Classical and Quantum Integrability
September, 2–6, 2019, Dijon, France

On dressing factors of 2-dimensional Toda field theories and multicomponent MKdV equations

Vladimir S. Gerdjikov

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria
e-mail: vgerjikov@math.bas.bg

It is my pleasure to thank the Organizing Committee for the hospitality.

Based on:

- V. S. Gerdjikov, G. G. Grahovski. Multi-component NLS Models on Symmetric Spaces: Spectral Properties versus Representations Theory. SIGMA **6** (2010), 044, 29 pages.

- V. S. Gerdjikov. On soliton interactions of vector nonlinear Schrödinger equations. AIP **1404** 57–67 (2011).

- V. S. Gerdjikov. Derivative Nonlinear Schrödinger Equations with $\mathbb{Z}_N$ and $\mathbb{D}_N$–Reductions. Romanian Journal of Physics, **58**, Nos. 5-6, 573-582 (2013).

- V S Gerdjikov, A B Yanovski. CBC systems with Mikhailov reductions by Coxeter Automorphism. I. Spectral Theory of the Recursion Operators. Studies in Applied Mathematics **134** (2), 145–180 (2015).

Plan:

Vector NLS equations

N**- soliton solutions via dressing method**

N**-soliton interactions and dressing**

Dressing factors and $\mathbb{Z}_h$**-reductions**

Conclusions

Two classes of vector NLS equations:

VNLS1 (Manakov model, Manakov, 1974):

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^{\,\dagger}, \vec{q})\vec{q}(x,t) = 0,$$

describes the propagation of birefringent EM pulses in fiber optics;

VNLS2 (Kulish and Sklyanin, 1983):

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^{\,\dagger}, \vec{q})\vec{q}(x,t) - (\vec{q}, s_0\vec{q})s_0\vec{q}^{\,*}(x,t) = 0,$$

$$s_0 = \sum_{k=1}^{2r-1} (-1)^k E_{k,2r-k}, \qquad (E_{kn})_{ij} = \delta_{ik}\delta_{nj}.$$

for $r = 2$ and $r = 3$ describes Bose-Einstein condensate with spin $F = 1$ and $F = 2$, respectively.

Hamiltonians:

$$H_{\rm VNLS1} = \int_{-\infty}^{\infty} dx \left((\partial_x \vec{q}^{\,\dagger}, \partial_x \vec{q}) - (\vec{q}^{\,\dagger}, \vec{q})^2 \right),$$

$$H_{\rm VNLS2} = \int_{-\infty}^{\infty} dx \left((\partial_x \vec{q}^{\,\dagger}, \partial_x \vec{q}) - (\vec{q}^{\,\dagger}, \vec{q})^2 + \frac{1}{2}(\vec{q}^{\,\dagger}, s_0\vec{q}^{\,*})(\vec{q}^{\,T}, s_0\vec{q}) \right),$$

Lax representations

VNLS1:

$$
\begin{aligned}
L\psi(x,t,\lambda) &\equiv i\partial_x\psi + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0. \\
M\psi(x,t,\lambda) &\equiv i\partial_t\psi + (V_0(x,t) + \lambda V_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0, \\
V_1(x,t) &= Q(x,t), \qquad V_0(x,t) = i\,\mathrm{ad}_J^{-1}\frac{dQ}{dx} + \frac{1}{2}\left[\mathrm{ad}_J^{-1}Q, Q(x,t)\right].
\end{aligned}
$$

Structure of the potentials and the scattering matrices

VNLS1

$$
Q(x,t) = \begin{pmatrix} 0 & \vec{q}^T \\ \vec{q}^* & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix}.
$$

$$
T(\lambda,t) = \begin{pmatrix} T_{11} & -\vec{b}^{-,T} \\ \vec{b}^+ & \mathbf{T}_{22} \end{pmatrix}, \qquad \hat{T}(\lambda,t) = \begin{pmatrix} T_{11}^* & \vec{b}^{+,\dagger} \\ -\vec{b}^{-,*} & \mathbf{T}_{22}^\dagger \end{pmatrix}.
$$

VNLS2

$$Q(x,t) = \begin{pmatrix} 0 & \vec{q}^T \\ \vec{q}^* & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix}$$

$$Q(x,t) = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{q}^* & 0 & s_0\vec{q} \\ 0 & \vec{q}^\dagger s_0 & 0 \end{pmatrix}, \qquad J = \mathrm{diag}(1, 0, \dots 0, -1).$$

Orthogonality: $X \in so(2r+1)$ if $X + S_0 X^T S_0 = 0$ where

$$S_0 = \sum_{k=1}^{2r+1} (-1)^{k+1} E_{k,2r+2-k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (E_{kn})_{ij} = \delta_{ik}\delta_{nj}$$

The direct and the inverse scattering problem for L

The Jost solutions of L are defined by:

$$\lim_{x\to-\infty} \phi(x,t,\lambda) e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to\infty} \psi(x,t,\lambda) e^{i\lambda Jx} = \mathbb{1}$$

and the scattering matrix $T(\lambda,t) \equiv \psi^{-1}\phi(x,t,\lambda)$. The typical reduction $Q(x,t) = Q^{\dagger}(x,t)$ satisfied by () imposes on $T(\lambda,t)$ the constraint $T^{\dagger}(\lambda,t) = \hat{T}(\lambda,t)$ for real values of $\lambda \in \mathbb{R}$, i.e.

