Integrable Lagrangians and Picard modular forms

E.V. Ferapontov

Department of Mathematical Sciences, Loughborough University, UK
E.V.Ferapontov@lboro.ac.uk

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Plan:

- Integrable Lagrangians \( \int f(v_{x_1}, v_{x_2}, v_{x_3}) \, dx_1 \, dx_2 \, dx_3 \)
- Integrability conditions
- Lagrangian densities \( f = v_{x_1} v_{x_2} g(v_{x_3}) \)
- Lagrangian densities \( f = v_{x_1} g(v_{x_2}, v_{x_3}) \)
- Generic Lagrangian densities \( f(v_{x_1}, v_{x_2}, v_{x_3}) \)

References:


Integrable Lagrangians \( \int f(v_{x_1}, v_{x_2}, v_{x_3}) \, dx_1 dx_2 dx_3 \)

Euler-Lagrange equation:

\[
(f_{v_{x_1}})_{x_1} + (f_{v_{x_2}})_{x_2} + (f_{v_{x_3}})_{x_3} = 0.
\]

Examples:

Dispersionless Kadomtsev-Petviashvili (dKP) equation

\[
v_{x_1 x_3} - v_{x_1} v_{x_1 x_1} - v_{x_2 x_2} = 0, \quad f = v_{x_1} v_{x_2} - \frac{1}{3} v_{x_1}^3 - v_{x_2}^2.
\]

Boyer-Finley (BF) equation

\[
v_{x_1 x_1} + v_{x_2 x_2} - e^{v_{x_3}} v_{x_3 x_3} = 0, \quad f = v_{x_1}^2 + v_{x_2}^2 - 2 e^{v_{x_3}}.
\]
Main problem

- The parameter space of integrable Lagrangian densities $f$ is 20-dimensional.
- Integrability conditions for $f$ are invariant under a 20-dimensional symmetry group which acts on the parameter space with an open orbit.

**Question:** What is the master-Lagrangian corresponding to the open orbit?

We will see that it is related to the theory of Picard modular forms.
Three equivalent approaches to integrability

- The method of hydrodynamic reductions based on the requirement that the equation possesses infinitely many multi-phase solutions of special type.
- The method of dispersionless Lax pairs based on the representation of the equation as the compatibility condition of two Hamilton-Jacobi type equations.
- Integrability ‘on solutions’ based on the condition that the characteristic variety of the equation defines a conformal structure which is Einstein-Weyl on every solution.

All three approaches lead to the same set of integrability conditions for the Lagrangian density $f$. 
Hydrodynamic reductions: example of dKP

First-order form of dKP equation $v_{x_1 x_3} - v_{x_1} v_{x_1} - v_{x_2} = 0$ (set $u = v_{x_1}$):

$$u_{x_3} - uu_{x_1} - w_{x_2} = 0, \quad u_{x_2} - w_{x_1} = 0.$$  

Look for $N$-phase solutions: $u = u(R^1, \ldots, R^n)$, $w = w(R^1, \ldots, R^n)$ where

$$R^{i}_{x_3} = \lambda^i(R) R^i_{x_1}, \quad R^{i}_{x_2} = \mu^i(R) R^i_{x_1}.$$  

The substitution of $u, w$ into the above first-order system implies

$$\partial_i w = \mu^i \partial_i u, \quad \lambda^i = u + (\mu^i)^2,$$

as well as the following equations for $u(R)$ and $\mu^i(R)$ (Gibbons-Tsarev system):

$$\partial_j \mu^i = \frac{\partial_j u}{\mu^j - \mu^i}, \quad \partial_i \partial_j u = 2 \frac{\partial_i u \partial_j u}{(\mu^j - \mu^i)^2}.$$  

In involution! General solution depends on $n$ arbitrary functions of one variable.

