \alpha_\ell)$... DDP formula...

We stop our comments on resurgence here, to move on to something else.

We obtained representations of $\varphi_\infty^\pm$ and $\chi^\pm$ as series of elementary non-trivial resurgent series. Can we do the same for $Y_+ = \frac{\chi^+}{\varphi^+}$ and $Y_- = \frac{\chi^-}{\varphi^-} + 2\mathrm{i}\eta$?

$$\begin{aligned}
\varphi^+ &= \sum \left(\partial_\infty^{-1}(\partial + 2\mathrm{i}\eta)^{-1}\mathcal{K}\right)^n 1 &&= \sum M^{(-+)^n} \\
\chi^+ &= (\partial + 2\mathrm{i}\eta)^{-1}\mathcal{K} \sum \left(\partial_\infty^{-1}(\partial + 2\mathrm{i}\eta)^{-1}\mathcal{K}\right)^n 1 &&= \sum M^{+(-+)^n} \\
\varphi^- &= -2\mathrm{i}\eta(\partial - 2\mathrm{i}\eta)^{-1} \sum \left(\partial_\infty^{-1}\mathcal{K}(\partial - 2\mathrm{i}\eta)^{-1}\right)^n 1 &&= \sum M^{-(+-)^n} \\
\chi^- &= -2\mathrm{i}\eta \sum \left(\partial_\infty^{-1}\mathcal{K}(\partial + 2\mathrm{i}\eta)^{-1}\right)^n 1 &&= \sum M^{(+-)^n}
\end{aligned}$$

with

$$M^\varnothing := 1 \text{ and } M^{\varepsilon_1\cdots\varepsilon_r} := \left(\partial + 2\mathrm{i}\eta(\varepsilon_1 + \cdots + \varepsilon_r)\right)_\infty^{-1}\left(b_{\varepsilon_1} M^{\varepsilon_2\cdots\varepsilon_r}\right)$$

where $b_+ = K$ and $b_- = 1$. This defines a "mould" M.

Family of series indexed by words $\varepsilon = \varepsilon_1 \cdots \varepsilon_r$ on the alphabet $\{+, -\}$.

Here, only a slice of mould M: only alternate words...

$$M^{\varnothing} := 1 \ \text{and} \ M^{\varepsilon_1\cdots\varepsilon_r} := \big(\partial + 2\mathrm{i}\eta(\varepsilon_1 + \cdots + \varepsilon_r)\big)^{-1}_{\infty}\big(b_{\varepsilon_1} M^{\varepsilon_2\cdots\varepsilon_r}\big)$$

where $b_+ = 1$ and $b_- = K$.

$$\varphi^+ = \sum M^{(-+)^n},\ \chi^+ = \sum M^{+(-+)^n},\ \varphi^- = \sum M^{-(+-)^n},\ \chi^- = \sum M^{(+-)^n}.$$

CLAIM: Define $B_+ = \frac{\partial}{\partial y}$ and $B_- = -y^2\frac{\partial}{\partial y}$. Then

$$Y_+ = \sum \beta^+_{\varepsilon_1\cdots\varepsilon_r} M^{\varepsilon_1\cdots\varepsilon_r}, \quad Y_- = \sum \beta^-_{\varepsilon_1\cdots\varepsilon_r} M^{\varepsilon_1\cdots\varepsilon_r},$$

with $\beta^+_{\varepsilon_1\cdots\varepsilon_r} = B_{\varepsilon_r}\cdots B_{\varepsilon_1} y$ if $\varepsilon_1 + \cdots + \varepsilon_r = 1$ and 0 else,

and $\beta^-_{\varepsilon_1\cdots\varepsilon_r} = (-1)^{r+1}\beta^+_{(-\varepsilon_1)\cdots(-\varepsilon_r)}$ (zero unless $\varepsilon_1 + \cdots + \varepsilon_r = -1$).