Let us now consider $Q(x,t)$ and J as in (). The Jost solutions and the scattering matrix take values in the group $SO(2r+1)$ we can use the following block-matrix structure of $T(\lambda,t)$

$$T(\lambda,t) = \begin{pmatrix} m_1^+ & -\vec{b}^{-T} & \mathbf{c}_1^- \\ \vec{b}^+ & \mathbf{T}_{22} & -s_0\vec{B}^- \\ \mathbf{c}_1^+ & \vec{B}^{+T}s_0 & m_1^- \end{pmatrix}, \qquad \hat{T}(\lambda,t) = \begin{pmatrix} m_1^- & \vec{B}^{-T} & \mathbf{c}_1^- \\ -\vec{B}^+ & \hat{\mathbf{T}}_{22} & s_0\vec{b}^- \\ \mathbf{c}_1^+ & -\vec{b}^{+T}s_0 & m_1^+ \end{pmatrix},$$

where $\vec{b}^{\pm}(\lambda,t)$ and $\vec{B}^{\pm}(\lambda,t)$ are $2r-1$-component vectors, $\mathbf{T}_{22}(\lambda)$ is $(2r-1)\times(2r-1)$ block matrix, and $m_1^{\pm}(\lambda)$, and $\mathbf{c}_1^{\pm}(\lambda)$ are scalar functions. Below we often use 'hat' to denote the matrix inverse, i.e. $\hat{X} \equiv X^{-1}$.

From the Lax representation it follows that if $\vec{q}(x,t)$ satisfies the VNLS then the scattering matrix $T(\lambda,t)$ satisfies

$$i\frac{dT}{dt} - \lambda^2[J, T(\lambda,t)] = 0,$$

or in components:

$$i\frac{d\vec{b}^{\pm}}{dt} \pm 2\lambda^2 \vec{b}^{\pm}(t,\lambda) = 0, \quad i\frac{dT_{11}^{\pm}}{dt} = 0, \quad i\frac{d\mathbf{T}_{22}}{dt} = 0,$$

for the VNLS1 and

$$i\frac{d\vec{b}^{\pm}}{dt} \pm \lambda^2 \vec{b}^{\pm}(t,\lambda) = 0, \quad i\frac{d\vec{B}^{\pm}}{dt} \pm \lambda^2 \vec{B}^{\pm}(t,\lambda) = 0, \quad i\frac{dm_1^{\pm}}{dt} = 0, \quad i\frac{d\mathbf{m}_2^{\pm}}{dt} = 0,$$

for the VNLS2.

The fundamental analytic solution and the Riemann-Hilbert problem

$$\chi^{\pm}(x,t,\lambda) = \phi(x,t,\lambda) S_J^{\pm}(t,\lambda) = \psi(x,t,\lambda) T_J^{\mp}(t,\lambda) D_J^{\pm}(\lambda),$$

where $T_J^{\mp}(t,\lambda)$, $D_J^{\pm}(\lambda)$, $T_J^{\mp}(t,\lambda)$ are the Gauss factors of $T(\lambda,t)$:

$$T(\lambda,t) = T_J^- D_J^+ \hat{S}_J^+, \qquad T(\lambda,t) = T_J^+ D_J^- \hat{S}_J^-,$$

$$S_J^+(t,\lambda) = \begin{pmatrix} 1 & \vec{\tau}^{+,T} & d^+ \\ 0 & \mathbb{1} & s_0\vec{\tau}^+ \\ 0 & 0 & 1 \end{pmatrix}, \qquad S_J^-(t,\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ \vec{\tau}^- & \mathbb{1} & 0 \\ d^- & \vec{\tau}^{-,T}s_0 & 1 \end{pmatrix},$$

$$T_J^+(t,\lambda) = \begin{pmatrix} 1 & \vec{\rho}^{+,T} & c^+ \\ 0 & \mathbb{1} & s_0\vec{\rho}^+ \\ 0 & 0 & 1 \end{pmatrix}, \qquad S_J^-(t,\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ \vec{\rho}^- & \mathbb{1} & 0 \\ c^- & \vec{\rho}^{-,T}s_0 & 1 \end{pmatrix},$$

$$D_J^+(\lambda) = \operatorname{diag}(m_1^+, m_2^+, (m_1^+)^{-1}), \qquad D_J^-(\lambda) = \operatorname{diag}((m_1^-)^{-1}, m_2^-, m_1^-),$$

where $d^{\pm}(\lambda) = \frac{1}{2}(\vec{\tau}^{\pm,T}s_0\vec{\tau}^{\pm})$ and $c^{\pm}(\lambda) = \frac{1}{2}(\vec{\rho}^{\pm,T}s_0\vec{\rho}^{\pm})$. The functions $m_1^{\pm}$ and $\mathbf{m}_2^{\pm}$ are analytic for $\lambda \in \mathbb{C}_{\pm}$.
The reflection coefficients $\vec{\rho}^{\pm}$ and $\vec{\tau}^{\pm}$ are expressed by:

$$\vec{\rho}^- = \frac{\vec{B}^-}{m_1^-}, \qquad \vec{\tau}^- = \frac{\vec{B}^+}{m_1^-}, \qquad \vec{\rho}^+ = \frac{\vec{b}^+}{m_1^+}, \qquad \vec{\tau}^+ = \frac{\vec{b}^-}{m_1^+}.$$

The FAS $\chi^{\pm}(x,t,\lambda)$ are related by:

$$\chi^+(x,t,\lambda) = \chi^-(x,t,\lambda)G_{0,J}(\lambda,t), \qquad G_{0,J}(\lambda,t) = \hat{S}_J^-(\lambda,t)S_J^+(\lambda,t)$$