Dispersionless Lax pairs: example of dKP

The dKP equation \(v_{x_1 x_3} - v_{x_1} v_{x_1 x_1} - v_{x_2 x_2} = 0\) possesses dispersionless Lax representation

\[
S_{x_2} = \frac{1}{2} S_{x_1}^2 + v_{x_1}, \quad S_{x_3} = \frac{1}{3} S_{x_1}^3 + v_{x_1} S_{x_1} + v_{x_2}.
\]

In parametric form:

\[
S_{x_1} = p, \quad S_{x_2} = \frac{1}{2} p^2 + v_{x_1}, \quad S_{x_3} = \frac{1}{3} p^3 + v_{x_1} p + v_{x_2}.
\]


**Observation** Integrability by the method of hydrodynamic reductions is equivalent to the existence of a dispersionless Lax representation.
Integrability via Einstein-Weyl geometry: example of dKP

Einstein-Weyl geometry is a triple \((\mathcal{D}, g, \omega)\) where \(\mathcal{D}\) is a symmetric connection, \(g\) is a conformal structure and \(\omega\) is a covector such that

\[
\mathcal{D}_k g_{ij} = \omega_k g_{ij}, \quad R_{(ij)} = \Lambda g_{ij}.
\]

Here \(R_{(ij)}\) is the symmetrised Ricci tensor of \(\mathcal{D}\) and \(\Lambda\) is some function (the first set of equations defines \(\mathcal{D}\) uniquely, so it is sufficient to specify \(g\) and \(\omega\) only).

Every solution of the dKP equation \(v_{x_1 x_3} - v_{x_1} v_{x_1 x_1} - v_{x_2 x_2} = 0\) carries Einstein-Weyl geometry (Dunajski, Mason, Tod):

\[
g = 4 dx_1 dx_3 - dx_2^2 + 4v_{x_1} dx_3^2, \quad \omega = -4v_{x_1} x_1 dx_3.
\]

**Observation** Integrability is equivalent to the Einstein-Weyl property of the characteristic conformal structure of the equation.

Integrability conditions

For a non-degenerate Lagrangian, the Euler-Lagrange equation is integrable (by either of the techniques mentioned above) if and only if the Lagrangian density $f$ satisfies the relation

$$d^4 f = d^3 f \frac{dH}{H} + \frac{3}{H} \det(dM).$$

Here $d^3 f$ and $d^4 f$ are the symmetric differentials of $f$ while the Hessian $H$ and the $4 \times 4$ augmented Hessian matrix $M$ are defined as

$$H = \det \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{pmatrix}, \quad M = \begin{pmatrix} 0 & f_x & f_y & f_z \\ f_x & f_{xx} & f_{xy} & f_{xz} \\ f_y & f_{xy} & f_{yy} & f_{yz} \\ f_z & f_{xz} & f_{yz} & f_{zz} \end{pmatrix}.$$  

Here $(x, y, z) = (v_{x_1}, v_{x_2}, v_{x_3})$. The non-degeneracy condition is equivalent to $H \neq 0$. The system for $f$ is in involution, and its solution space is 20-dimensional.
Weierstrass sigma function $\sigma$ and integers $B_k$

Let $\sigma$ be the Weierstrass sigma function of the elliptic curve $y^2 = 4x^3 - \frac{1}{2}$ (case $g_2 = 0, \ g_3 = \frac{1}{2}$). This function satisfies the fourth-order ODE

$$\sigma\sigma''' - 4\sigma'\sigma'' + 3(\sigma'')^2 = 0$$

and possesses a power series expansion

$$\sigma(z) = \sum_{k \geq 0} B_k \frac{z^{6k+1}}{(6k + 1)!}$$

where $B_k$ are certain integers.
**Lagrangian density** \( \mathcal{L} = u_{x_1} u_{x_2} u_{x_3} \)

Euler-Lagrange equation:

\[
u_{x_3} u_{x_1 x_2} + u_{x_2} u_{x_1 x_3} + u_{x_1} u_{x_2 x_3} = 0.
\]

Parametric Lax representation:

\[
\frac{S_{x_1}}{u_{x_1}} = \zeta(p) + \frac{\wp'(p) + \lambda}{2\wp(p)}, \quad \frac{S_{x_2}}{u_{x_2}} = \zeta(p) + \frac{\wp'(p) - \lambda}{2\wp(p)}, \quad \frac{S_{x_3}}{u_{x_3}} = \zeta(p),
\]

where \((\wp')^2 = 4\wp^3 + \lambda^2\) and \(\zeta' = -\wp\) (Weierstrass \(\wp\) and \(\zeta\) functions).
Lagrangian densities \( f = v x_1 v x_2 g (v x_3) \)

Euler-Lagrange equation:

\[
(v x_2 g(v x_3))_{x_1} + (v x_1 g(v x_3))_{x_2} + (v x_1 v x_2 g'(v x_3))_{x_3} = 0.
\]