$B_{\varepsilon_r}\cdots B_{\varepsilon_1}$ is homogeneous of degree $-(\varepsilon_1 + \cdots + \varepsilon_r)$, hence $\beta^+_{\varepsilon_1\cdots\varepsilon_r} \in \mathbb{Z}$

$$\beta^+_{+} = 1,\ \beta^+_{++-} = -2,\ \beta^+_{-++} = \beta^+_{+-+} = 0,\ \text{etc.}$$

Remark: one can check $\sum \beta^+_{(-\varepsilon_1)\cdots(-\varepsilon_r)} M^{\varepsilon_1\cdots\varepsilon_r} = \dfrac{1}{-2\mathrm{i}\eta + Y_-}$.

Explanation: The key point is a certain family of quadratic relations.

An R-valued mould on $\mathscr{A}$ is just the collection of values

$$V^{a_1 \cdots a_n} = \mathcal{V}_n(a_1, \dots, a_n)$$

of a sequence of functions $\mathcal{V}_0, \mathcal{V}_1, \dots$, with $\mathcal{V}_n \colon \mathscr{A}^n \to R$.

Quadratic relations called "symmetrality": $\mathcal{V}_0 = 1_R$ and, for all p, q,

$$\mathcal{V}_p(b_1, \dots, b_p)\mathcal{V}_q(c_1, \dots, c_q) = \sum_{\substack{I \subset \{1,\dots,n\} \\ |I|=p,\ J:=\{1,\dots,n\}\setminus I}} \mathcal{V}_n\big((b,c)_{I,J}\big)$$

$\uparrow$ the n-tuple a such that $a_{\{I\}} = b$ and $a_{\{J\}} = c$ (with $n := p + q$)

Example: $\mathcal{V}_n(a_1, \dots, a_n) = \frac{1}{(a_1+\cdots+a_2)(a_2+\cdots+a_r)\cdots a_r}$ with $\mathscr{A} = \mathbb{Z}_{>0}$ (or...)

Equivalently: $V^{\varnothing} = 1_R$ and, for all words b, c, $\quad V^b V^c = \sum_{a \in \operatorname{shuffle}(b,c)} V^a.$

Our mould M on $\mathscr{A} = \{+, -\}$ is symmetral (easy proof by induction).

What symmetrality is good for... (SE) was written for φ^+ in the form of a system

$$\left| \begin{aligned} &\partial\varphi = \chi \\ &\partial\chi = -2\mathrm{i}\eta\chi + K\varphi \end{aligned} \right.$$

Equivalently, consider $L := \partial - 2\mathrm{i}\eta y_2 \frac{\partial}{\partial y_2} + b_+\, y_1 \frac{\partial}{\partial y_2} + b_-\, y_2 \frac{\partial}{\partial y_1}$ operator acting in $R[[y_1, y_2]]$, where $R := \mathcal{O}(\dot{\mathbb{C}}_2)[[\eta^{-1}]]$.

Let $L_0 := \partial - 2\mathrm{i}\eta y_2 \frac{\partial}{\partial y_2}$, $\bar{B}_+ := y_1 \frac{\partial}{\partial y_2}$, $\bar{B}_- := y_2 \frac{\partial}{\partial y_1}$, so

$$L = L_0 + b_+ \bar{B}_+ + b_- \bar{B}_-.$$

All these are *derivations*. Consider the operator $\Theta := \sum M^{\varepsilon_1 \cdots \varepsilon_r} \bar{B}_{\varepsilon_r} \cdots \bar{B}_{\varepsilon_1}$. The inductive definition of M says that

$$\Theta L = L_0 \Theta.$$

Because M is symmetral, Θ is an *automorphism*, thus $\Theta f = f \circ \theta$ with $\theta(y_1, y_2) = (\Theta y_1, \Theta y_2)$. In fact

$$\theta(y_1, y_2) = (y_1\varphi^+ + y_2\varphi^-,\, y_1\chi^+ + y_2\chi^-).$$

$$\bar{B}_+ = y_1 \tfrac{\partial}{\partial y_2}, \quad \bar{B}_- := y_2 \tfrac{\partial}{\partial y_1}, \quad \Theta := \sum M^{\varepsilon_1 \cdots \varepsilon_r} \bar{B}_{\varepsilon_r} \cdots \bar{B}_{\varepsilon_1},$$

$$\Theta y_1 = y_1 \varphi^+ + y_2 \varphi^-, \quad \Theta y_2 = y_1 \chi^+ + y_2 \chi^-.$$