Below for convenience we introduce $\xi^{\pm}(x,\lambda) = \chi^{\pm}(x,\lambda)e^{i\lambda Jx}$ which satisfy the equation:

$$i\frac{d\xi^{\pm}}{dx} + Q(x)\xi^{\pm}(x,\lambda) - \lambda[J,\xi^{\pm}(x,\lambda)] = 0,$$

and the relation

$$\lim_{\lambda\to\infty} \xi^{\pm}(x,t,\lambda) = \mathbb{1},$$

Then $\xi^{\pm}(x,\lambda)$ satisfy the RHP's

$$\xi^{+}(x,t,\lambda) = \xi^{-}(x,t,\lambda)G_J(x,t,\lambda), \qquad \lim_{\lambda\to\infty} \xi^{-}(x,t,\lambda) = \mathbb{1},$$

$$G_J(x,t,\lambda) = e^{-i\lambda J(x+\lambda t)}G_{0,J}(\lambda)e^{i\lambda J(x+\lambda t)}.$$

The construction of the soliton solutions

Applying the Zakharov-Shabat dressing method for the VNLS. The symmetric space is $SU(n+1)/S(U(1)\times U(n))$ and the dressing factor is:

(Zakharov, Shabat Functional annal. and Appl. (1974)):

$$u_{1s}(x,t,\lambda) = \mathbb{1} + (c_1(\lambda) - 1)P_1(x,t), \qquad \hat{u}_{1s}(x,t,\lambda) = \mathbb{1} + (c_1^{-1}(\lambda) - 1)P_1(x,t),$$

$$u_{Ns}(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{N} \frac{A_k(x,t)}{\lambda - \lambda_k^+}, \qquad \hat{u}_{Ns}(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{N} \frac{B_k(x,t)}{\lambda - \lambda_k^-},$$

For the VNLS2 one needs an appropriate modification. The symmetric space is $SO(2n+1)/(SO(2n-1) \times SO(2))$. We will say that $u(x,t,\lambda) \in SO(2n+1)$ if:

$$\hat{u}(x,t,\lambda) = S_0 u^T(x,t,\lambda) S_0, \qquad S_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

For one-soliton solution we need TWO poles (Zakharov, Mikhailov CMP

(1980))

$$u(x,t,\lambda) = \mathbb{1} + \left((c_1(\lambda) - 1) P_1(x,t) + \left(\frac{1}{c_1(\lambda)} - 1 \right) \bar{P}_1(x,t) \right),$$

$$\bar{P}_1(x,t) = S_0 P_1(x,t)^T S_0, \qquad c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-}, \quad P_1 \bar{P}_1 = 0 \quad P_1^2 = P_1.$$

We can rewrite $u(x,t,\lambda)$ in the form:

$$u(x,t,\lambda) = \exp\left(\ln c_1(\lambda) (P_1(x,t) - \bar{P}_1(x,t)) \right). \qquad (1)$$

Note: Since $P_1 - \bar{P}_1 \in so(2n+1)$, then $u(x,t,\lambda) \in SO(2n+1)$;
Since $P_1 \bar{P}_1 = 0$ and $P_1^2 = P_1$, $\bar{P}_1^2 = \bar{P}_1$ then $u(x,t,\lambda)$ has only simple poles! VSG (2011).

For N-solitons we need $2N$ poles (Zakharov, Mikhailov CMP (1980)):

$$u(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{N} \left(\frac{A_k(x,t)}{\lambda - \lambda_k^+} + \frac{B_k(x,t)}{\lambda - \lambda_k^-} \right), \qquad \hat{u}(x,t,\lambda) = S_0 u^T(x,t,\lambda) S_0.$$

From $Q(x,t) = Q^\dagger(x,t)$ it follows that $\lambda_k^- = (\lambda_k^+)^*$.

The residues of u admit the following decomposition

$$A_k(x,t) = X_k(x,t)F_k^T(x,t), \qquad B_k(x,t) = Y_k(x,t)G_k^T(x,t),$$

where all matrices involved for simplicity are supposed to be of rank 1. For the pure solitonic case the factors F_k and G_k can be expressed by the trivial fundamental solutions

$$\chi_0^{\pm}(x,t,\lambda) = e^{-i\lambda(x+\lambda t)J},$$

$$F_k^T(x,t) = F_{k,0}^T[\chi_0^+(x,t,\lambda_k^+)]^{-1}, \qquad G_k^T(x,t) = G_{k,0}^T[\chi_0^-(x,t,\lambda_k^-)]^{-1}.$$

$$F_{k,0}^T S_0 F_{k,0} = 0, \qquad G_{k,0}^T S_0 G_{k,0} = 0.$$

For the vectors $X_k(x,t)$ and $Y_k(x,t)$ we obtain an algebraic system of equations

$$S_0 F_k = \sum_{l\neq k} \frac{X_l F_l^T S_0 F_k}{\lambda_l^+ - \lambda_k^+} + \sum_l \frac{Y_l G_l^T S_0 F_k}{\lambda_l^- - \lambda_k^+},$$

$$S_0 G_k = \sum_l \frac{X_l F_l^T S_0 G_k}{\lambda_l^+ - \lambda_k^-} + \sum_{l\neq k} \frac{Y_l G_l^T S_0 G_k}{\lambda_l^- - \lambda_k^-}.$$