Integrability condition for \( g(z) \):

\[
g''''(g^2 g'' - 2g(g')^2) - 9(g')^2(g'')^2 + 2gg'g''' + 8(g')^3g''' - g^2(g''')^2 = 0.
\]

\( GL(2, \mathbb{R}) \)-invariance:

\[
\tilde{z} = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \tilde{g} = (\gamma z + \delta)g.
\]

This invariance allows one to linearise the integrability condition for \( g(z) \).
Auxiliary hypergeometric equation

Consider the auxiliary hypergeometric equation

$$u(1 - u)h_{uu} + (1 - 2u)h_u - \frac{2}{9}h = 0,$$

parameters $(1/3, 2/3, 1)$. The geometry behind this equation is a 1-parameter family of genus 2 trigonal curves

$$r^3 = q(q - 1)(q - u)^2$$

supplied with the holomorphic differential $\omega = dq/r$. The corresponding periods, $h = \int_a^b \omega$ where $a, b \in \{0, 1, \infty, u\}$, form a 2-dimensional vector space and satisfy the above (Picard-Fuchs) hypergeometric equation.
**Generic solution \( g(z) \)**

The generic solution \( g(z) \) can be represented in any of the three equivalent forms:

1. **Parametric form:**

   \[
   z = \frac{h_1(u)}{h_2(u)}, \quad g = h_2(u)
   \]

   where \( h_i \) are two linearly independent solutions of the hypergeometric equation. \( GL(2, \mathbb{R}) \)-invariance corresponds to the freedom in the choice of basis \( h_i \).

2. **Theta representation:**

   \[
   g(z) = \sum_{(k,l) \in \mathbb{Z}^2} e^{2\pi i (k^2 + kl + l^2)} z = 1 + 6q + 6q^3 + 6q^4 + 12q^4 + ... , \quad q = e^{2\pi i z}.
   \]

3. **Power series:**

   \[
   g(z) = \sum_{k \geq 0} B_k^2 \frac{z^{6k+1}}{(6k + 1)!}
   \]

   where \( B_k \) are integers (same as in the expansion of \( \sigma \) function).
Lagrangian densities \( f = v_1 g(v_2, v_3) \)

Euler-Lagrange equation

\[
(g)_{x_1} + (v_1 g v_2)_{x_2} + (v_1 g v_3)_{x_3} = 0.
\]

Integrability conditions lead to an involutive system of five PDEs for \( g(y, z) \) which are invariant under the 10-dimensional symmetry group:

\[
\tilde{y} = \frac{l_1(y, z)}{l(y, z)}, \quad \tilde{z} = \frac{l_2(y, z)}{l(y, z)}, \quad \tilde{g} = \alpha g + \beta,
\]

where \( l, l_1, l_2 \) are arbitrary (inhomogeneous) linear forms. This invariance allows one to linearise the integrability conditions for \( g(y, z) \).
**Auxiliary hypergeometric system**

Consider the auxiliary (Appell) hypergeometric system

\[ h_{u_1 u_2} = \frac{1}{3} \frac{h_{u_1} - h_{u_2}}{u_1 - u_2}, \]

\[ h_{u_1 u_1} = -\frac{h}{9u_1(u_1 - 1)} + \frac{h_{u_2}}{3(u_1 - u_2)} \frac{u_2(u_2 - 1)}{u_1(u_1 - 1)} - \frac{h_{u_1}}{3} \left( \frac{1}{u_1 - u_2} + \frac{2}{u_1} + \frac{2}{u_1 - 1} \right), \]

\[ h_{u_2 u_2} = -\frac{h}{9u_2(u_2 - 1)} + \frac{h_{u_1}}{3(u_2 - u_1)} \frac{u_1(u_1 - 1)}{u_2(u_2 - 1)} - \frac{h_{u_2}}{3} \left( \frac{1}{u_2 - u_1} + \frac{2}{u_2} + \frac{2}{u_2 - 1} \right). \]

The geometry behind this system is the family of genus 3 Picard trigonal curves

\[ r^3 = q(q - 1)(q - u_1)(q - u_2) \]

supplied with the holomorphic differential \( \omega = dq/r \). The corresponding periods,
\( h = \int_a^b \omega \) where \( a, b \in \{0, 1, \infty, u_1, u_2\} \), form a 3-dimensional vector space and satisfy the above (Picard-Fuchs) hypergeometric system.
**Generic solution** $g(y, z)$