Now the general solution to (RE') is

$$Y = \frac{\sigma_+ \chi^+ + \sigma_- \mathrm{e}^{-2\mathrm{i}\eta} \chi^-}{\sigma_+ \varphi^+ + \sigma_- \mathrm{e}^{-2\mathrm{i}\eta} \varphi^-} = \frac{\chi^+ + \sigma \mathrm{e}^{-2\mathrm{i}\eta} \chi^-}{\varphi^+ + \sigma \mathrm{e}^{-2\mathrm{i}\eta} \varphi^-} = \theta_{\mathrm{RE}}(\sigma \mathrm{e}^{-2\mathrm{i}\eta}), \quad \sigma = \tfrac{\sigma_-}{\sigma_+}$$

$$\theta_{\mathrm{RE}}(y) = \frac{\chi^+ + y\, \chi^-}{\varphi^+ + y\, \varphi^-}, \quad y = \tfrac{y_2}{y_1}.$$

Because Θ is an automorphism (symmetrality),

$$\theta_{\mathrm{RE}}(y) = \frac{\chi^+ + \frac{y_2}{y_1} \chi^-}{\varphi^+ + \frac{y_2}{y_1} \varphi^-} = \frac{\Theta y_2}{\Theta y_1} = \Theta\big(\tfrac{y_2}{y_1}\big) = \sum M^{\varepsilon_1 \cdots \varepsilon_r} \bar{B}_{\varepsilon_r} \cdots \bar{B}_{\varepsilon_1}\big(\tfrac{y_2}{y_1}\big).$$

We get $\bar{B}_+\big(\tfrac{y_2}{y_1}\big) = 1 = B_+ y$ and $\bar{B}_-\big(\tfrac{y_2}{y_1}\big) = -\big(\tfrac{y_2}{y_1}\big)^2 = B_- y$, and since $B_{\varepsilon_r} \cdots B_{\varepsilon_1} y = \beta_\varepsilon\, y^{-(\varepsilon_1 + \cdots + \varepsilon_r)+1}$, we find

$$Y_+ = \theta_{\mathrm{RE}}(0) = \sum M^{\varepsilon_1 \cdots \varepsilon_r} \beta_\varepsilon^+.$$

Food for thought... $Y_\pm = \sum \beta_\varepsilon^\pm M^\varepsilon$. Recursive definition of M $\rightsquigarrow$

$$M^\varepsilon = \big(\partial + 2\mathrm{i}\eta(\varepsilon_1 + \cdots + \varepsilon_r)\big)^{-1}\Big(\\ b_{\varepsilon_1}\big(\partial + 2\mathrm{i}\eta(\varepsilon_2 + \cdots + \varepsilon_r)\big)^{-1}\Big(\\ \cdots b_{\varepsilon_{r-1}}\big(\partial + 2\mathrm{i}\eta\varepsilon_r\big)^{-1} b_{\varepsilon_r}\Big)\cdots\Big)$$

using ∂_∞^{-1} whenever a suffix $\varepsilon_j \cdots \varepsilon_r$ of ε has vanishing sum.

"Resonance level": $\ell(\varepsilon) = n$ number of zero-sum suffixes
$\rightsquigarrow$ M^ε involves an n-fold integration from ∞.

Group together the words with the same resonance level:
$W_n^\pm := \sum_{\ell(\varepsilon)=n} \beta_\varepsilon^\pm M^\varepsilon$ is the "n-point component" in $Y_\pm$

$$Y_\pm = \sum_n W_n^\pm \quad \rightsquigarrow \quad \text{WKB solution } \psi_\infty^\pm = \mathrm{e}^{\pm\mathrm{i}\eta A(x)}\, Q^{-1/4}\, \mathrm{e}^{\partial_\infty^{-1} Y_\pm}.$$

THANKS !