Then the N-soliton solution can be recovered from $u(x,t,\lambda)$ by:

$$Q_{\rm Ns} = \lim_{\lambda\to\infty} \lambda(J - uJu^{-1}(x,t,\lambda)) = \left[J, \sum_{k=1}^{N} (A_k + B_k) \right].$$

We parametrize F_k and G_k by:

$$F_k(x,t) = S_0 |n_k(x,t)\rangle = \left(\begin{array}{c} n_{k0;1} e^{-z_k + i\phi_k} \\ -\sqrt{2} s_0 \vec{\nu}_{0k} \\ n_{k0;2r+1} e^{z_k - i\phi_k} \end{array} \right),$$

$$G_k(x,t) = |n_k^*(x,t)\rangle = \left(\begin{array}{c} n_{k0;1}^* e^{z_k + i\phi_k} \\ \sqrt{2} \vec{\nu}_{0k}^{\,*} \\ n_{k0;2r+1}^* e^{-z_k - i\phi_k} \end{array} \right),$$

where $\vec{\nu}_{0k}$ are constant $2r-1$-component polarization vectors and

$$z_j = \nu_j (x + 2\mu_j t), \qquad \phi_j = \mu_j x + (\mu_j^2 - \nu_j^2) t,$$
$$\langle n_j^T(x,t) | S_0 | n_j(x,t)\rangle = 0, \qquad \text{or} \qquad (\vec{\nu}_{0,j} s_0 \vec{\nu}_{0,j}) = 1.$$

For $N = 2$ this system was solved explicitly and the 2-soliton solution of the VNLS2 is:

$$Q_{2\mathrm{s}}(x,t) = [J, A_1 + A_2 + B_1 + B_2] = \frac{1}{Z}[J, C(x,t) - S_0 C^T(x,t) S_0],$$

$$\begin{aligned} C(x,t) &= \frac{\kappa_{22}}{\lambda_2^+ - \lambda_2^-}|n_1\rangle\langle n_1^\dagger| - \frac{\kappa_{12}}{\lambda_2^+ - \lambda_1^-}|n_1\rangle\langle n_2^\dagger| - \frac{\kappa_{21}}{\lambda_1^+ - \lambda_2^-}|n_2\rangle\langle n_1^\dagger| \\ &+ \frac{\kappa_{11}}{\lambda_1^+ - \lambda_1^-}|n_2\rangle\langle n_2^\dagger| - \frac{f_{12}^*}{\lambda_1^- - \lambda_2^-}|n_1\rangle\langle n_2|S_0 - \frac{f_{12}}{\lambda_1^+ - \lambda_2^+}S_0|n_2^*\rangle\langle n_1^\dagger|. \end{aligned}$$

where

$$Z(x,t) = \left(\frac{|f_{12}|^2}{|\lambda_2^+ - \lambda_1^+|^2} - \frac{\kappa_{12}\kappa_{21}}{|\lambda_2^+ - \lambda_1^-|^2} + \frac{\kappa_{11}\kappa_{22}}{4\nu_1\nu_2} \right),$$

$$\kappa_{ij}(x,t) = \langle n_i^\dagger | n_j \rangle, \qquad f_{ij}(x,t) = f_{ji}(x,t) = \langle n_i | S_0 | n_j \rangle.$$

Similarly one can derive the N-soliton solutions.

Next in we can calculated the asymptotics of the 2-soliton solution along the trajectory of the first soliton. To this end we keep $z_1(x,t)$ fixed and let $\tau = z_2 - z_1$ tend to $\pm\infty$. This is possible if $\mu_1 \neq \mu_2$, i.e the two solitons have different velocities. For definiteness we assume that $\mu_2 > \mu_1$.

$$\lim_{\tau\to\infty} \vec{q}_{2\mathrm{s}}(x,t;z_1,z_2;\phi_1,\phi_2) = \vec{q}_{1\mathrm{s}}(x,t;z_1+r_+,\phi_1-\alpha_+),$$
$$\lim_{\tau\to-\infty} \vec{q}_{2\mathrm{s}}(x,t;z_1,z_2;\phi_1,\phi_2) = \vec{q}_{1\mathrm{s}}(x,t;z_1-r_+,\phi_1+\alpha_+),$$

where $\vec{q}_{1\mathrm{s}}$ is the one-soliton solution

$$\vec{q}_{1\mathrm{s}}(x,t;z_1,\phi_1) = -\frac{i\sqrt{2}\nu_1 e^{-i(\phi_1)}\left(e^{-z_1} s_0|\vec{\nu}_{01}\rangle + e^{z_1}|\vec{\nu}^*_{01}\rangle\right)}{\cosh(2z_1) + (\vec{\nu}^\dagger_{01},\vec{\nu}_{01})},$$

and the shifts of its arguments

$$r_+ = \ln\left|\frac{\lambda_1^+ - \lambda_2^+}{\lambda_1^+ - \lambda_2^-}\right|, \qquad \alpha_+ = \arg\frac{\lambda_1^+ - \lambda_2^+}{\lambda_1^+ - \lambda_2^-}.$$

are expressed in terms of the discrete eigenvalues $\lambda_j^\pm$ only.