The generic solution $g(y, z)$ can be represented in any of the 3 equivalent forms:

1. **Parametric form:**

   $$y = \frac{h_1(u_1, u_2)}{h_3(u_1, u_2)}, \quad z = \frac{h_2(u_1, u_2)}{h_3(u_1, u_2)}, \quad g = F(s), \quad s = \frac{u_1(u_2 - 1)}{u_2(u_1 - 1)}$$

   where $h_i$ are three linearly independent solutions of the hypergeometric system and $F' = [s(s - 1)]^{-2/3}$.

2. **Theta representation:**

   $$g(y, z) = y + \sum_{(k,l) \in \mathbb{Z}^2 \setminus 0} \frac{\sigma((k + \epsilon l)y)}{k + \epsilon l} e^{2\pi i (k^2 + kl + l^2)} z, \quad \epsilon = e^{\pi i / 3}.$$  

3. **Power series:**

   $$g(y, z) = \sum_{j,k \geq 0} B_j B_k B_{j+k} \frac{y^{6j+1}}{(6j + 1)!} \frac{z^{6k+1}}{(6k + 1)!}.$$
Relation to Picard modular forms

The period map

\[ y = \frac{h_1(u_1, u_2)}{h_3(u_1, u_2)}, \quad z = \frac{h_2(u_1, u_2)}{h_3(u_1, u_2)}, \]

was inverted by Picard (1883):

\[ u_1 = \frac{\varphi_1(y, z)}{\varphi_0(y, z)}, \quad u_2 = \frac{\varphi_2(y, z)}{\varphi_0(y, z)}, \]

where \( \varphi_\nu \) are single-valued modular forms on a 2-dimensional complex ball \( 2\text{Re}y + |z|^2 < 0 \) with respect to the Picard modular group

\[ \Gamma[\sqrt{-3}] = \{ g \in U(2, 1; \mathbb{Z}[\rho]) : g \equiv 1 (\text{mod } \sqrt{-3}) \}, \quad \rho = e^{2\pi i/3}. \]

Picard modular forms were extensively studied by Holzapfel, Feustel, Finis, Shiga, Cléry and van der Geer.
Differential $dg$ via Picard modular forms

There is a simple expression of the differential $dg$ in terms of $\varphi_\nu$:

$$dg = \frac{\varphi_1 \varphi_2 (\varphi_2 - \varphi_1) d\varphi_0 + \varphi_0 \varphi_2 (\varphi_0 - \varphi_2) d\varphi_1 + \varphi_0 \varphi_1 (\varphi_1 - \varphi_0) d\varphi_2}{\zeta^2}$$

where $\zeta$ is a modular form defined as

$$\zeta^3 = \varphi_0 \varphi_1 \varphi_2 (\varphi_1 - \varphi_0) (\varphi_2 - \varphi_0) (\varphi_2 - \varphi_1).$$

Up to a constant factor, the differential $dg$ coincides with the Eisenstein series $E_{1,1}$ which was known before from the theory of vector-valued Picard modular forms.


**Generic Lagrangian densities** $f(v_{x_1}, v_{x_2}, v_{x_3})$

Euler-Lagrange equation:

$$(f_{v_{x_1}})_1 + (f_{v_{x_2}})_2 + (f_{v_{x_3}})_3 = 0.$$ 

Integrability conditions lead to a system of fifteen PDEs for $f(x, y, z)$ which are invariant under 20-dimensional symmetry group:

$$\tilde{x} = \frac{l_1(x, y, z)}{l(x, y, z)}, \quad \tilde{y} = \frac{l_2(x, y, z)}{l(x, y, z)}, \quad \tilde{z} = \frac{l_3(x, y, z)}{l(x, y, z)}, \quad \tilde{f} = \frac{f}{l(x, y, z)},$$

as well as obvious symmetries of the form

$$\tilde{f} = \epsilon f + \alpha x + \beta y + \gamma z + \delta,$$

where $l, l_1, l_2, l_3$ are arbitrary (inhomogeneous) linear forms. This invariance allows one to linearise the integrability conditions for $f(x, y, z)$. 
**Auxiliary hypergeometric system**