N-soliton interaction for the BD.I VNLS

An alternative derivation of the N-soliton solutions.

$$\chi^{\pm}_{(1)}(x,\lambda) = u_1(x,t,\lambda)\chi^{\pm}_{(0)}(x,t,\lambda)\hat{u}_1^{-}(\lambda),$$

$$u_1(x,\lambda) = \mathbb{1} + (c_1(\lambda)-1)P_1(x,t) + (c_1^{-1}(\lambda)-1)\bar{P}_1(x,t),$$

$$c_1(\lambda) = \frac{\lambda-\lambda_1^+}{\lambda-\lambda_1^-}, \qquad P_1 = \frac{|n_1\rangle\langle n_1|}{\langle n_1|n_1\rangle},$$

$$|n_1\rangle = \chi^{+}_{(0)}(x,t,\lambda_1^+)|n_{10}\rangle, \quad \langle n_1| = \langle n_{10}|\hat{\chi}^{-}_{(0)}(x,t,\lambda_1^-).$$

Then

$$Q_{(1)} = Q_{(0)} - 2i\nu_1[J, P_1(x,t) - \bar{P}_1(x,t)].$$

We will choose $Q_{(0)} = 0$; therefore

$$\chi^{\pm}_{(0)}(x,t,\lambda) = e^{-i(\lambda x+\lambda^2 t)J}.$$

Now we construct the N-soliton solution by applying N times the above procedure with the result:

$$u_{\mathrm{Ns}}(x,\lambda) = u_N(x,t,\lambda)\cdots u_1(x,t,\lambda)\hat{u}_N^-(\lambda)\cdots\hat{u}_1^-(\lambda),$$

where

$$u_s(x,\lambda) = \mathbb{1} + (c_s(\lambda)-1)\mathbf{P}_s(x,t) + (c_s^{-1}(\lambda)-1)\bar{\mathbf{P}}_s(x,t),$$

$$|\mathbf{n}_s\rangle = u_{s-1}(x,\lambda_s^+)\cdots u_1(x,\lambda_s^+)\hat{u}_1^-(x,\lambda_s^+)\cdots\hat{u}_{s-1}^-(x,\lambda_s^+)|n_s\rangle,$$

$$\langle\mathbf{n}_s| = \langle n_s|u_{s-1}^-(x,\lambda_s^-)\cdots u_1^-(x\lambda_s^-)\hat{u}_1(x,\lambda_s^-)\cdots\hat{u}_{s-1}(x,\lambda_s^-),$$

$$c_s(\lambda) = \frac{\lambda-\lambda_s^+}{\lambda-\lambda_s^-}, \qquad \mathbf{P}_s = \frac{|\mathbf{n}_s\rangle\langle\mathbf{n}_s|}{\langle\mathbf{n}|\mathbf{n}\rangle},$$

$$|n_s\rangle = \chi_{(0)}^+(x,t,\lambda_s^+)|n_{s0}\rangle = e^{(z_s-i\phi_s)J}|n_{s0}\rangle, \qquad z_s = \nu_s(x+2\mu_s t),$$

$$\langle n_s| = \langle n_{s0}|\hat{\chi}_{(0)}^-(x,t,\lambda_s^-) = \langle n_{s0}|e^{(z_s+i\phi_s)J}, \qquad \phi_s = \mu_s x + (\mu_s^2-\nu_s^2)t.$$

Lemma 1 *The asymptotics of* $\mathbf{P}_s(x,t)$ *for* $z_s \to \pm\infty$ *are given by:*

$$\lim_{z_s\to\infty} \mathbf{P}_s(x,t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \lim_{z_s\to-\infty} \mathbf{P}_s(x,t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Thus

$$\lim_{z_s\to\infty} u_s(x,t,\lambda) \equiv u_s^+(\lambda) = e^{J\ln c_s(\lambda)}, \qquad \lim_{z_s\to-\infty} u_s(x,t,\lambda) \equiv u_s^-(\lambda) = e^{-J\ln c_s(\lambda)}.$$

Assume that any two solitons move with different speeds.

$$\mu_k \neq \mu_s, \qquad \text{for} \qquad k \neq s.$$

Calculate the asymptotics of $u_{\rm Ns}(x,t)$ for taking $t \to \infty$ and $t \to -\infty$ keeping z_N fixed: For $\mu_s > \mu_N$ we get the result:

$$\lim_{\substack{t\to\infty \\ z_N=\text{fix}}} P_s(x,t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lim_{\substack{t\to-\infty \\ z_N=\text{fix}}} P_s(x,t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

For $\mu_s < \mu_N$ we have:

$$\lim_{\substack{t\to\infty \\ z_N=\mathrm{fix}}} P_s(x,t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \lim_{\substack{t\to-\infty \\ z_N=\mathrm{fix}}} P_s(x,t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Thus if $\mu_s > \mu_N$

$$\lim_{\substack{t\to\infty \\ z_N=\mathrm{fix}}} u_s(x,t) \equiv u_s^+(\lambda) = e^{J\ln c_s(\lambda)}, \qquad \lim_{\substack{t\to-\infty \\ z_N=\mathrm{fix}}} u_s(x,t) \equiv u_s^-(\lambda) = e^{-J\ln c_s(\lambda)}$$

$$= \begin{pmatrix} c_1(\lambda) & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & 1/c_1(\lambda) \end{pmatrix}, \qquad = \begin{pmatrix} 1/c_1(\lambda) & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & c_1(\lambda) \end{pmatrix};$$

for $\mu_s < \mu_N$:

$$\lim_{\substack{t\to\infty \\ z_N=\mathrm{fix}}} u_s(x,t) \equiv u_s^-(\lambda) = e^{-J\ln c_s(\lambda)}, \qquad \lim_{\substack{t\to-\infty \\ z_N=\mathrm{fix}}} u_s(x,t) \equiv u_s^+(\lambda) = e^{J\ln c_s(\lambda)},$$

$$= \begin{pmatrix} 1/c_1(\lambda) & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & c_1(\lambda) \end{pmatrix}, \qquad = \begin{pmatrix} c_1(\lambda) & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & 1/c_1(\lambda) \end{pmatrix}.$$

Finally we evaluate the limits of $\mathbf{P}_N(x,t)$ with the result:

$$\lim_{\substack{t\to\infty \\ z_N=\text{fix}}} P_N(x,t) = \frac{|\mathbf{n}_N^+\rangle\langle n_N^+|}{\langle n_N^+|n_N^+\rangle}, \qquad \lim_{\substack{t\to-\infty \\ z_N=\text{fix}}} P_s(x,t) = \frac{|\mathbf{n}_N^-\rangle\langle n_N^-|}{\langle n_N^-|n_N^-\rangle},$$

$$|\mathbf{n}_N^+\rangle = \prod_{s\in\sigma_\mu^+} (u^+(\lambda_N^+))^2|n_N\rangle, \qquad |\mathbf{n}_N^-\rangle = \prod_{s\in\sigma_\mu^-} (u^+(\lambda_N^+))^2|n_N\rangle,$$

where

$$\sigma_\mu^+ \equiv \{\mu_s,\ \mu_s > \mu_N\}_{s=1}^{N-1}, \qquad \sigma_\mu^- \equiv \{\mu_s,\ \mu_s < \mu_N\}_{s=1}^{N-1}.$$

Thus we find:

$$\lim_{\substack{t\to\infty \\ z_N=\text{fix}}} Q_{\rm Ns}(x,t) = Q_{\rm 1s}(z_N^+,\phi_N^+), \qquad \lim_{\substack{t\to-\infty \\ z_N=\text{fix}}} Q_{\rm Ns}(x,t) = Q_{\rm 1s}(z_N^-,\phi_N^-),$$

$$z_N^+ = z_N + 2 \sum_{s \in \sigma_\mu^+} \ln \left| \frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-} \right|, \qquad z_N^- = z_N - 2 \sum_{s \in \sigma_\mu^-} \ln \left| \frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-} \right|$$

$$\phi_N^+ = \phi_N + 2 \sum_{s \in \sigma_\mu^+} \arg \frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-}, \qquad \phi_N^- = \phi_N - 2 \sum_{s \in \sigma_\mu^-} \arg \frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-}.$$

The N-soliton interactions of VNLS2 are like for the scalar NLS case; the shifts of relative center-of-mass coordinates and of the phases:

$$z_N^+ - z_N^- = 2 \sum_{s \in \sigma_\mu^+} \ln \left| \frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-} \right| - 2 \sum_{s \in \sigma_\mu^-} \ln \left| \frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-} \right|,$$

$$\phi_N^+ - \phi_N^- = 2 \sum_{s \in \sigma_\mu^+} \arg \frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-} - 2 \sum_{s \in \sigma_\mu^-} \arg \frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-},$$

The polarization vectors $\vec{\nu}_{s0}$ are preserved.

Lax operators with $\mathbb{Z}_h$ and $\mathbb{D}_h$ symmetries

Let us fix up $\mathfrak{g} \simeq so(7)$, or $\mathfrak{g}_2$. The Coxeter number $h = 6$.

Lax pair:

$$\begin{aligned} L\psi &\equiv i\frac{\partial \psi}{\partial x} + (i\phi_x - \lambda \mathcal{J}_0)\psi(x,t,\lambda) = 0, \\ M\psi &\equiv i\frac{\partial \psi}{\partial x} - (i\phi_t + \frac{1}{\lambda}\mathcal{K}_0)\psi(x,t,\lambda) = 0, \end{aligned} \tag{2}$$

$$C^{-1}\phi_x C = \phi_x, \quad C^{-1}(\mathcal{J}_0)C = \omega\mathcal{J}_0, \quad C^{-1}(\mathcal{K}_0)C = \omega^{-1}\mathcal{K}_0, \quad \omega = e^{2\pi i/6}.$$

$$\mathcal{J}_0 = \operatorname{diag}(1, \omega, \omega^5, 0, -\omega^5, -\omega, -1), \qquad \mathcal{K}_0 = \mathcal{J}_0^*.$$

$$\phi = \frac{i\sqrt{3}}{6} \begin{pmatrix} 0 & -A_1 & A_1 & -\sqrt{2}A_1 & -A_2 & -A_2 & 0 \\ A_1 & 0 & -A_2 & -\sqrt{2}A_1 & A_1 & 0 & -A_2 \\ -A_1 & A_2 & 0 & -\sqrt{2}A_1 & 0 & A_1 & A_2 \\ \sqrt{2}A_1 & \sqrt{2}A_1 & \sqrt{2}A_1 & 0 & -\sqrt{2}A_1 & \sqrt{2}A_1 & -\sqrt{2}A_1 \\ A_2 & -A_1 & 0 & \sqrt{2}A_1 & 0 & -A_1 & -A_1 \\ A_2 & 0 & -A_1 & -\sqrt{2}A_1 & A_2 & 0 & -A_1 \\ 0 & A_2 & -A_2 & \sqrt{2}A_1 & A_2 & A_2 & 0 \end{pmatrix},$$

where $A_2 = 2\phi_1 - 3\phi_2$, $A_1 = 2\phi_0 - \phi_2$.