Consider the auxiliary (Appell) hypergeometric system

\[ h_{u_i u_j} = \frac{1}{3} \frac{h_{u_i} - h_{u_j}}{u_i - u_j}, \]

\[ h_{u_i u_i} = -\frac{2}{9} \frac{h}{u_i(u_i - 1)} - \frac{1}{3u_i(u_i - 1)} \sum_{j \neq i}^{3} \frac{u_j(u_j - 1)}{u_j - u_i} h_{u_j} + \]

\[ -\frac{1}{3} \left( \sum_{j \neq i}^{3} \frac{1}{u_i - u_j} + \frac{2}{u_i} + \frac{2}{u_i - 1} \right) h_{u_i}. \]

The geometry behind this system is the family of genus 4 Picard trigonal curves

\[ r^3 = q(q - 1)(q - u_1)(q - u_2)(q - u_3) \]

supplied with the holomorphic differential \( \omega = dq/r \). The corresponding periods, \( h = \int_{a}^{b} \omega \) where \( a, b \in \{0, 1, \infty, u_1, u_2, u_3\} \), form a 4-dimensional vector space and satisfy the above (Picard-Fuchs) hypergeometric system.
Inhomogeneous hypergeometric extension

We will also need the inhomogeneous hypergeometric system

\[ F_{u_i u_j} = \frac{1}{3} \frac{F_{u_i} - F_{u_j}}{u_i - u_j} + \epsilon_{ijk} \frac{u_k (u_k - 1)(u_i - u_j)}{U^{2/3}}, \]

\[ F_{u_i u_i} = -\frac{2}{9} \frac{F}{u_i (u_i - 1)} - \frac{1}{3u_i (u_i - 1)} \sum_{j \neq i} \frac{u_j (u_j - 1)}{u_j - u_i} F_{u_j} + \]

\[ -\frac{1}{3} \left( \sum_{j \neq i} \frac{1}{u_i - u_j} + \frac{2}{u_i} + \frac{2}{u_i - 1} \right) F_{u_i}, \]

where \( \epsilon_{ijk} \) is the totally antisymmetric tensor and

\[ U = u_1 u_2 u_3 (u_1 - 1)(u_2 - 1)(u_3 - 1)(u_1 - u_2)(u_2 - u_3)(u_3 - u_1). \]

The inhomogeneous system for \( F \) is in involution.
**Generic solution** \( f(x, y, z) \)

The generic solution \( f(x, y, z) \) can be represented in any of the 3 equivalent forms:

1. **Parametric form:**
   \[
   x = \frac{h_1}{h_4}, \quad y = \frac{h_2}{h_4}, \quad z = \frac{h_3}{h_4}, \quad f = \frac{F}{h_4}
   \]
   where \( h_i(u_1, u_2, u_3) \) are four independent solutions of the hypergeometric system and \( F(u_1, u_2, u_3) \) is a solution of the inhomogeneous system.

2. **Theta representation:**
   \[
   f(x, y, z) = xy + \sum_{(k, l) \in \mathbb{Z}^2 \setminus 0} \frac{\sigma((k + \epsilon l)x)\sigma((k + \epsilon l)y)}{(k + \epsilon l)^2} e^{2\pi i(k^2 + kl + l^2)z}.
   \]

3. **Power series:**
   \[
   f(x, y, z) = \sum_{i, j, k \geq 0} B_i B_j B_k B_{i+j+k} \frac{x^{6i+1}}{(6i + 1)!} \frac{y^{6j+1}}{(6j + 1)!} \frac{z^{6k+1}}{(6k + 1)!}.
   \]
Concluding remarks

- There exists integrable Lagrangians whose densities $f$ are polynomial, or can be expressed in terms of elementary functions. It would be interesting to clarify how these (and similar) examples can be recovered as degenerations of the ‘master-Lagrangian’, and to describe singular orbits of lower dimension. This should be related to understanding degenerations/compactifications of the moduli space of Picard curves.

- Although our parametric, theta and power series representations for integrable densities possess straightforward generalisations to dimensions higher than 3, the relation to integrable Lagrangians will be lost: one can show that in higher dimensions every integrable first-order Lagrangian density $f(v_x)$ is necessarily of the form $f = \frac{Q(v_x)}{l(v_x)}$ where $Q$ and $l$ are arbitrary quadratic and linear functions of the first-order derivatives. Thus, the occurrence of modular forms is the essentially 3-dimensional phenomenon.