The 2-dimensional Toda field theory for $\mathfrak{g}_2$ has Lagrangian:

$$\begin{gathered} \mathcal{L} = (\vec{\phi}_x, \vec{\phi}_t) - \frac{1}{2}\left(e^{2(\beta_1,\vec{\phi})} + e^{2(\beta_2,\vec{\phi})} + e^{2(\beta_0,\vec{\phi})}\right), \\ \beta_1 = e_2 - e_3, \qquad \beta_2 = e_1 - e_2 + 2e_3, \qquad \beta_0 = -(2e_1 + e_2 + e_3), \end{gathered} \tag{3}$$

Reduction groups and Contours of RHP

The contour of RHP must be invariant with respect to the Reduction group G_R.

It splits the complex λ-plane into $|G_R|$ regions Ω_k; one of them Ω_1 is fundamental.

The RHP takes the form:

$$\xi_k(x,t,\lambda) = \xi_s(x,t,\lambda) G_{k,s}(x,t,\lambda), \qquad \lambda \in \Omega_k \cup \Omega_s.$$

$$i \frac{\partial G_{k,s}}{\partial x} = [J(\lambda), G_{k,s}(x,t,\lambda)], \qquad i \frac{\partial G_{k,s}}{\partial t} = [K(\lambda), G_{k,s}(x,t,\lambda)],$$

where $J(\lambda)$ and $K(\lambda)$ are also invariant with respect to the Reduction group G_R.

We parametrize the solution $\xi_k(x,t,\lambda)$ only in Ω_1.

Figure 1: The contour for the RHP of L with $\mathbb{Z}_6$-symmetry.

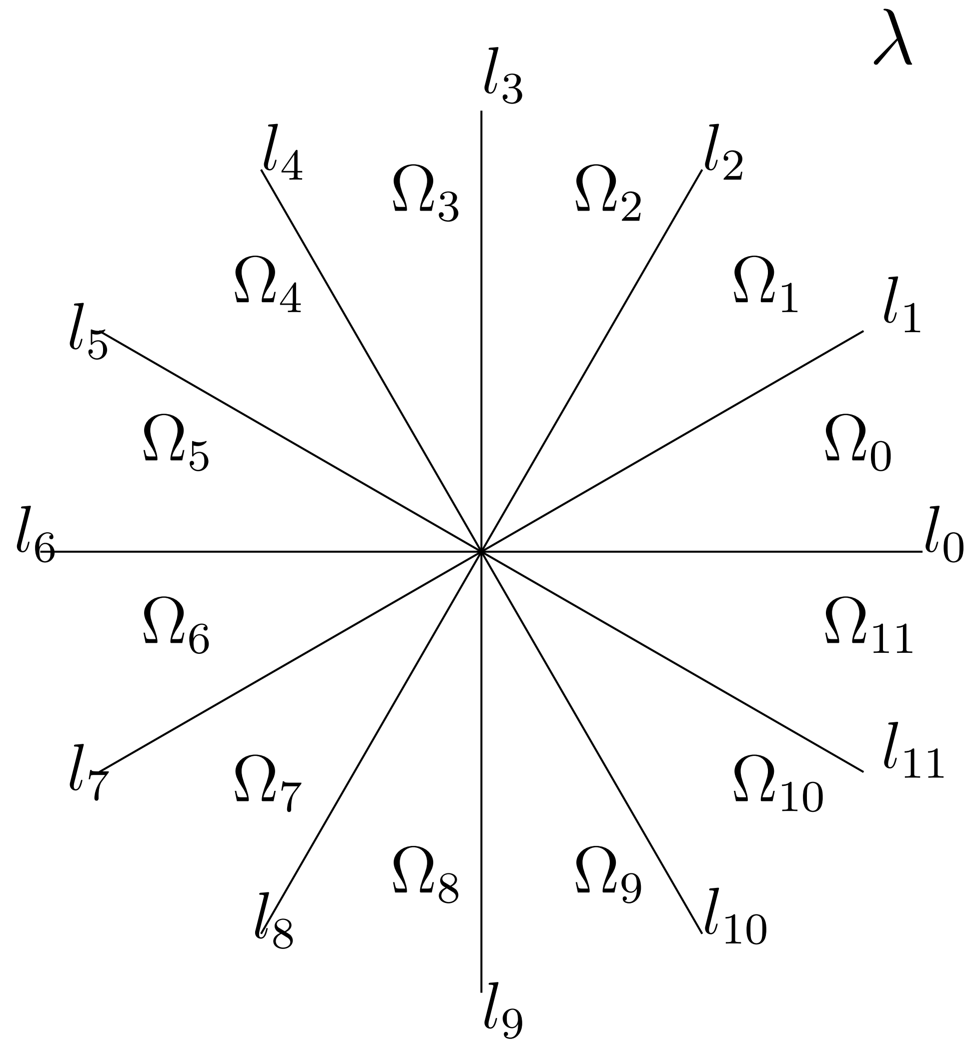
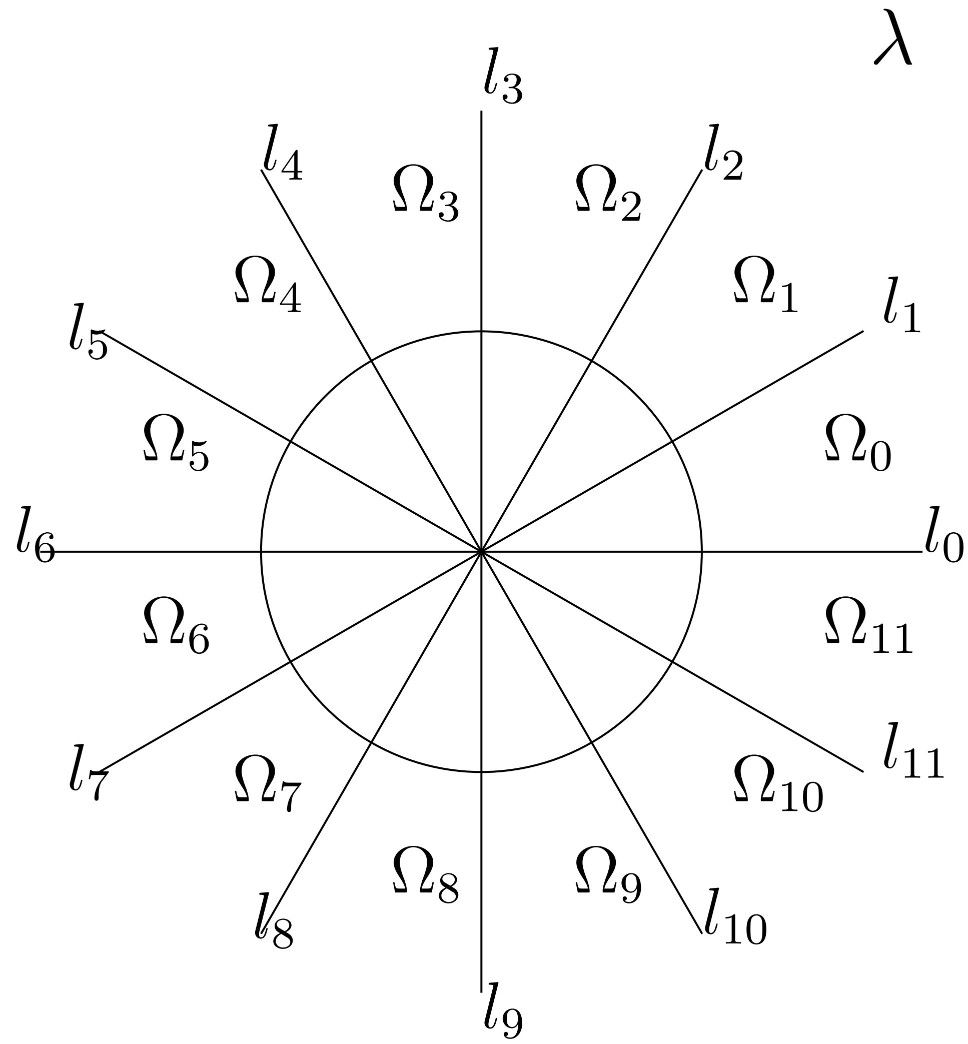


Figure 2: The contour for the RHP with $\mathbb{D}_6$-symmetry.



Dressing factors and $\mathbb{Z}_h$-reductions

The dressing factor must be a group element. In addition we need an anzatz which will result in that $u(x,t,\lambda)$ is a comparatively simple rational function of λ, which must be $\mathbb{Z}_6$-invariant. There is a chance to satisfy all the above requirements if we choose:

$$\begin{gathered} u(x,t,\lambda) = \exp\left(\sum_{s=0}^{2} \ln c_0(\lambda\omega^s)(P_s - \bar{P}_s) \right), \\ c_0(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda + \lambda_1^+}, \qquad c_1(\lambda) = \frac{\lambda - \lambda_1^-}{\lambda + \lambda_1^-}, \qquad \lambda_1^- = -(\lambda_1^+)^*, \\ C^{-1} P_s C = P_{s+1}, \qquad P_k P_m = \delta_{km} P_k, \qquad \bar{P}_k = S_0 P_k^T S_0. \end{gathered} \tag{4}$$

Here P_s and $\tilde{P}_s$ are mutually orthogonal projectors:

$$P_s P_k = P_k \delta_{sk}, \qquad P_s \bar{P}_k = 0. \tag{5}$$

$$
\begin{aligned}
u(x,t,\lambda) &= \mathbb{1} + \sum_{s=0}^{2} \left((c_0(\lambda\omega^s) - 1)P_s + (c_0^{-1}(\lambda\omega^s) - 1)\bar{P}_s \right) \\
u^{-1}(x,t,\lambda) &= \mathbb{1} + \sum_{s=0}^{2} \left((c_0(\lambda\omega^s) - 1)\bar{P}_s + (c_0^{-1}(\lambda\omega^s) - 1)P_s \right).
\end{aligned}
\tag{6}
$$

The equation for the dressing factor:

$$
i\frac{\partial u}{\partial x} + (i\phi_x - \lambda\mathcal{J}_0)u - u(i\phi_{0,x} - \lambda\mathcal{J}_0) = 0. \tag{7}
$$

If $\phi_{0,x} = 0$ the one-soliton potential will be determined from:

$$
i\phi_x = -2\lambda_1^+ \left[\mathcal{J}_0, \sum_{s=0}^{2} \omega^{-s}(P_s - \bar{P}_s) \right]. \tag{8}
$$

Conclusions

The important result for VNLS2 consists in the following:

- N-soliton interactions are purely elastic and always split into sequences of elementary 2-soliton interactions;
- the effect of each 2-soliton interaction consists in shifts of the relative center of mass and relative phases of each of the solitons;
- there are no non-trivial 3-soliton interactions;
- The polarization vectors $\vec{\nu}_{s0}$ are preserved.
- Method for constructing dressing factors for systems with $\mathbb{Z}_h$ reductions is proposed.

Thank you for your attention!